**MTH 309-4** **Linear Algebra I** **F11**

**Homework 8/Solutions**

| Section | Exercises |
|---|---|
| 6.2 | 1,2,9,12,16,21 |

(**Section 6.2 Exercise 2**). *For each of the following functions, either show the function is onto by choosing an arbitrary element of the codomain and finding an element of the domain that the function maps to the chosen element, or show the function is not onto by finding an element of the codomain that is not in the image of the function.*

*(a)* $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = \frac{1}{3}x - 2$.

*(b)* $p : \mathbb{R} \to \mathbb{R}$ *defined by* $p(x) = x^2 - 3x + 2$.

*(c)* $s : \mathbb{R} \to \mathbb{R}$ *defined by* $s(x) = (e^x - e^{-x})/2$.

*(d)* $W : \mathbb{R} \to \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ *defined by* $W(t) = (\cos t, \sin t)$.

*(e)* $L : \mathbb{R}^3 \to \mathbb{R}^3$ *defined by* $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + y - z \\ -x + 2z \\ x + y + z \end{pmatrix}$.

(a): Let $x, y \in \mathbb{R}$. Then

$$\begin{array}{rccc} & f(x) & = & y \\ \iff & \frac{1}{3}x - 2 & = & y \\ \iff & \frac{1}{3}x = y + 2 & & \\ \iff & \overset{x}{=} 3(y + 2) & & \end{array}$$

If follows that $f\big(3(y + 2)\big) = y$ and so $f$ is onto.

(b) Let $x \in \mathbb{R}$. Then $p(x) = (x^2 - 3x + 2) = (x - \frac{3}{2})^2 - (\frac{3}{2})^2 + 2 = (x\frac{3}{2})^2 - \frac{1}{4} \leq \frac{1}{4}$. So $f(x) \neq -1$ for all $x \in \mathbb{R}$ and so $f$ is not onto.

(c) Let $u \in \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Then $u = (x, y)$ for some $x, y \in \mathbb{R}$ with $x^2 + y^2 = 1$. Let $(r, t)$ be the polar coordinates of $(x, y)$. So $x = r\cos t$ and $y = r \sin t$. Since $x^2 + y^2 = 1$ we get $r = \sqrt{x^2 + y^2} = 1$ and so

$$u = (x, y) = (r\cos t, r\sin t) = (\cos t, \sin t) = W(t)$$

Thus $W$ is onto.

(d) Let $x, y \in \mathbb{R}$ and put $z = e^x$. Then $z > 0$. We have $z + \frac{1}{z} = 2y$ iff $z^2 - 2yz = 1$ iff $(z-y)^2 = 1 + y^2$ and iff $z = y \pm \sqrt{1+y^2}$. Since $z$ is positive this holds if and only if $z = y + \sqrt{1+y^2}$.

It follows that $(e^x - e^{-x})/2 = y$ iff $z + \frac{1}{z} = 2y$ and iff $x = \ln\left(y + \sqrt{1+y^2}\right)$. Hence $s$ is onto.

(e) Let $u \in \mathbb{R}^3$. Then $u = (a, b, c)$ for some $a, b, c \in \mathbb{R}^3$. To show that $L$ is onto we need the find $(x, y, z) \in \mathbb{R}^3$ with $L(x, y, z) = (a, b, c)$. This is a linear system of equation and we can solve it with the Gauss Jordan algorithm

$$\begin{bmatrix} 2 & 1 & -1 & a \\ -1 & 0 & 2 & b \\ 1 & 1 & 1 & c \end{bmatrix} \xrightarrow[\substack{-2R3+R1 \to R1 \\ R1 \leftrightarrow R3}]{R3+R2 \to R2} \begin{bmatrix} 1 & 1 & 1 & c \\ 0 & 1 & 3 & b+c \\ 0 & -1 & -3 & a-2c \end{bmatrix} \xrightarrow[R2+R3 \to R3]{-R2+R1 \to R1} \begin{bmatrix} 1 & 0 & -2 & -b \\ 0 & 1 & 3 & b+c \\ 0 & 0 & 0 & a+b-c \end{bmatrix}$$

If $a + b - c \neq 0$, this has no solutions. So for example $(1, 0, 0)$ is not in the image of $L$.

(**Section 6.2 Exercise 12**). *Prove that the composition of 1-1 functions is 1-1.*

Let $f : I \to J$ and $g : J \to K$ be 1-1 functions. Let $a, b \in I$ with

$$(g \circ f)(a) = (g \circ f)(b).$$

Then the definition of composition yields

$$g\Big(f(a)\Big) = g\Big(f(b)\Big).$$

Since $g$ is 1-1 this implies

$$f(a) = f(b)$$

Since $f$ is 1-1 we conclude that $a = b$. So $g \circ f$ is 1-1.

(**Section 6.2 Exercise 16**). *For an element $v_0$ of a vector space $V$, consider the translation function $\tau_{v_0} : V \to V$ defined by $\tau_{v_0}(v) = v + v_0$. Show that $\tau_{v_0}$ is invertible. Show that $\tau_{v_0}^{-1}$ is also a translation function.*

Let $v, w \in V$. Then

$$\begin{array}{ccl} & \tau_{v_0}(v) = w & \\ \iff & v + v_0 = w & \text{– definition of } \tau_{v_0} \\ \iff & (v + v_0) + (-v_0) = w + (-v_0) & \text{– Cancellation Theorem 1.8} \\ \iff & v = w + (-v_0) & \text{– Lemma N1.3.1} \\ \iff & v = \tau_{-v_0}(w) & \text{– definition of } \tau_{-v_0} \end{array}$$

Thus by Lemma A.5.6 in the appendix of the notes, $\tau_{-v_0}$ is an inverse of $\tau_{v_0}$. So $\tau_{v_0}$ is invertible and $\tau_{v_0}^{-1}$ is the translation function $\tau_{-v_0}$.

**A.** *Let $I$ be a set, $V$ a vector space and $F(I,V)$ the set of function from $I$ to $V$. For $r \in \mathbb{R}$ and $f, g \in F(I,V)$ define the the functions $f+g$ and $rf$ from $I$ to $V$ by*

$$(f+g)(i) = f(i) + g(i) \text{ and } (rf)(i) = r\big(f(i)\big)$$

*for all $i \in I$. Prove that $F(I,V)$ with these operations is a vector space.*

Properties (i),(ii) and (iii) of a vector space hold by definition of $F(I,V)$. We will now verify the eight axioms of a vector space one by one. From Lemma A.2.2 in the appendix of the notes we have

(*) Let $f, g \in F(I,V)$. Then $f = g$ if and only if $f(i) = g(i)$ for all $i \in I$.

Let $f, g, h \in F(I,V)$, $a, b \in \mathbb{R}$ and $i \in I$.

**Ax 1**: We have

$$\begin{aligned}
(f+g)(i) &= f(i) + g(i) && \text{−Definition of '+' for functions} \\
&= g(i) + f(i) && \text{−Axiom 1 of } V \\
&= (g+f)(i) && \text{−Definition of '+' for functions}
\end{aligned}$$

So $f + g = g + f$ by (*) and **Ax 1** is proved.

**Ax 2**: We have

$$\begin{aligned}
\Big((f+g)+h\Big)(i) &= (f+g)(i) + h(i) && \text{−Definition of '+' for functions} \\
&= \Big(f(i) + g(i)\Big) + h(i) && \text{−Definition of '+' for functions} \\
&= f(i) + \Big(g(i) + h(i)\Big) && \text{−Axiom 2 of } V \\
&= f(i) + (g+h)(i) && \text{−Definition of '+' for functions} \\
&= \Big(f + (g+h)\Big)(i) && \text{−Definition of '+' for functions}
\end{aligned}$$

So $(f+g)+h = f+(g+h)$ by (*) and **Ax 2** is proved.

**Ax 3**: Define a function, denoted by $0^*$, in $F(I,V)$ by $0^*(i) = \mathbf{0}$ for all $i$ in $I$. We will show that $0^*$ is an additive identity:

$$\begin{aligned}(f+0^*)(i) &= f(i)+0^*(i) && \text{–Definition of '+' for functions}\\ &= f(i)+0 && \text{–Definition of } 0^*\\ &= f(i) && \text{–Axiom 3 of } V\end{aligned}$$

So $f+0^*=f$ by (*) and **Ax 3** is proved.

**Ax 4** Define a function, denoted by $-f$, in $F(I,V)$ by $(-f)(i)=-(f(i))$ for all $i$. We will show that $-f$ is an additive inverse of $f$.

$$\begin{aligned}\Big(f+(-f)\Big)(i) &= f(i)+(-f)(i) && \text{–Definition of '+' for functions}\\ &= f(i)+(-(f(i)) && \text{–Definition of } -f\\ &= \mathbf{0} && \text{–Axiom 4 of } V\\ &= 0^*(i) && \text{–Definition of } 0^*\end{aligned}$$

So $f+(-f)=0^*$ by (*) and **Ax 4** is proved.
**Ax 5**: We have

$$\begin{aligned}\Big(a(f+g)\Big)(i) &= a\Big((f+g)(i)\Big) && \text{–Definition of multiplications for functions}\\ &= a\Big(f(i)+g(i)\Big) && \text{–Definition of '+' for functions}\\ &= a\big(f(i)\big)+a\big(g(i)\big) && \text{–Axiom 5 of } V\\ &= (af)(i)+(ag)(i) && \text{–Definition of multiplications for functions}\\ &= (af+ag)(i) && \text{–Definition of '+' for functions}\end{aligned}$$

So $a(f+g)=af+ag$ by (*) and **Ax 5** is proved.

**Ax 6**: We have

$$\begin{aligned}\big((a+b)f\big)(i) &= \big(a+b\big)\big(f(i)\big) && \text{–Definition of multiplications for functions}\\ &= a\big(f(i)\big)+b\big(f(i)\big) && \text{–Axiom 6 of } V\\ &= (af)(i)+(bf)(i) && \text{–Definition of multiplications for functions}\\ &= (af+bf)(i) && \text{–Definition of '+' for functions}\end{aligned}$$

So $(a+b)f=af+bf$ by (*) and **Ax 6** is proved.

**Ax 7**: We have

$$\begin{aligned}
\big((ab)f\big)(i) &= (ab)\big(f(i)\big) && \text{– Definition of multiplications for functions}\\
&= a\Big(b\big(f(i)\big)\Big) && \text{– Axiom 7 of } V\\
&= a\Big((bf)(i)\Big) && \text{– Definition of multiplications for functions}\\
&= \Big(a(bf)\Big)(i) && \text{– Definition of multiplications for functions}
\end{aligned}$$

So $(ab)f = a(bf)$ by (*) and **Ax 7** is proved.

**Ax 8**

$$\begin{aligned}
(1f)(i) &= 1\big(f(i)\big) && \text{– Definition of multiplications for functions}\\
&= f(i) && \text{– Axiom 8 of } V
\end{aligned}$$

So $1f = f$ by (*) and **Ax 8** is proved.

**B.** *Let $T : V \to W$ be linear and let $Y$ be a subspace of $W$. Put*

$$X = \{x \in V \mid T(x) \in Y\}$$

*Show that $X$ is a subspace of $V$.*

Observe that by definition of $X$:

(*) Let $v \in V$. Then $v \in X$ if and only if $T(v) \in Y$.

We will now verify the three conditions of the subspace theorem.

(1) Since $T$ is linear, Theorem 6.2 yields $T(\mathbf{0}) = \mathbf{0}$. Since $Y$ is subspace of $W$, the subspace theorem gives $\mathbf{0} \in Y$. So $T(\mathbf{0}) \in Y$ and thus $\mathbf{0} \in X$ by (*). So Condition (1) of the subspace theorem holds.

(2) Let $a, b \in X$. By (*), $T(a) \in Y$ and $T(b) \in Y$. Since $Y$ is subspace of $W$, the subspace theorem gives $T(a) + T(b) \in Y$. Since $T$ is linear, $T(a + b) = T(a) + T(b)$ and so $T(a + b) \in Y$. Thus by (*), $a + b \in X$. So Condition (2) of the subspace theorem holds.

(3) Let $a \in X$ and $r \in \mathbb{R}$. By (*), $T(a) \in Y$. Since $Y$ is subspace of $W$, the subspace theorem gives $r\big(T(a)\big) \in Y$. Since $T$ is linear, $r\big(T(a)\big) = T(ra)$ and so $T(ra) \in Y$. Thus by (*), $ra \in X$. So Condition (3) of the subspace theorem holds.

We verified the three conditions of the subspace theorem and so $X$ is a subspace of $V$.