Homework 5/Solutions

Section	Exercises
3.1	2cd,4ce,9,11,12
3.2	2,5,7,9,14
3.3	3,5,6,14,17

(**Section 3.1 Exercise 4** ce)**.** *Write the polynomials defined by the following formulas as linear combination of* (p_1, p_2, p_3) *where*

$$p_1 = x + 1, \quad p_2 = x^2 + x, \quad x^3 + x^2$$

(c) $4x^3 - 7x^2 - 3x + 8$ *(e)* $(x + 1)(x^2 + x)$

Note that $p_1 = x + 1$, $p_2 = x(x + 1)$ and $p_3 = x^2(x + 1)$.

((c)): $4x^3-7x^2-3x+8 = (4x^2-11x+8)(x+1) = 4(x^2(x+1))-11(x(x+1))+8(x+1) = 4p_3 - 11p_2 + 8p_1$.

((e)): $(x + 1)(x^2 + x) = x^2(x + 1) + x(x + 1) = 1p_3 + 1p_2$.

(**Section 3.1 Exercise 12**)**.** *Let* $\mathbf{V}$ *be a vector space and* v, w_1, w_2, x_1 *and* x_2 *vectors in* V*. Suppose that* v *is a linear combination of* w_1 *and* w_2 *and that* w_1 *and* w_2 *are linear combinations of* x_1 *and* x_2*. Show that* v *is a linear combination of* x_1 *and* x_2*.*

Since v is a linear combination of w_1 and w_1, there exist $a_1, a_2 \in \mathbb{R}$ with $v = a_1w_1+a_2w_2$. Let $1 \leq i \leq 2$. Since w_i is a linear combination of x_1 and x_2 there exist b_{i1} and b_{i2} in $\mathbb{R}$ with $w_i = b_{i1}x_1 + b_{i2}x_2$. Thus

$$v = a_1w_1+a_2w_2 = a_1(b_{11}x_1+b_{12}x_2)+a_2(b_{21}x_1+b_{22}x_2) = (a_1b_{11}+a_2b_{21})x_1+(a_1b_{12}+a_2b_{22})x_2$$

and so v is a linear combination of x_1 and x_2.

(**Section 3.2 Exercise 5**)**.** *Show that the list*

$$\left(\begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix} \right)$$

does not span $\mathbb{M}(2, 3)$*.*

Note that $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is in the span of the four matrices if and only if

$$x_1 \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} + x_4 \begin{bmatrix} 0 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for some $x_1, x_2, x_3, x_4 \in \mathbb{R}$. Using the definitions of addition and scalar multiplication for matrices this is equivalent to

$$\begin{bmatrix} 2x_1 + 1x_2 - 1x_3 + 0x_4 & 1x_1 + 0x_2 + 2x_3 - 2x_4 & 0x_1 + 1x_2 + 1x_3 + 1x_4 \\ -1x_1 + 1x_2 + 0x_3 + 1x_4 & 1x_1 + 1x_2 + 0x_3 - 1x_4 & 0x_1 + 1x_2 + 1x_3 + 2x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for some $x_1, x_2, x_3, x_4 \in \mathbb{R}$. We use the Gauss Jordan algorithm to solve this system of six linear equation in four unknowns:

$$\begin{bmatrix} 2 & 1 & -1 & 0 & 1 \\ 1 & 0 & 2 & -2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{R2 \leftrightarrow R1} \begin{bmatrix} 1 & 0 & 2 & -2 & 0 \\ 2 & 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{\substack{R2 - 2R1 \to R2 \\ R4 + R1 \to R4 \\ R5 - R1 \to R5}} \begin{bmatrix} 1 & 0 & 2 & -2 & 0 \\ 0 & 1 & -5 & 4 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix}$$

$$\xrightarrow{R2 \leftrightarrow R3} \begin{bmatrix} 1 & 0 & 2 & -2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -5 & 4 & 1 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{\substack{R3 - R2 \to R3 \\ R4 - R2 \to R4 \\ R5 - R2 \to R5 \\ R6 - R2 \to R6}} \begin{bmatrix} 1 & 0 & 2 & -2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -6 & 3 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The last two rows of the last matrix show that $x_3 = 0$ and $x_4 = 0$. But according to row 3, $-6x_3 + 3x_4 = 1$ and so $0 = 1$. Thus the system of equations does not have a solutions and so $\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$ is not in the span of the four initial matrices. Hence the four matrices do not span $\mathbb{M}(2,3)$.

(**Section 3.3 Exercise 3**). *Show that*

$$\Big((1,0,0,0,0,0),(1,2,0,0,0,0),(0,1,2,3,0,0),(0,0,1,2,3,4)\Big).$$

is a linearly independent list in $\mathbb{R}^6$.

Let $(x_1,x_2,x_3,x_4) \in \mathbb{R}^4$ with

$$x_1(1,0,0,0,0,0)+x_2(1,2,0,0,0,0)+x_3(0,1,2,3,0,0)+x_4(0,0,1,2,3,4)=(0,0,0,0,0,0)$$

Then

$$\begin{array}{ccccccccc} x_1 & + & x_2 & & & & & = 0 \\ & & 2x_2 & + & x_3 & & & = 0 \\ & & & & 2x_3 & + & x_4 & = 0 \\ & & & & 3x_3 & + & 2x_4 & = 0 \\ & & & & & & 3x_4 & = 0 \\ & & & & & & 4x_4 & = 0 \end{array}$$

Either of the last two equations shows that $x_4 = 0$. Either of the third and fourth equation then shows that $x_3 = 0$. The second equation then shows that $x_2 = 0$ and finally the first equation shows that $x_1 = 0$. So the given list in $\mathbb{R}^6$ is indeed linearly independent.

(**Section 3.3 Exercise 6**). *Show that the polynomials*

$$p_0 = x^3, \quad p_1 = (x-1)^3, \quad p_2 = (x-2)^3, \quad p_3 = (x-3)^3$$

form a linearly independent list in $\mathbb{P}_3$.

Let $a,b,c,d \in \mathbb{R}$ with

$$ap_0 + bp_1 + cp_2 + dp_3 = 0$$

Note that $p_0 = x^3, p_1 = x^3-3x^2+3x-1, p_2 = x^3-6x^2+12x-8$ and $p_3 = x^3-9x^2+27x-27$. Hence

$$\begin{array}{cccccccc} a & + & b & + & c & + & d & = 0 \\ & - & 3b & - & 6c & - & 9d & = 0 \\ & & 3b & + & 12c & + & 27d & = 0 \\ & - & b & - & 8c & - & 27d & = 0 \end{array}$$

We use the Gauss Jordan algorithm to solve this homogeneous linear system of equations.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -6 & -9 \\ 0 & 3 & 12 & 27 \\ 0 & -1 & -8 & -27 \end{bmatrix} \xrightarrow[\substack{\frac{1}{3}R3 \to R3 \\ -R4 \to R4}]{-\frac{1}{3}R2 \to R2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \\ 0 & 1 & 8 & 27 \end{bmatrix} \xrightarrow[R4 - R2 \to R4]{R3 - R2 \to R3} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 6 & 24 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{6}R4 \to R4]{\frac{1}{2}R3 \to R3} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{R4 - R3 \to R4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus the system has no free variables and so $a = 0, b = 0, c = 0, d = 0$ is the only solutions. Hence (p_0, p_1, p_2, p_3) is linearly independent.