Homework 6/Solutions

Section	Exercises
3.6	1ab,2defi,4b,5af
3.4	1,2,7,8,9,12,13

(**Section 3.6 Exercise 2** defi). *Consider the basis*

$$B = \left(\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right)$$

for $\mathbb{R}^3$. *Find the following coordinate vectors:*

d. $\left[\begin{pmatrix} 5 \\ 6 \\ 2 \end{pmatrix} \right]_B$ *e.* $\left[\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \right]_B$ *f.* $\left[\begin{pmatrix} 5 \\ 6 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \right]_B$ *i.* $\left[\left[\begin{pmatrix} 5 \\ 6 \\ 2 \end{pmatrix} \right]_B \right]_B$

We solve d. and e. simultaneously :

$$\begin{bmatrix} 2 & 1 & 0 & 5 & 2 \\ 0 & 1 & 1 & 6 & 1 \\ 3 & 1 & -1 & 2 & 5 \end{bmatrix} \xrightarrow[R1 \leftrightarrow R3]{R3 - R1 \to R3} \begin{bmatrix} 1 & 0 & -1 & -3 & 3 \\ 0 & 1 & 1 & 6 & 1 \\ 2 & 1 & 0 & 5 & 2 \end{bmatrix} \xrightarrow{R3 - 2R1 \to R3} \begin{bmatrix} 1 & 0 & -1 & -3 & 3 \\ 0 & 1 & 1 & 6 & 1 \\ 0 & 1 & 2 & 11 & -4 \end{bmatrix}$$

$$\xrightarrow{R3 - R2 \to R3} \begin{bmatrix} 1 & 0 & -1 & -3 & 3 \\ 0 & 1 & 1 & 6 & 1 \\ 0 & 0 & 1 & 5 & -5 \end{bmatrix} \xrightarrow[R2 - R3 \to R2]{R1 + R3 \to R1} \begin{bmatrix} 1 & 0 & 0 & 2 & -2 \\ 0 & 1 & 0 & 1 & 6 \\ 0 & 0 & 1 & 5 & -5 \end{bmatrix}$$

Thus

$$\left[\begin{pmatrix} 5 \\ 6 \\ 2 \end{pmatrix} \right]_B = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \text{ and } \left[\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \right]_B = \begin{pmatrix} -2 \\ 6 \\ -5 \end{pmatrix}$$

For f. we compute

$$\left[\begin{pmatrix}5\\6\\2\end{pmatrix}+\begin{pmatrix}2\\1\\5\end{pmatrix}\right]_B=\left[\begin{pmatrix}7\\7\\7\end{pmatrix}\right]=\left[7\begin{pmatrix}1\\1\\1\end{pmatrix}\right]_B=\begin{pmatrix}0\\7\\0\end{pmatrix}$$

where the last equality holds since $(1,1,1)$ is the second member of B.

Using d. and e. we compute

$$\left[\left[\begin{pmatrix}5\\6\\2\end{pmatrix}\right]_B\right]_B=\left[\begin{pmatrix}2\\1\\5\end{pmatrix}\right]_B=\begin{pmatrix}-2\\6\\-5\end{pmatrix}$$

(**Section 3.6 Exercise 5**af)**.** *Use the polynomials*

$$p_1=x^2+1,\quad p_2=x^2+x+2,\quad p_3=3x-1$$

to form a basis $B=(p_1,p_2,p_3)$ *of* $\mathbb{P}_2$*. Compute the following coordinate vectors:*

a. $[x^2+x+1]_B$ *f.* $\left[p_3^2\right]_B$

Since $x^2+x+1=0p_1+1p_2+0p_3$, $[x^2+x+1]_B=(0,1,0)$.

We have $p_3^2=(3x-1)^2=9x^2-6x+1$. Let $a,b,c\in\mathbb{R}^3$. Then

$$a(x^2+1)+b(x^2+x+2)+c(3x-1)=9x^2-6x+1$$

if and only if

$$a+b=9, b+3c=-6 \text{ and } a+2b-c=1$$

We use the Gauss Jordan algorithm to solve this linear system of equations:

$$\begin{bmatrix}1&1&0&9\\0&1&3&-6\\1&2&-1&1\end{bmatrix}\xrightarrow[R1-R2\to R1]{R3-R1\to R3}\begin{bmatrix}1&0&-3&15\\0&1&3&-6\\0&1&-1&-8\end{bmatrix}\xrightarrow{R3-R2\to R3}$$

$$\begin{bmatrix}1&0&-3&15\\0&1&3&-6\\0&0&-4&-2\end{bmatrix}\xrightarrow{-\frac{1}{4}R_3\to R_3}\begin{bmatrix}1&0&-3&15\\0&1&3&-6\\0&0&1&\frac{1}{2}\end{bmatrix}\xrightarrow[R2-3R3\to R2]{R1+3R3\to R1}\begin{bmatrix}1&0&0&\frac{33}{2}\\0&1&0&-\frac{15}{2}\\0&0&1&\frac{1}{2}\end{bmatrix}$$

So $\left[p_3^2\right]_B=\frac{1}{2}(33,-15,1)$.

(Section 3.4 Exercise 7). *Show that the polynomials*

$$p_1 = x^2 + 1, \quad p_2 = 2x^2 + x - 1, \quad p_3 = x^2 + x$$

form a basis for $\mathbb{P}_2$.

Let $p = rx^2 + sx + t \in \mathbb{P}_2$ and $a, b, c \in \mathbb{R}$. Then

$$ap_1 + bp_2 + cp_3 = p$$

if and only if

$$a + 2b + c = r, \quad b + c = s, \quad \text{and} \quad a - b = t$$

We use the Gauss Jordan algorithm to solve this linear system of equations

$$\begin{bmatrix} 1 & 2 & 1 & r \\ 0 & 1 & 1 & s \\ 1 & -1 & 0 & t \end{bmatrix} \xrightarrow{R3 - R1 \to R3} \begin{bmatrix} 1 & 2 & 1 & r \\ 0 & 1 & 1 & s \\ 0 & -3 & -1 & -r+t \end{bmatrix} \xrightarrow[R3 + 3R2 \to R3]{R1 - 2R2 \to R1} \begin{bmatrix} 1 & 0 & -1 & r-2s \\ 0 & 1 & 1 & s \\ 0 & 0 & 2 & -r+3s+t \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R3 \to R3} \begin{bmatrix} 1 & 0 & -1 & r-2s \\ 0 & 1 & 1 & s \\ 0 & 0 & 1 & -\frac{1}{2}r + \frac{3}{2}s + \frac{1}{2}t \end{bmatrix} \xrightarrow[R2 - R3 \to R2]{R1 + R3 \to R3} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2}r - \frac{1}{2}s + \frac{1}{2}t \\ 0 & 1 & 0 & \frac{1}{2}r - \frac{1}{2}s - \frac{1}{2}t \\ 0 & 0 & 1 & -\frac{1}{2}r + \frac{3}{2}s + \frac{1}{2}t \end{bmatrix}$$

So for each $p \in \mathbb{P}_2$ there exist unique $a, b, c \in \mathbb{R}$ with $p = ap_1 + bp_2 + cp_3$. Thus by Theorem 3.17 (p_1, p_2, p_3) is a basis for $\mathbb{P}_2$.

(Section 3.4 Exercise 13). *(a) Prove that if (v, w) is a basis for a vector space, then $(2v, w)$ is a basis for the vector space.*

(b) Suppose (v, w) is a basis for a vector space. For what scalars a and b will (av, bw) be a basis for the vector space? Prove your claim.

(c) Generalize your claim in part b to bases with more than two elements.

(c) Let $(v_1, \ldots, v_n)$ be basis for the vector space $\mathbf{V}$. Let $(a_1, \ldots, a_n)$ be a list in $\mathbb{R}$. We will show that $(a_1v_1, \ldots, a_nv_n)$ is a basis for $\mathbf{V}$ if and only if $a_i \neq 0$ for all $1 \leq i \leq n$.

Suppose first that $(a_1v_1, \ldots, a_nv_n)$ is a basis. Then $(a_1v_1, \ldots, a_nv_n)$ is linearly independent. Hence by Theorem 3.5 $a_iv_i \notin \text{span}(a_1v_1, \ldots, a_{i-1}v_{i-1}, \ldots a_nv_n)$ for all $1 \leq i \leq n$. Hence $a_iv_i \neq \mathbf{0}$ and so by Theorem 1.4, $a_i \neq 0$.

Suppose next that $a_i \neq 0$ for all $1 \leq i \leq n$. Let $v \in V$. Since $(v_1, \ldots, v_n)$ is a basis Theorem 3.17 shows that

(*) there exist a unique $(r_1, \ldots, r_n) \in \mathbb{R}$ with $r_1 v_1 + \ldots + r_n v_n = v$.

Let $(s_1, \ldots, s_n) \in \mathbb{R}$. Then

$$\begin{aligned} & s_1(a_1 v_1) + \ldots + s_n(a_n v_n) = v & \\ \iff\ & (s_1 a_1) v_1 + \ldots + (s_n a_n) v_n = v & \text{– Axiom 7} \\ \iff\ & s_1 a_1 = r_1, \ldots, s_n a_n = r_n & \text{– (*)} \\ \iff\ & s_1 = r_1 a_1^{-1}, \ldots, s_n = r_n a_n^{-1} & \text{– } a_i \neq 0, \text{ Property of } \mathbb{R} \end{aligned}$$

So for each $v \in V$ there exist a unique $(s_1, \ldots, s_n) \in \mathbb{R}$ with $s_1(a_1 v_1) + \ldots s_n(a_n v_n) = v$. Thus by Theorem 3.17, $(a_1 v_1, \ldots, a_n v_n)$ is basis for $\mathbf{V}$.

(a) Note that $(2v, w) = (2v, 1w)$, $2 \neq 0$ and $1 \neq 0$. Thus (a) follows from (c).

(b) By (c), (av, bw) is a basis if and only if $a \neq 0$ and $b \neq 0$.

A. *Let* $\mathbf{V}$ *be a vector space and* $(v_1, \ldots, v_n)$ *a linearly independent list in* $\mathbf{V}$ *. Show that the following two statements are equivalent:*

(a) $(v_1, \ldots, v_n)$ *is a basis for* $\mathbf{V}$*.*

(b) $(v_1, \ldots, v_n, v)$ *is linearly dependent in* $\mathbf{V}$ *for all* $v \in V$*.*

(a) $\Longrightarrow$ (b): Suppose first that $(v_1, \ldots, v_n)$ is a basis for $\mathbf{V}$ and let $v \in V$. Then $(v_1, \ldots, v_n)$ spans $\mathbf{V}$ and so $v \in \operatorname{span}(v_1, \ldots, v_n)$. Thus by Theorem 3.5, $(v_1, \ldots, v_n, v)$ is linearly dependent.

(b) $\Longrightarrow$ (a): Suppose that (b) holds and let $v \in V$. Then $(v_1, \ldots, v_n, v)$ is linearly dependent and so there exist $r_1, \ldots, r_n, r \in \mathbb{R}$, not all zero, such that

$$r_1 v_1 + \ldots + r_n v_n + rv = \mathbf{0}$$

Suppose that $r = 0$. Then $r_1 v_1 + \ldots + r_n v_n = \mathbf{0}$. Since $(v_1, \ldots, v_n)$ is linearly independent, we conclude that $r_1 = 0, r_2 = 0, \ldots, r_n = 0$. Since also $r = 0$ this contradicts the choice of $r_1, \ldots, r_n, r$.

Thus $r \neq 0$. Hence by Lemma N3.3.2, $v \in \operatorname{span}(v_1, \ldots, v_n)$. Since this holds for all $v \in V$, $(v_1, \ldots, v_n)$ spans V. Since $(v_1, \ldots, v_n)$ is also linearly independent, it is a basis.