MTH 309-4 **Linear Algebra I** **F11**

Homework 3/Solutions

Section	Exercises
1.3	1,6,8,12,14
1.4	1,4,11,12
1.5	1,2,8,10

(**Section 1.3 Exercise 6**). *Prove Theorem 1.5, part b:* $r\mathbf{0} = \mathbf{0}$. *In other words, multiplying any real number* r *times the additive identity vector* $\mathbf{0}$ *yields the additive identity vector* $\mathbf{0}$.

Let $\mathbf{V}$ be a vector space and $r \in \mathbb{R}$. By Axiom 4 $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and so by substitution $r(\mathbf{0} + \mathbf{0}) = r\mathbf{0}$. Thus Axiom 5 gives $r\mathbf{0} + r\mathbf{0} = r\mathbf{0}$. Hence by Theorem 1.2(b) applied with $v = r\mathbf{0}$ and $w = r\mathbf{0}$, $r\mathbf{0} = \mathbf{0}$.

(**Section 1.3 Exercise 12**). *Prove Theorem 1.5, part i: if* $v \neq \mathbf{0}$ *and* $rv = sv$, *then* $r = s$.

Let V be vector space, $v \in V$ with $v \neq \mathbf{0}$ and $r, s \in \mathbb{R}$ with $rv = sv$. We have

$$
\begin{aligned}
& rv = sv \\
\implies\ & rv + (-(sv)) = sv + (-(sv)) && \text{– substitution} \\
\implies\ & rv + (-(sv)) = \mathbf{0} && \text{– Axiom 4} \\
\implies\ & rv + (-s)v = \mathbf{0} && \text{– Theorem 1.5g} \\
\implies\ & \big(r + (-s)\big)v = \mathbf{0} && \text{– Axiom 6} \\
\implies\ & r + (-s) = 0 \text{ or } v = \mathbf{0} && \text{– Theorem 1.5c} \\
\implies\ & r + (-s) = 0 && \text{– since } v \neq \mathbf{0} \text{ by assumption} \\
\implies\ & r = s && \text{– Property of } \mathbb{R}
\end{aligned}
$$

(**Section 1.4 Exercise 12**). *Prove Theorem 1.7, part n:* $(r - s)v = rv - (sv)$.

Let V be vector space, $v \in V$ and $r, s \in \mathbb{R}$. We compute

$$
\begin{aligned}
(r - s)v &= \big(r + (-s)\big)v && \text{– Property of } \mathbb{R} \\
&= rv + (-s)v && \text{– Axiom 6} \\
&= rv + (-(sv)) && \text{– Theorem 1.5g} \\
&= rv - (sv) && \text{– Definition of substraction}
\end{aligned}
$$

(**Section 1.5 Exercise 1**). *Determine whether we obtain a vector space from* $\mathbb{R}^2$ *with operations defined by*

$$\begin{aligned} (v_1, v_2) + (w_1, w_2) &= (v_2 + w_2, v_1 + w_1) \\ r(v_1, v_2) &= (rv_1, rv_2) \end{aligned}$$

We will show that Axiom 2 fails. Let $(v_1, v_2), (w_1, w_2), (x_1, x_2) \in \mathbb{R}^3$.
Then

$$\Big((v_1, v_2)+(w_1, w_2)\Big)+\Big(x_1, x_2\Big) = (v_2+w_2, v_1+w_1)+(x_1, x_2) = \Big((v_1+w_1)+x_2, (v_2+w_2)+x_1\Big)$$

and

$$\Big(v_1, v_2\Big)+\Big((w_1, w_2)+(x_1, x_2)\Big) = (v_1, v_2)+(w_2+x_2, w_1+x_1) = \Big(v_2+(w_1+x_1), v_1+(w_2+x_2)\Big)$$

Note that these two elements in $\mathbb{R}^2$ are usually different. For example if we choose $v_1 = 1$ and $v_2 = w_1 = w_2 = x_1 = x_2 = 0$, then the first element is $(1, 0)$ and the second is $(0, 1)$. Thus the addition is not associative and so $\mathbb{R}^2$ with these operations is not a vector space.

(**Section 1.5 Exercise 8**). *Determine whether we obtain a vector space from the following subset of* $\mathbb{R}^2$ *with the standard operations:*

$$S = \big\{(v_1, v_2) \in \mathbb{R}^2 \mid v_1 \textit{ and } v_2 \textit{ are integers}\big\}$$

This is not a vector space. Note that $(1, 1) \in S$ but $\frac{1}{2}(1, 1) = (\frac{1}{2}, \frac{1}{2}) \notin S$. So property (iii) of a vector space (Closure of multiplication) fails.