MTH 309-4 **Linear Algebra I** **F11**

Homework 2/Solutions

Section	Exercises
2.2	2,3,5,7
2.3	2,7
1.2	8

(**Section 2.2 Exercise 2**). *For each type of row operation show that there is a row operation that will undo it. That is, if M is transformed into M' by a certain row operation, determined a row operation that can be applied to M' to yield M.*

1. Suppose M' is obtained by interchanging row i and row j of M. Then interchanging row i and j of M' yields M.

2. Suppose that M' is obtained by multiplying row i of M by a non-zero real number c. Then multiplying row i of M' by $\frac{1}{c}$ yields M.

3. Suppose that M' is obtained by adding c times row i of M to row j of M. Then adding $-c$-times row i of M' to row j of M' yields M.

So $R_i \leftrightarrow R_j$ is undone by $R_i \leftrightarrow R_j$; $cR_i \to R_i$ is undone by $\frac{1}{c}R_i \to R_i$; and $R_j + cR_i \to R_j$ is undone by $R_j - cR_i \to R_j$.

(**Section 2.2 Exercise 3**). *If two row operation are applied in succession to transform the matrix M into the matrix M', describe the row operations that will transform M' back to M.*

Let S be the first row operation and T the second row operations used to transform M into M'. So if M'' is the matrix obtained from M via S, then M' is the matrix obtained from M'' via T. Let S^* be the row operations which undoes S and T^* be the row operation which undoes T (see **Section 2.2 Exercise 2**). Then M can be obtain from M' by first applying T^* and then S^*. Indeed, since T transforms M'' to M', T^* transform M' to M''. And since S transforms M to M'', S^* will transform M'' into M.

The preceding proof can be summarized in the following diagram

$$M \xrightarrow{S} M'' \xrightarrow{T} M'$$
$$M \xleftarrow{S^*} M'' \xleftarrow{T^*} M'$$

(**Section 2.3 Exercise 7**). *Determine the solutions set for each of following systems of linear equation.*

$$
\begin{array}{lrcrcrcrcrcr}
a. & x_1 & + & 2x_2 & - & x_3 & + & x_4 & - & 2x_5 & = & 7 \\
 & 2x_1 & - & x_2 & + & x_3 & + & x_4 & & & = & 3 \\
 & x_1 & - & 3x_2 & + & 2x_3 & & & + & 2x_5 & = & -4 \\
b. & x_1 & + & 2x_2 & - & x_3 & + & x_4 & - & 2x_5 & = & 6 \\
 & 2x_1 & - & x_2 & + & x_3 & + & x_4 & & & = & 3 \\
 & x_1 & - & 3x_2 & + & 2x_3 & & & + & 2x_5 & = & -4 \\
c. & x_1 & + & 2x_2 & - & x_3 & + & x_4 & - & 2x_5 & = & 7 \\
 & 2x_1 & - & x_2 & + & x_3 & + & x_4 & & & = & 3 \\
 & x_1 & - & 3x_2 & + & 2x_3 & + & x_4 & + & 2x_5 & = & -4 \\
d. & x_1 & + & 2x_2 & - & x_3 & + & x_4 & - & 2x_5 & = & 0 \\
 & 2x_1 & - & x_2 & + & x_3 & + & x_4 & & & = & 0 \\
 & x_1 & - & 3x_2 & + & 2x_3 & & & + & 2x_5 & = & 0
\end{array}
$$

We will solve systems a,b and d simultaneously by using a matrix with two augmented columns:

$$
\begin{bmatrix} 1 & 2 & -1 & 1 & -2 & 7 & 6 \\ 2 & -1 & 1 & 1 & 0 & 3 & 3 \\ 1 & -3 & 2 & 0 & 2 & -4 & -4 \end{bmatrix}
\xrightarrow[R3-R1\to R3]{R2-2R1\to R2}
\begin{bmatrix} 1 & 2 & -1 & 1 & -2 & 7 & 6 \\ 0 & -5 & 3 & -1 & 4 & -11 & -9 \\ 0 & -5 & 3 & -1 & 4 & -11 & -10 \end{bmatrix}
$$

$$
\xrightarrow[\substack{R1+\frac{2}{5}R2\to R1 \\ -\frac{1}{5}R2\to R2}]{R3-R2\to R3}
\begin{bmatrix} 1 & 0 & \frac{1}{5} & \frac{3}{5} & -\frac{2}{5} & \frac{13}{5} & * \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5} & -\frac{4}{5} & \frac{11}{5} & * \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}
$$

The -1 in the last columns shows that system b does not have a solution.

To solve system a, we ignore the last column of the above matrix and observe that x_1 and x_2 are the lead variables and x_3, x_4, x_5 are the free variables. Moving the free variables to the right we get

$$\begin{array}{ccccccccccc}
x_1 & = & - & \frac{1}{5}x_3 & - & \frac{3}{5}x_4 & + & \frac{2}{5}x_5 & + & \frac{13}{5} \\
x_2 & = & + & \frac{3}{5}x_3 & - & \frac{1}{5}x_4 & + & \frac{4}{5}x_5 & + & \frac{11}{5} \\
x_3 & = & + & 1x_3 & + & 0x_4 & + & 0x_5 & + & 0 \\
x_4 & = & + & 0x_3 & + & 1x_4 & + & 0x_5 & + & 0 \\
x_5 & = & + & 0x_3 & + & 0x_4 & + & 1x_5 & + & 0
\end{array}$$

Thus the solution set for system a is

$$S = \left\{ x_3 \begin{pmatrix} -\frac{1}{5} \\ \frac{3}{5} \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -\frac{3}{5} \\ -\frac{1}{5} \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} \frac{2}{5} \\ \frac{4}{5} \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{13}{5} \\ \frac{11}{5} \\ 0 \\ 0 \\ 0 \end{pmatrix} \middle| x_3, x_4, x_5 \in \mathbb{R} \right\}$$

So the solutions set for the homogeneous system d is

$$S = \left\{ x_3 \begin{pmatrix} -\frac{1}{5} \\ \frac{3}{5} \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -\frac{3}{5} \\ -\frac{1}{5} \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} \frac{2}{5} \\ \frac{4}{5} \\ 0 \\ 0 \\ 1 \end{pmatrix} \middle| x_3, x_4, x_5 \in \mathbb{R} \right\}$$

It remains to solve system c:

$$\begin{bmatrix} 1 & 2 & -1 & 1 & -2 & 7 \\ 2 & -1 & 1 & 1 & 0 & 3 \\ 1 & -3 & 2 & 1 & 2 & -4 \end{bmatrix} \xrightarrow[R3 - R1 \to R3]{R2 - 2R1 \to R2} \begin{bmatrix} 1 & 2 & -1 & 1 & -2 & 7 \\ 0 & -5 & 3 & -1 & 4 & -11 \\ 0 & -5 & 3 & 0 & 4 & -11 \end{bmatrix}$$

$$\xrightarrow[\substack{R1 + \frac{2}{5}R2 \to R1 \\ -\frac{1}{5}R2 \to R2}]{R3 - R2 \to R2} \begin{bmatrix} 1 & 0 & \frac{1}{5} & \frac{3}{5} & -\frac{2}{5} & \frac{13}{5} \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5} & -\frac{4}{5} & \frac{11}{5} \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow[R2 - \frac{1}{5}R3 \to R2]{R1 - \frac{3}{5}R3 \to R1} \begin{bmatrix} 1 & 0 & \frac{1}{5} & 0 & -\frac{2}{5} & \frac{13}{5} \\ 0 & 1 & -\frac{3}{5} & 0 & -\frac{4}{5} & \frac{11}{5} \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

So x_1, x_2 and x_4 are the lead variables and x_3, x_5 are the free variables. Moving the free variables to the right we get

$$\begin{array}{rcrcrcrcr} x_1 & = & - & \frac{1}{5}x_3 & + & \frac{2}{5}x_5 & + & \frac{13}{5} \\ x_2 & = & + & \frac{3}{5}x_3 & + & \frac{4}{5}x_5 & + & \frac{11}{5} \\ x_3 & = & + & 1x_3 & + & 0x_5 & + & 0 \\ x_4 & = & + & 0x_3 & + & 0x_5 & + & 0 \\ x_5 & = & + & 0x_3 & + & 1x_5 & + & 0 \end{array}$$

Thus the solution set for system c is

$$S = \left\{ x_3 \begin{pmatrix} -\frac{1}{5} \\ \frac{3}{5}5 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} \frac{2}{5} \\ \frac{4}{5} \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{13}{5} \\ \frac{11}{5} \\ 0 \\ 0 \\ 0 \end{pmatrix} \middle| \, x_4, x_5 \in \mathbb{R} \right\}$$

(**Section 2.3 Exercise 7**). *Which of the following subsets of* $\mathbb{R}$ *are closed under ordinary multiplication? In each case, prove that the set is closed or provide an explicit counter-example.*

(a) $[5, \infty)$.

(b) $[0, 1)$.

(c) $(-1, 0]$.

(d) $\{-1, 1, 0\}$

(e) $\{1, 2, 4, 8, 16, \ldots\}$.

(a) Let $a, b \in [5, \infty)$. Then $a \geq 5$ and $b \geq 5$. In particular, $a \geq 1$ and $b > 0$. So $ab \geq 1b = b \geq 5$ and thus $ab \in [5, \infty)$. Hence $[5, \infty)$ is closed under multiplication.

(b) Let $a, b \in [0, 1]$. The $0 \leq a \leq 1$ and $0 \leq b \leq 1$. Since $b \geq 0$ we get $0 = 0b \leq ab \leq 1b \leq 1$ and so $0 \leq ab \leq 1$ and $ab \in [0, 1]$. Thus $[0, 1]$ is closed under multiplication.

(c) $-\frac{1}{2} \in (-1, 0]$ but $(-\frac{1}{2})(-\frac{1}{2}) = \frac{1}{4} \notin (-1, 0]$. So $(-1, 0]$ is not closed under multiplication.

(d) Let $a, b \in \{-1, 1, 0\}$. If $a = 0$ or $b = 0$, then $ab = 0$ and if $a \neq 0$ and $b \neq 0$, then $ab = 1$ or $ab = -1$. In either case $ab \in \{-1, 1, 0\}$ and so $\{-1, 1, 0\}$ is closed under multiplication.

(e) Let $a, b \in \{1, 2, 4, 8, 16, \ldots\}$. Then $a = 2^n$ and $b = 2^m$ for some non-negative integers n and m. Thus $ab = 2^n 2^m = 2^{n+m}$. Note that $n + m$ is a non-negative integer and so $ab \in \{1, 2, 4, 8, 16, \ldots\}$. Thus $\{1, 2, 4, 8, 16, \ldots\}$ is closed under multiplication.

A. *Define*

$$\oplus : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (v, w) \to \max(v, w)$$

and

$$\odot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (a, v) \to av$$

Which of the eight axioms of a vector space hold for $(\mathbb{R}, \oplus, \odot)$*? Is* $(\mathbb{R}, \oplus, \odot)$ *a vector space?*

Let $v, w, x, a, b \in \mathbb{R}$.
(Ax 1) $v \oplus w = \max(v, w) = \max(w, v) = w \oplus v$. So Ax 1 holds.

(Ax 2) $(v \oplus w) \oplus x = \max\Big(\max(v, w), x\Big) = \max(v, w, x) = \max\Big(v, \max(w.x)\Big) = v \oplus (w \oplus x)$. So Ax 2 holds.

(Ax 3) Let $z \in \mathbb{R}$. Then $(z-1) \oplus z = \max(z-1, z) = z \neq z-1$ and so z is not an additive identity. Thus Ax 3 does not hold.

(Ax 4) Since there does not exist an additive identity, Ax 4 is not applicable..

(Ax 5) $(-1) \odot (0 \oplus 1) = -1 \cdot \max(0, 1) = -1 \cdot 1 = -1$ and $\Big((-1) \odot 0\Big) \oplus \Big((-1) \odot 1\Big) = \max(-1 \cdot 0, -1 \cdot 1) = \max(0, -1) = 0$. Since $-1 \neq 0$, Ax 5 does not hold.

(Ax 6) $(1+1) \odot 1 = 2 \cdot 1 = 2$ and $(1 \odot 1) \oplus (1 \odot 1) = \max(1 \cdot 1, 1 \cdot 1) = \max(1, 1) = 2$. Since $1 \neq 2$, Ax 6 does not hold.

(Ax 7) $(ab) \odot v = (ab)c = a(bc) = a \odot (b \odot c)$ so Ax 7 holds.

(Ax 8) $1 \odot v = 1v = v$ and so Ax 8 holds.
In summary: Ax 1,2,7 and 8 holds. Ax 3,5 and 6 fail and Ax 4 is not applicable. Hence $\mathbb{R}$ with these operations is not a vector space.

B. *Define*

$$\oplus : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (v, w) \to \sqrt[3]{v^3 + w^3}$$

and

$$\odot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (a, v) \to \sqrt[3]{a}\, v$$

Which of the eight axioms of a vector space hold for $(\mathbb{R}, \oplus, \odot)$*? Is* $(\mathbb{R}, \oplus, \odot)$ *a vector space?*

Let $v, w, x, a, b \in \mathbb{R}$. Then
(Ax 1) $v \oplus w = \sqrt[3]{v^3 + w^3} = \sqrt[3]{w^3 + v^3} = w \oplus v$ and so Ax 1 holds.

(Ax 2) $(v \oplus w) \oplus x = \sqrt[3]{\left(\sqrt[3]{v^3 + w^3}\right)^3 + x^3} = \sqrt[3]{(v^3 + w^3) + x^3)} = \sqrt[3]{v^3 + (w^3 + x^3)} = \sqrt[3]{v^3 + \left(\sqrt[3]{w^3 + x^3}\right)^3} = v \oplus (w \oplus x)$ and so Ax 2 holds.

(Ax 3) $v \oplus 0 = \sqrt[3]{v^3 + 0^3} = \sqrt[3]{v^3} = v$ and so 0 is an additive identity. Hence Ax 3 holds.

(Ax 4) $v \oplus (-v) = \sqrt[3]{v^3 + (-v)^3} = \sqrt[3]{v^3 - v^3} = \sqrt[3]{0} = 0$. Hence Ax 4 holds.

(Ax 5) $a \odot (v \oplus w) = \sqrt[3]{a}\sqrt[3]{v^3 + w^3} = \sqrt[3]{a(v^3 + w^3)} = \sqrt[3]{av^3 + aw^3} = \sqrt[3]{(\sqrt[3]{a}v)^3 + (\sqrt[3]{a}w)^3} = (a \odot v) \oplus (a \odot w)$ and so Ax 5 holds.

(Ax 6) $(a + b) \odot v = \sqrt[3]{a + b}v = \sqrt[3]{a + b}\sqrt[3]{v^3} = \sqrt[3]{(a + b)v^3} = \sqrt[3]{av^3 + bv^3} = \sqrt[3]{\left(\sqrt[3]{a}v\right)^3 + \left(\sqrt[3]{b}v\right)^3} = (a \odot v) \oplus (b \odot 3)$ and so Ax 6 holds.

(Ax 7) $(ab) \odot v = \sqrt[3]{ab}\, v = (\sqrt[3]{a}\sqrt[3]{b})v = \sqrt[3]{a}(\sqrt[3]{b}v) = a \odot (b \odot v)$ and so Ax 7 holds.

(Ax 8) $1 \odot v = \sqrt[3]{1}v = 1v = v$ and so Ax 8 holds.

We proved that all eight axioms holds and so $\mathbb{R}$ with these operations is a vector space.

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