MTH 309-4 **Linear Algebra I** **F11**

Homework 4/Solutions

Section	Exercises
1.6	2,4,8
1.7	5b, 7(for axioms 5-8), 10
1.8	2,4,6,9a,12,19
7.1	4,5,8,12

(**Section 1.6 Exercise 4**). *(a) Find values of the scalars r and s so that*

$$r\begin{bmatrix} 2 & -5 & 0 \\ -3 & 1 & 4 \end{bmatrix} + s\begin{bmatrix} 7 & 0 & 8 \\ 1 & -2 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Find values of the scalars r and s so that

$$r\begin{bmatrix} 2 & -5 & 0 \\ -3 & 1 & 4 \end{bmatrix} + s\begin{bmatrix} 7 & 0 & 8 \\ 1 & -2 & 7 \end{bmatrix} = \begin{bmatrix} -1 & -15 & -8 \\ -10 & 5 & 5 \end{bmatrix}$$

(c) Are there solutions to the equations in parts a and b other then the ones you found.

(a) Looking at the (1,3)-entry we see that $0r + 8s = 0$. So $s = 0$. Looking at the (1,2)-entry we see that $-5r + 0s = 0$. So $r = 0$. Clearly $r = s = 0$ is a solutions and so $r = 0, s = 0$ is the unique solution.

(b) Looking at the (1,3)-entry we see that $0r + 8s = -8$. So $s = -1$. Looking at the (1,2)-entry we see that $-5r + 0s = -15$. So $r = 3$. Thus $r = 3$ and $s = -1$ is the only possible solutions. We have

$$3\begin{bmatrix} 2 & -5 & 0 \\ -3 & 1 & 4 \end{bmatrix} - 1\begin{bmatrix} 7 & 0 & 8 \\ 1 & -2 & 7 \end{bmatrix} = \begin{bmatrix} 6-7 & -15-0 & 0-8 \\ -9-1 & 3+2 & 12-7 \end{bmatrix} = \begin{bmatrix} -1 & -15 & -8 \\ -10 & 5 & 5 \end{bmatrix}$$

and so $r = 3$ and $s = -1$ is the unique solution.

(c) No, since we proved in (a) and (b) that the solutions are unique.

(**Section 1.7 Exercise 7** for axioms 5-8). *Verify that the two operations defined on $F(I)$ satisfie Axiom 5-8 of a vector space.*

Let $f, g \in F(I)$, $i \in I$ and $a, b \in \mathbb{R}$.

Ax 5: We have

$$\begin{aligned}
\Big(a(f+g)\Big)(i) &= a\Big((f+g)(i)\Big) && \text{–Definition of multiplications for functions}\\
&= a\Big(f(i)+g(i)\Big) && \text{–Definition of '+' for functions}\\
&= a\big(f(i)\big)+a\big(g(i)\big) && \text{– Property of } \mathbb{R}\\
&= (af)(i)+(ag)(i) && \text{–Definition of multiplications for functions}\\
&= (af+ag)(i) && \text{–Definition of '+' for functions}
\end{aligned}$$

So $a(f+g) = af + ag$ by (*) and **Ax 5** is proved.

Ax 6: We have

$$\begin{aligned}
\big((a+b)f\big)(i) &= \big(a+b\big)\big(f(i)\big) && \text{–Definition of multiplications for functions}\\
&= a\big(f(i)\big)+b\big(f(i)\big) && \text{– Property of } \mathbb{R}\\
&= (af)(i)+(bf)(i) && \text{–Definition of multiplications for functions}\\
&= (af+bf)(i) && \text{–Definition of '+' for functions}
\end{aligned}$$

So $(a+b)f = af + bf$ by (*) and **Ax 6** is proved.

Ax 7: We have

$$\begin{aligned}
\big((ab)f\big)(i) &= \big(ab\big)\big(f(i)\big) && \text{– Definition of multiplications for functions}\\
&= a\Big(b\big(f(i)\big)\Big) && \text{– Property of } \mathbb{R}\\
&= a\Big((bf)(i)\Big) && \text{– Definition of multiplications for functions}\\
&= \Big(a(bf)\Big)(i) && \text{– Definition of multiplications for functions}
\end{aligned}$$

So $(ab)f = a(bf)$ by (*) and **Ax 7** is proved.

Ax 8

$$\begin{aligned}
(1f)(i) &= 1\big(f(i)\big) && \text{– Definition of multiplications for functions}\\
&= f(i) && \text{– Property of } \mathbb{R}
\end{aligned}$$

So $1f = f$ by (*) and **Ax 8** is proved.

(**Section 6.1 Exercise 2**). *Show that the line* $S = \{(x, y) \in \mathbb{R}^2 \mid y = 2x + 1\}$ *is not a subspace of* $\mathbb{R}^2$*.*

We have $0 \neq 2 \cdot 0 + 1$ and so $(0,0) \notin \mathbb{R}^2$. Since $(0,0)$ is the additive identity of $\mathbb{R}^2$, condition (1) of the Subspace Theorem is not fulfilled and so S is not a subspace.

(**Section 6.1 Exercise 6**). *Show that*

$$S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a = b \text{ and } b + 2c = 0 \right\}$$

is a subspace of $\mathbb{M}(2,2)$

By definition of S we have

(*) Let $a, b, c, d \in \mathbb{R}$ then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S \quad \text{if and only if} \quad a = b \text{ and } b + 2c = 0.$$

We will now verify the three conditions of the subspace theorem

(1) Since $0 = 0$ and $0 + 2 \cdot 0 = 0$, (*) shows that

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$$

So Condition (1) of the Subspace Theorem holds.

(2) Let

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S \text{ and } \begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{bmatrix} \in S.$$

Then by (*)

$$a = b, b + 2c = 0, \tilde{a} = \tilde{b} \text{ and } \tilde{b} + 2\tilde{c} = 0.$$

Thus

$$a + \tilde{a} = b + \tilde{b} \text{ and } (b + \tilde{b}) + 2(c + \tilde{c}) = (b + 2c) + (\tilde{b} + 2\tilde{c}) = 0 + 0 = 0.$$

So using (*) and the definition of addition of matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{bmatrix} = \begin{bmatrix} a + \tilde{a} & b + \tilde{b} \\ c + \tilde{c} & d + \tilde{d} \end{bmatrix} \in S.$$

So Condition (2) of the Subspace Theorem holds.

(3) Let $r \in \mathbb{R}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S$.

Then by (*)

$$a = b \text{ and b+2c=0.}$$

Thus

$$ra = rb \text{ and } rb + 2rc = r(b + 2c) = r0 = 0.$$

So using (*) and the definition of scalar multiplication for matrices

$$r \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix} \in S.$$

So Condition (3) of the Subspace Theorem holds.

We verified all three conditions of the Subspace Theorem and so S is a subspace of $\mathbb{M}(2, 2)$.

(**Section 1.8 Exercise 19**). *Suppose* **S** *and* **T** *are subspaces of a vector space* **V**. *Show that the intersection* $S \cap T$ *also a subspace of* **V**.

Recall that $S \cap T = \{v \mid v \in S \text{ and } v \in T\}$. So

(*) For any object v, $v \in S \cap T$ if and only if $v \in S$ and $v \in T$.

We will verify the three conditions of the Subspace Theorem.

(1) Since S and T are subspaces of V, the Subspace Theorem shows that $\mathbf{0} \in S$ and $\mathbf{0} \in T$. So by (*) $\mathbf{0} \in S \cap T$.
So Condition (1) of the Subspace Theorem holds.

(2) Let $x, y \in S \cap T$. Then by (*), $x, y \in S$ and $x, y \in T$. Since S and T are subspaces of V, the Subspace Theorem shows that $x + y \in S$ and $x + y \in T$. So by (*) $x + y \in S \cap T$.
So Condition (2) of the Subspace Theorem holds.

(3) Let $x \in S \cap T$ and $r \in \mathbb{R}$. Then by (*), $x \in S$ and $x \in T$. Since S and T are subspaces of V, the Subspace Theorem shows that $rx \in S$ and $rx \in T$. So by (*) $rx \in S \cap T$.
So Condition (3) of the Subspace Theorem holds.

We proved that the three conditions of the Subspace Theorem hold for $S \cap T$ and so $S \cap T$ is a subspace of V.

(Section 7.1 Exercise 4). *Use induction to prove* $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$ *for all positive integers* n.

Let S_n be the statement

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

S_1 says $1^2 = \frac{1^1 \cdot 2^2}{4}$ and so S_1 is true.
Suppose that S_n is true. Then

$$\begin{aligned}
\textstyle\sum_{k=1}^{n+1} k^3 &= \left(\textstyle\sum_{k=1}^{n} k^3\right) + (n+1)^3 && \text{– definition of } \textstyle\sum \\
&= \tfrac{n^2(n+1)^2}{4} + (n+1)^3 && \text{– since } S_n \text{ holds} \\
&= (n+1)^2\left(\tfrac{n^2}{4} + (n+1)\right) \\
&= (n+1)^2\left(\tfrac{n^2+4n+4}{4}\right) \\
&= (n+1)^2\left(\tfrac{(n+2)^2}{4}\right) \\
&= \tfrac{\left(n+1\right)^2\left((n+1)+1\right)^2}{4}
\end{aligned}$$

Thus S_{n+1} holds.
We proved that S_1 holds and that S_n implies S_{n+1}. So the principal of induction shows that S_n holds for all positive integers n.