MTH 309-4 **Linear Algebra I** **F11**

Homework 7/Solutions

Section	Exercises
3.5	6,9,21
6.1	1,2,4,6,9,10,19

(**Section 3.5 Exercise 21**). *Let* $\mathbf{S}$ *be a subspace of a finite dimensional vector space* $\mathbf{V}$*. Prove that if* $\dim \mathbf{S} = \dim \mathbf{V}$*, then* $S = V$*.*

Let $n = \dim \mathbf{V}$. Then also $n = \dim \mathbf{S}$ and so by N3.4.4 $\mathbf{S}$ has a basis $(s_1, \ldots, s_n)$ of length n. Since $n = \dim \mathbf{V}$, N3.5.4 implies that $(s_1, \ldots, s_n)$ is also basis for $\mathbf{V}$. Since bases are spanning lists we conclude

$$V = \operatorname{span}(s_1, \ldots, s_n) = S.$$

(**Section 6.1 Exercise 1**). *Show that* $P : \mathbb{R}^3 \to \mathbb{R}^3$ *defined by* $P\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ *is linear. Find a matrix* A *such that* $P\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{pmatrix} x \\ y \\ z \end{pmatrix}$*.*

Put $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = P\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Hence $P = L_A$ and so by N6.1.7, P is linear.

A. *Let* V *be a vector space and* U *and* W *subspaces of* V *with* $U \cap W = \{\mathbf{0}\}$*. Let* $(u_1, \ldots, u_n)$ *be a linearly independent list in* U *and* $(w_1, \ldots, w_m)$ *a linearly independent list in* W*. Show that*

$$(u_1, \ldots, u_n, w_1, \ldots, w_m)$$

is a linearly independent list in V*.*

Let $r_1, \ldots, r_n, s_1, \ldots, s_m$ be in $\mathbb{R}$ with

$$(*) \qquad r_1u_1 + \ldots + r_nu_n + s_1w_1 + \ldots + s_mw_m = \mathbf{0}$$

Put $u = r_1u_1 + \ldots + r_nu_n$ and $w = s_1w_1 + \ldots s_mw_m$. Since U and W are subspaces, Theorem 3.3 shows that $u \in U$ and $w \in W$. By (*), $u + w = \mathbf{0}$ and so $u = -w$. Since W

is a subspace $u = -w = (-1)w \in W$. So $u \in U \cap W$ and since $U \cap W = \{\mathbf{0}\}$ we conclude that $u = \mathbf{0}$. From $u + w = \mathbf{0}$ we also get $w = \mathbf{0}$. Thus by definition of u and w:

$$r_1u_1 + \dots r_nu_n = \mathbf{0} \text{ and } s_1w_1 + \dots s_mw_m = \mathbf{0}$$

Since $(u_1, \dots, u_n)$ and $(w_1, \dots, w_m)$ are linearly independent this implies $r_1 = 0, r_2 = 0, \dots, r_n = 0$ and $s_1 = 0, \dots, s_m = 0$. So $(u_1, \dots, u_n, w_1, \dots, w_m)$ is linearly independent.

(**Section 6.1 Exercise 19**). *Suppose the linear function* $L : \mathbb{M}(2,2) \to \mathbb{M}(2,2)$ *satisfies*

$$L\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right) = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \quad L\left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right) = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$$

$$L\left(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right) = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \quad L\left(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right) = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

Show that $L\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\right) = \begin{bmatrix}a & c\\b & d\end{bmatrix}$ *for all matrices* $\begin{bmatrix}a & b\\c & d\end{bmatrix}$ *in* $\mathbb{M}(2,2)$.

By definition of scalar multiplication and addition for matrices:

$$\begin{bmatrix}a & b\\c & d\end{bmatrix} = a\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + b\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + c\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + d\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

Applying L to both sides of this equation and using Theorem 6.2(c) we conclude

$$L\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\right) = a\,L\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right) + b\,L\left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right) + c\,L\left(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right) + d\,L\left(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right)$$

and so by the assumption of the exercise:

$$L\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\right) = a\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + b\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + c\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + d\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

So using the definition of scalar multiplication and addition for matrices:

$$L\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\right) = \begin{bmatrix}a & c\\b & d\end{bmatrix}$$

B. *Let*

$$A = L\begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 1 & 3 & 4 & 1 & 2 \\ 2 & 5 & 7 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

Find bases for $\mathrm{Col}A, \mathrm{Row}A$ *and* $\mathrm{Nul}A$.

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 1 & 3 & 4 & 1 & 2 \\ 2 & 5 & 7 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix} \begin{matrix} R2 - R1 \to R1 \\ R3 - 2R1 \to R2 \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix} \begin{matrix} R1 - 2R2 \to R1 \\ R3 - R2 \to R3 \\ R4 - R2 \to R4 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & -2 & -4 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix}$$

$$\begin{matrix} R1 - R4 \to R1 \\ R2 - R4 \to R2 \\ -R_3 \to R_3 \\ R3 \leftrightarrow R4 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & -4 & -6 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R1 + 6R4 \to R4 \\ R3 - 2R4 \to R3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & -4 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The lead variables are x_1, x_2, x_3, x_5 and so by Theorem N3.7.5 columns $1, 2, 3$ and 5 of A form a basis for $\mathrm{Col}A$. So

$$\left(\begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 7 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 4 \end{pmatrix} \right)$$

is a basis for $\mathrm{Col}A$.

By Theorem N3.7.5 non-zero rows of B form a basis for $\mathrm{Col}A$. Hence

$$\Big((1, 0, 0, -4, 0), (0, 1, 0, -1, 0), (0, 0, 1, 2, 0), (0, 0, 0, 0, 1) \Big)$$

is a basis for $\mathrm{Row}A$.

Using the reduced row echelon form to solve for the lead variables we obtain

$$x_1 = 4x_4, \quad x_2 = 1x_4, \quad x_3 = -2x_4, \quad x_4 = 1x_4, \quad x_5 = 0x_4,$$

and so by Theorem N3.7.5

$$\Big((4,1,-2,1,0)\Big)$$

is a basis for Nul$A$

C. *Consider the following vectors in* $\mathbb{R}^6$*:*

$$u_1 = \begin{pmatrix}1\\0\\1\\0\\1\\0\end{pmatrix}, u_2 = \begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix}, u_3 = \begin{pmatrix}3\\1\\3\\1\\3\\1\end{pmatrix}, w_1 = \begin{pmatrix}1\\1\\1\\0\\0\\0\end{pmatrix}, w_2 = \begin{pmatrix}0\\0\\0\\1\\1\\1\end{pmatrix}, \text{ and } w_3 = \begin{pmatrix}1\\2\\3\\4\\5\\6\end{pmatrix}.$$

Put $U = \text{span}(u_1, u_2, u_3)$ *and* $W = \text{span}(u_1, u_2, u_3, w_1, w_2, w_3)$*. Find a sublist* $(x_1, \ldots, x_n)$ *of* (u_1, u_2, u_3) *and a sublist* $(y_1, \ldots, y_m)$ *of* (w_1, w_2, w_3)*, such that* $(x_1, \ldots, x_n)$ *is basis for* U *and* $(x_1, \ldots, x_n, y_1, \ldots, y_m)$ *is a basis for* W*.*

We compute the reduced row echelon form of the matrix A formed by $u_1, u_2, u_3, w_1, w_2, w_3$:

$$A = \begin{bmatrix}1&1&3&1&0&1\\0&1&1&1&0&2\\1&1&3&1&0&3\\0&1&1&0&1&4\\1&1&3&0&1&5\\0&1&1&0&1&6\end{bmatrix} \xrightarrow[R5-R1\to R5]{R3-R1\to R3} \begin{bmatrix}1&1&3&1&0&1\\0&1&1&1&0&2\\0&0&0&0&0&2\\0&1&1&0&1&4\\0&0&0&-1&1&4\\0&1&1&0&1&6\end{bmatrix} \xrightarrow[\substack{R4-R2\to R4\\ R6-R2\to R2}]{\substack{R1-R2\to R1\\ \frac{1}{2}R3\to R3}} \begin{bmatrix}1&0&2&0&0&-1\\0&1&1&1&0&2\\0&0&0&0&0&1\\0&0&0&-1&1&2\\0&0&0&-1&1&4\\0&0&0&-1&1&4\end{bmatrix}$$

$$\xrightarrow[\substack{R5-R4\to R5\\ -R4\to R4}]{\substack{R2+R4\to R2\\ R6-R5\to R6}} \begin{bmatrix}1&0&2&0&0&-1\\0&1&1&0&1&4\\0&0&0&0&0&1\\0&0&0&1&-1&-2\\0&0&0&0&0&2\\0&0&0&0&0&0\end{bmatrix} \xrightarrow[\substack{R5-2R3\to R5\\ R3\leftrightarrow R4}]{\substack{R1+R3\to R1\\ R2-4R3\to R2\\ R4+2R3\to R4}} \begin{bmatrix}1&0&2&0&0&0\\0&1&1&0&1&0\\0&0&0&1&-1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$$

So the leading variables are x_1, x_2, x_4 and x_6. By Theorem 3.7.5 (applied to the first three columns of A), (a_1, a_2) is a basis for $\text{Col}[a_1, a_2, a_3]$ and by Theorem 3.7.5 applied to A, (a_1, a_2, a_4) is basis for $\text{Col}A$. Since $a_i = u_i$ and $a_{i+3} = w_i$ for $1 \leq i \leq 3$ we conclude that (u_1, u_2) is a basis for $\text{Col}[u_1, u_2, u_3] = \text{span}(u_1, u_2, u_2) = U$, and (u_1, u_2, w_1) is a basis for $\text{Col}A = \text{Col}[u_1, u_2, u_3, w_1, w_2, w_3] = \text{span}(u_1, u_2, u_3, w_1, w_2, w_3) = W$.