MTH 309-4 **Linear Algebra I** **F11**

Homework 1/Solutions

Section	Exercises
1.1	2, 5a, 7, 8abcd
2.1	1, 2, 3

(**Section 1.1 Exercise 2**). *Suppose R, S and T are sets. Prove the following properties.*

(a) $S \cup T = T \cup S$.

(b) $R \cup (S \cup T) = (R \cup S) \cup T$.

(c) $S \cup \emptyset = S$.

Let x be an object.
(a):

$$
\begin{array}{rcl}
 & x \in S \cup T & \\
\iff & (x \in S) \text{ or } (x \in T) & - \text{ definition of } \cup \\
\iff & (x \in T) \text{ or } (x \in S) & - \text{ N1.1.1(14)} \\
\iff & x \in T \cup S & - \text{ definition of } \cup
\end{array}
$$

We proved that $x \in S \cup T$ if and only if $x \in T \cup S$. So $S \cup T = T \cup S$ by Theorem N1.1.3.

(b)

$$
\begin{array}{rcl}
 & x \in R \cup (S \cup T) & \\
\iff & (x \in R) \text{ or } (x \in S \cup T) & - \text{ definition of } \cup \\
\iff & \Big(x \in R\Big) \text{ or } \Big((x \in S) \text{ or } (x \in T)\Big) & - \text{ definition of } \cup \\
\iff & \Big((x \in R) \text{ or } (x \in S)\Big) \text{ or } \Big(x \in T\Big) & - \text{ N1.1.1(25)} \\
\iff & (x \in R \cup S) \text{ or } (x \in T) & - \text{ definition of } \cup \\
\iff & x \in (R \cup S) \cup T & - \text{ definition of } \cup
\end{array}
$$

We proved that $x \in R \cup (S \cup T)$ if and only if $x \in (R \cup S) \cup T$. So $R \cup (S \cup T) = (R \cup S) \cup T$. by Theorem N1.1.3.(c)

$$\begin{array}{lll} & x \in S \cup \emptyset & \\ \iff & (x \in S) \text{ or } (x \in \emptyset) & - \text{ definition of } \cup \\ \iff & x \in S & - \text{ N1.1.1(8) and the fact that } x \in \emptyset \text{ is false by definition of } \emptyset \end{array}$$

We proved that $x \in S \cup \emptyset$ if and only if $x \in S$. So $S \cup \emptyset = S$ by Theorem N1.1.3.

(**Section 1.1 Exercise 5**a). *There is an analogy between the containment relation $\subseteq$ in set theory and the order relation $\leq$ among real numbers. You are undoubtedly familiar with order properties for real numbers x,y,z, such as:*

$x \leq x$

If $x \leq y$ and $y \leq x$, then $x = y$.

If $x \leq y$ and $y \leq z$, then $x \leq z$.

Either $x \leq y$ or $y \leq x$

(a) Replace x, y, z by sets, and $\leq$ by $\subseteq$. Which statements remain true? Provide a proof or a counter example.

Let R, S and T be sets. Replacing x by R, y by S, z by T and $\leq$ by $\subseteq$ we obtain the following four statements:

1. $R \subseteq R$.
2. If $R \subseteq S$ and $S \subseteq R$, then $R = S$.
3. If $R \subseteq S$ and $S \subseteq T$, then $R \subseteq T$.
4. Either $R \subseteq S$ or $S \subseteq R$.

By Lemma N1.1.5, $R = S$ if and only if $R \subseteq S$ and $S \subseteq R$. This shows that (2) holds. Since $R = R$, it also shows that $R \subseteq R$. So (1) holds.

Suppose that $R \subseteq S$ and $S \subseteq T$. Then by definition of $\subseteq$, $x \in R$ implies $x \in S$; and $x \in S$ implies $x \in T$. Thus by Lemma N1.1.1(28), $x \in R$ implies $x \in T$. So $R \subseteq T$ by definition of $\subseteq$. Thus (3) is true.

(4) is false. For example $\{0\} \nsubseteq \{1\}$ and $\{1\} \nsubseteq \{0\}$.

To summarize (1),(2) and (3) are true, and (4) is false.

(**Section 1.1 Exercise 7**). *Is it possible to add two sets? For sets of real numbers, we might define addition of sets in terms of addition of the elements in the sets. Let us introduce the following meaning to the symbol $\oplus$ for adding a set A of real numbers to another set B of real numbers:*

$$A \oplus B = \{a + b \mid a \in A \text{ and } b \in B\}.$$

(a) List the elements in the set $\{1, 2, 3\} \oplus \{5, 10\}$.

(b) List the elements in the set $\{1, 2, 3\} \oplus \{5, 6\}$.

(c) List the elements in the set $\{1, 2, 3\} \oplus \emptyset$.

(d) If a set A contains m real numbers and a set B contains n real numbers, can you predict the number of elements in $A \oplus B$. If you run into difficulties, can you determine the minimum and maximum numbers of elements possible in $A \oplus B$.

(a) $\{1, 2, 3\} \oplus \{5, 10\} = \{1 + 5, 2 + 5, 3 + 5, 1 + 10, 2 + 10, 3 + 10\} = \{6, 7, 8, 11, 12, 13\}$.

(b) $\{1, 2, 3\} \oplus \{5, 6\} = \{1+5, 2+5, 3+5, 1+6, 2+6, 3+6\} = \{6, 7, 8, 7, 8, 9\} = \{6, 7, 8, 9\}$.

(c) $\{1, 2, 3\} \oplus \emptyset = \emptyset$.

(d) Since there are m choices for $a \in A$ and n choices for $b \in B$, there are mn choices for the pair (a, b). Hence there are at most m sums of the form $a + b$ with $a \in A$ and $b \in B$. Thus $A \oplus B$ has at most mn elements.

If $n = 0$ or $m = 0$, then $nm = 0$ and so $A \oplus B$ has exactly $nm = 0$ elements in this case.

Suppose now that $n > 0$ and $m > 0$. We will show that $A \oplus B$ has at least $m + n - 1$ elements. Let $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, b_2 \ldots, b_n\}$ with

$$a_1 < a_2 < \ldots < a_{m-1} < a_m \quad \text{and} \quad b_1 < b_2 < \ldots < b_{n-1} < b_n.$$

Then

$$a_1 + b_1 < a_2 + b_1 < \ldots < a_m + b_1 < a_m + b_2 < \ldots < a_m + b_n.$$

and so we found $m + (n - 1) = m + n - 1$ distinct elements in $A \oplus B$. Hence $A \oplus B$ has at least $n + m - 1$ elements.

(**Section 2.1 Exercise 1**). *In each of the following systems, add a multiple of one equation to another equation to obtain an equivalent system with no more than one unknown in each equation. Write the solution set.*

(a) $$\begin{aligned} x + 2y &= 7 \\ y &= 2 \end{aligned}$$

(b) $$\begin{aligned} x \phantom{{}+6y} &= 2 \\ -2x + 6y &= 5 \end{aligned}$$

$$(c) \quad \begin{aligned} x - 3y \quad &= 2 \\ y \quad &= -1 \\ z &= 5 \end{aligned} \qquad\qquad (d) \quad \begin{aligned} x \quad + 7z &= 0 \\ y \quad &= -3 \\ z &= 2 \end{aligned}$$

(a) Adding -2-times equation 2 to equation 1 we get

$$\begin{aligned} x &= 3 \\ y &= 2 \end{aligned}.$$

So the solution set is $\{(3, 2)\}$.

(b) Adding 2-times equation 1 to equation 2 we get

$$\begin{aligned} x &= 2 \\ 6y &= 9 \end{aligned}$$

So the solution set is $\{(2, \frac{3}{2})\}$.

(c) Adding 3-times equation 2 to equation 1 we get

$$\begin{aligned} x &= -1 \\ y &= -1 \\ z &= 5 \end{aligned}$$

So the solution set is $\{(-1, -1, 5)\}$.

(d) Adding -7-times equation 3 to equation 1 we get

$$\begin{aligned} x &= -14 \\ y &= -3 \\ z &= 2 \end{aligned}$$

So the solution set is $\{(-14, -3, 2)\}$.