1. Let p be a polynomial of degree 2 in $\mathbb{P}_2$, let p' and p'' be the first and second derivative of p. Show that (p, p', p'') is a basis for $\mathbb{P}_2$.

Solution: Let $(a, b, c) \in \mathbb{R}^3$ with $ap + bp' + cp'' = 0$. Note that p has degree 2, p' has degree 1 and p'' has degree 0. So the coefficient of x^2 in $ap + bp' + cp'' = 0$ is a. Thus $a = 0$. Hence $bp' + cp'' = 0$ and looking at the coefficient of x we get $b = 0$. So $cp'' = 0$ and thus $c = 0$. We proved that $a = b = c = 0$ and so (p, p', p'') is linearly independent. Since $\mathbb{P}_2$ is three dimensional, any linearly independent list of length 3 in $\mathbb{P}_2$ is a basis for $\mathbb{P}_2$. So $\mathbb{P}_2$ is a basis.

2. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear function defined by

$$L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \end{pmatrix}$$

(a) Determine the kernel of L.

(b) Find the matrix A of L with respect to the standard bases.

(c) Find a basis for $\mathrm{Im}L$.

(d) Find a basis for $\ker L$.

Solution:

(a) $(x_1, x_2, x_3) \in \ker L$ if and only if $L(x_1, x_2, x_3) = \mathbf{0}$, if and only if $x_2 - x_1 = 0, x_3 - x_2 = 0$ and $x_3 - x_1 = 0$ and if and only if $x_1 = x_2, x_2 = x_3$ and $x_1 = x_3$. So

$$\ker L = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = x_2 = x_3\} = \{(a, a, a) \mid a \in \mathbb{R}\} = \{a(1, 1, 1) \mid a \in \mathbb{R}\}$$

(b) Solution 1: The standard basis for $\mathbb{R}^3$ is $E = (e_1, e_2, e_3)$. Also $[x]_E = x$ for all $x \in \mathbb{R}^3$. So

$$L(e_1)]_E = \left[\begin{pmatrix} 0-1 \\ 0-0 \\ 0-1 \end{pmatrix}\right]_E = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \qquad [L(e_2)]_E = \left[\begin{pmatrix} 1-0 \\ 0-1 \\ 0-0 \end{pmatrix}\right]_E = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$[L(e_3)]_E = \left[\begin{pmatrix} 0-0 \\ 1-0 \\ 1-0 \end{pmatrix}\right]_E = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \text{thus} \qquad A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

Solution 2:

$$L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \end{pmatrix} = x_1\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} + x_2\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and so the matrix for L with respect to the standard bases is

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

(c) Note that $L = L_A$ and so $\text{Im}L = \text{Col}A$. So we can use the Gauss Jordan algorithm to find a basis:

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow[-R1 \to R1]{R3 - R1 \to R3} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[-R2 \to R2]{R3 - R2 \to R3} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So the leading variables are x_1 and x_2 and the first two colums of A form a basis for $\text{Im}L$. Thus

$$\left(\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right)$$

is a basis for $\text{Im}L$.

(d) From (a) see that $\big((1, 1, 1)\big)$ spans $\ker L$. It also linearly independent and so $\big((1, 1, 1)\big)$ is a basis for $\ker L$.

3. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

   (a) Find all eigenvalues of A.

   (b) For each of the eigenvalues in a), find its corresponding eigenvectors.

   (c) Determine whether A is diagonalizable, and justify your answer.

   (d) If A is diagonalizable, find a matrix P such that $P^{-1}AP$ is a diagonal matrix.

Solution: (a)

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 2 & 0 & 0 \\ 0 & \lambda - 2 & 0 \\ -1 & -1 & \lambda - 1 \end{bmatrix} = (\lambda - 2)(\lambda - 2)(\lambda - 1)$$

and so the eigenvalues are 1 and 2

(b) We use the Gauss Jordan algorithm to find a basis for $\text{Nul}(\lambda I - A)$:

$\lambda = 1$:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & -1 & 0 \end{bmatrix} \xrightarrow[\substack{-R1 \to R1 \\ -R2 \to R2}]{R3 - R1 - R2 \to R3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So x_3 is free, $x_1 = 0, x_2 = 0, x_3 = x_3$ and

$$\text{E}_1(A) = \{x_3(0, 0, 1) \mid x_3 \in \mathbb{R}\}$$

$\lambda = 2$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \end{bmatrix} \xrightarrow[R1 \leftrightarrow R3]{-R3 \to R3} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So x_2 and x_3 are free, $x_1 = -x_2 + x_3, x_2 = x_2, x_3 = x_3$ and

$$\mathrm{E}_2(A) = \{x_1(-1,1,0) + x_3(1,0,1) \mid x_2, x_3 \in \mathbb{R}\}$$

(c) and (d) By (b), $\big((0,0,1)\big)$ is a basis for $E_A(1)$ and $\big((-1,1,0),(1,0,1)\big)$ is a basis for $E_A(2)$. Thus

$$\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right)$$

is a linearly independent list eigenvectors of A. Since this list has length 3 and $\dim \mathbb{R}^3 = 3$, this list is a basis of eigenvectors. So A is diagonalizable and

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \text{where} \quad P = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

4. Let A be an $m \times n$-matrix with $\dim \mathrm{Col}A = n$.
   (a) State the definition of linear independence for a list in a vector space V.
   (b) Show that L_A is 1-1.
   (c) Let $(x_1, x_2, \ldots, x_k)$ be a linearly independent list in $\mathbb{R}^n$ and put $y_i = Ax_i$ for all $1 \leq i \leq k$. Show that $(y_1, y_2, \ldots, y_k)$ is linearly independent in $\mathbb{R}^m$.

Solution: (a) A list $(v_1, v_2, \ldots, v_n)$ in a vector space V is linearly independent if for all $(r_1, \ldots, r_n) \in \mathbb{R}^n$,

$$r_1v_1 + r_2v_2 + \ldots + r_nv_n = 0 \Longrightarrow r_1 = 0, r_2 = 0, \ldots, r_n = 0.$$

(b) By the dimension formula, $\dim \mathrm{Nul}A + \dim \mathrm{Col}A = n$. Since $\dim \mathrm{Col}A = n$, this gives $\dim \mathrm{Nul}A = 0$. Thus $\ker L_A = \mathrm{Nul}A = \{\mathbf{0}\}$ and so L_A is 1-1.

(c) Let $(r_1, \ldots r_k) \in \mathbb{R}^n$ with $r_1y_1 + \ldots + r_ky_k = \mathbf{0}$. Note that $y_i = Ax_i = L_A(x_i)$ and so, since L_A is linear,

$$L_A(r_1x_1 + \ldots + r_kx_k) = r_1L_A(x_1) + \ldots + r_kL_A(x_k) = r_1y_1 + \ldots + r_ky_k = \mathbf{0} = L_A(\mathbf{0}).$$

Since L_A is 1-1 we conclude that $r_1x_1 + \ldots + r_kx_k = \mathbf{0}$. Since $(x_1, x_2, \ldots, x_k)$ is linearly independent this implies $r_1 = r_2 = \ldots r_k = 0$ and so $(y_1, \ldots, y_k)$ is linearly independent.

5. Consider the linear system of equations:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + 2x_2 + x_3 &= 2 \\ x_1 + x_2 + ax_3 &= b \end{aligned}$$

(a) Find all values for a and b such that the system has no solution.

(b) Find all values for a and b such that the system has a unique solution.

(c) Find all values for a and b such that the system has infinitely many solutions.

Solution:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 2 \\ 1 & 1 & a & b \end{bmatrix} \xrightarrow[\frac{1}{2}R2 \to R2]{R3 - R1 \to R3} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & 1 \\ 0 & 0 & a-1 & b-1 \end{bmatrix}$$

If $a \neq 1$, then $a - 1 \neq 0$, we can divide the last row by $a - 1$. It follows that all columns but the last contain a leading one, and the system has a unique solution.

If $a = 1$ and $b \neq 1$, then $a - 1 = 0$ and $b - 1 \neq 0$. So the last row shows that the system has no solution.

If $a = 1$ and $b = 1$, then $a - 1 = 0$ and $b - 1 = 0$. So the last two columns do not contain a leading one. Thus the system has a solutions and x_3 is a free variable. Thus the system has infinitely many solutions.

6. Let A be an upper triangular matrix (that is an $n \times n$-matrix with $a_{ij} = 0$ for all $1 \leq j < i \leq n$.) Prove by induction that the determinant of A is equal to the product of the diagonal entries of A.

Solution: If $n = 1$, then $\det(A) = a_{11}$ and the statement holds So suppose the statement holds for $n - 1$ and let A be an $n \times n$ upper triangular matrix. Since $a_{ni} = 0$ for all $1 \leq i < n$ we have

$$\det(A) = \sum_{i=1}^{n} (-1)^{n+i} a_{ni} \det(A_{ni}) = (-1)^{n+n} a_{nn} \det(A_{nn}) = a_{nn} \det A_{nn} = \det(A_{nn})\, a_{nn}$$

Observe that A_{nn} is an $(n-1) \times (n-1)$ upper triangular matrix and so by the induction assumption, $\det(A_{nn}) = a_{11}a_{22}\ldots a_{(n-1)(n-1)}$. Thus $\det A = a_{11}a_{22}\ldots a_{nn}$

We proved that the statements holds for 1 and if it holds for $n - 1$ it holds for n. Thus by the principal of induction it holds for all positive integers n.

7. Let A be an $m \times n$-matrix. Prove that $\mathrm{Nul}(A) = \{\mathbf{0}\}$ if and only if the columns of A are linearly independent. (Do not use any theorems)

Solution:

$$\begin{array}{lll}
 & \mathrm{Nul}(A) = \{\mathbf{0}\} & \\
\iff & \{x \in \mathbb{R}^n \mid Ax = \mathbf{0}\} = \{\mathbf{0}\} & \text{– definition of } \mathrm{Nul}(A) \\
\iff & \text{for all } x \in \mathbb{R}^n : \quad Ax = \mathbf{0} \iff x = \mathbf{0} & \text{– Equality of sets} \\
\iff & \text{for all } x_1, \ldots, x_n \in \mathbb{R} : \quad A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{0} \iff \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{0} & \text{– definition of } \mathbb{R}^n \\
\iff & \text{for all } x_1, \ldots, x_n \in \mathbb{R}, & \text{– definition of } Ax \\
 & x_1 a_1 + \ldots + x_n a_n = \mathbf{0} \iff x_1 = 0, \ldots, x_n = \mathbf{0} & \\
\iff & (a_1, \ldots, a_n) \text{ is linearly independent} & \text{– definition of linearly independent}
\end{array}$$

8. Let S be the subset of $\mathbb{P}_3$ defined by $S = \{p(x) \in \mathbb{P}_3 \mid p(2) = 0\}$. Show that S is a subspace of $\mathbb{P}_3$.

> **Solution:** Recall that the additive identity in $\mathbb{P}_2$ is the zero polynomial 0. We have $0(2) = 0$ and so $0 \in S$.
>
> Let $p, q \in S$. Then $p(2) = 0$ and $q(2) = 0$ and so $(p+q)(0) = p(0) + q(0) = 0 + 0 = 0$. Thus $p + q \in S$.
>
> Let $p \in S$ and $r \in \mathbb{R}$. Then $(rp)(0) = r\big(p(0)\big) = r0 = 0$ and so $rp \in S$.
>
> Hence all three conditions of the subspace theorem hold and so S is a subspace of $\mathbb{P}_3$.

9. Let $L : V \to W$ be a linear function from a vector space V to a vector space W.
   (a) State the definition of $\ker(L)$.
   (b) Prove that $\ker(L)$ is subspace of V. (Do not quote any theorems other than the Subspace Theorem)

> **Solution:** (a) $\ker(L) = \{v \in V \mid L(v) = \mathbf{0}\}$.
>
> (b) By definition of $\ker L$ we have
>
> (*) Let $v \in V$. Then $v \in \ker L$ if and only if $L(v) = \mathbf{0}$.
>
> By Theorem 6.2, $L(\mathbf{0}) = \mathbf{0}$ and so by (*), $\mathbf{0} \in \ker L$.
>
> Let $u, v \in \ker L$. Then by (*) $L(u) = \mathbf{0}$ and $L(v) = \mathbf{0}$. Since L is linear we get $L(u+v) = L(u) + L(v) = \mathbf{0} + \mathbf{0} = \mathbf{0}$ and so $u + v \in \ker L$ by (*)
>
> Let $v \in \ker L$ and $r \in \mathbb{R}$. Then by (*), $L(v) = \mathbf{0}$. . Since L is linear we get $L(rv) = rL(u) = r\mathbf{0} = \mathbf{0}$ and so $u + v \in \ker L$ by (*).
>
> We verified that all three conditions of the Subspace Theorem holds for S and so S is a subspace of V.

10. (a) State the definition of a linear function.
    (b) Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear function with

$$L(1,1,0) = (0,2,3), \quad L(1,0,0) = (0,2,0), \quad L(1,1,1) = (4,3,3)$$

Compute $L(0,1,0)$ and $L(0,0,1)$.

> **Solution:** (a) *A linear function is a function $T : V \to W$ where $\mathbf{V}$ and $\mathbf{W}$ are vector space and*
>
> *(i) $T(u+v) = T(u) + T(v)$ for all $u, v \in V$, and*
>
> *(ii) $T(rv) = r\big(T(v)\big)$ for all $r \in \mathbb{R}$ and $v \in V$.*
>
> (b) $L(0,0,1) = L\Big((1,1,1) + (-1)(1,1,0)\Big) = L(1,1,1) + (-1)L(1,1,0) = (4,3,3) - (0,2,3) = (4,1,0)$ and then
>
> $L(0,1,0) = L\Big((1,1,0) + (-1)(1,0,0)\Big) = L(1,1,0) + (-1)L(1,0,0) = (0,2,3) - (0,2,0) = (0,0,3)$.

11. Let A be a 3×5-matrix. Suppose the list of rows of A is linearly independent. Compute $\dim \operatorname{Nul} A$.

Solution: By definition $\operatorname{Row} A$ is spanned by the list of rows of A. By assumption the list of rows of A is linearly independent and so is a basis for $\operatorname{Row} A$. Since A has three rows this gives $\dim \operatorname{Row} A = 3$. Since $\dim \operatorname{Col} A = \dim \operatorname{Row} A$ we conclude that $\dim \operatorname{Col} A = 3$. By the dimension formula, $\dim \operatorname{Nul} A + \dim \operatorname{Col} A = 5$. Thus $\dim \operatorname{Nul} A = 5 - 3 = 2$.

12. Let A and B be $m \times m$ matrices with $AB = BA$. Use mathematical induction to show that $A^n B = BA^n$ for all positive integers n.

Solution: Let $S(n)$ be the statement, $A^n B = BA^n$. For $n = 1$ this says $AB = BA$ which is true by assumption.

Suppose that $S(n)$ holds. Then $A^n B = BA^n$. We compute

$$\begin{aligned}
A^{n+1}B &= (A^n A)B && - \text{ definition of } A^{n+1} \\
&= A^n(AB) && - \text{ matrix multiplication is associative} \\
&= A^n(BA) && - \text{ since } AB = BA \\
&= (A^n B)A && - \text{ matrix multiplication is associative} \\
&= (BA^n)A && - \text{ since } S(n) \text{ holds} \\
&= B(A^n A) && - \text{ matrix multiplication is associative} \\
&= BA^{n+1} && - \text{ definition of } A^{n+1}
\end{aligned}$$

So $S(n)$ implies $S(n+1)$. Thus by the principal of induction $S(n)$ holds for all positive integers n.

13. Let $X = \operatorname{Span}\Big((1,0,2,1), (1,2,3,0), (3,2,7,2), (0,0,1,1)\Big)$

(a) State the definitions of the span of a list of vectors in a vector space V.

(b) Find a basis for X.

(c) Is $(3,4,9,2) \in X$

(d) Is $X = \mathbb{R}^4$?

Solution: (a) Let $B = (v_1, \ldots, v_n)$ be a list in V. Then the span of B, denoted $\operatorname{span} B$, is the set of linear combinations of V, that is

$$\operatorname{span} B = \{r_1 v_1 + \ldots + r_n v_n \mid (r_1, \ldots, r_n) \in \mathbb{R}^n\}$$

(b) We compute the row echelon form of the matrix A whose rows are the above four vectors:

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 2 & 3 & 0 \\ 3 & 2 & 7 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[R3 - 3R1 \to R3]{R2 - R1 \to R2} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[\substack{\frac{1}{2}R2 \to R2 \\ R3 \leftrightarrow R4}]{R3 - R2 \to R2} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R2 - \frac{1}{2}R3 \to R2]{R1 - 2R3 \to R1} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So

$$\Big((1,0,0,-1),(0,1,0,-1),(0,0,1,1)\Big)$$

is a basis for X.

(c) $(3,4,9,2)$ is in X if and only if its is a linear combination of the basis we found in (c). Looking at the first three entries, we see that the coefficients have to be $(3,4,9)$. We compute

$$3(1,0,0,-1)+4(0,1,0,-1)+9(0,0,1,1)=(3,4,9,-3-4+9)=(3,4,9,2)$$

and so $(3,4,9,2)$ is in X.

(d) By (b) X is three dimensional. Since $\mathbb{R}^4$ is 4-dimensional, $X\neq\mathbb{R}^4$.

Solution: We give a second solution for (b) and (c). We will compute a row echelon form of the matrix B with columns the four vectors spanning X augmented by the vector in (c):

$$B=\begin{bmatrix}1&1&3&0&3\\0&2&2&0&4\\2&3&7&1&9\\1&0&2&1&2\end{bmatrix}\xrightarrow[\frac12R_2\to R_2]{\substack{R3-2R1\to R3\\R4-2R1\to R4}}\begin{bmatrix}1&1&3&0&3\\0&1&1&0&2\\0&1&1&1&3\\0&-1&-1&1&-1\end{bmatrix}\xrightarrow{\substack{R3-R1\to R3\\R4+R1\to R4}}\begin{bmatrix}1&1&3&0&3\\0&1&1&0&2\\0&0&0&1&1\\0&0&0&1&1\end{bmatrix}\xrightarrow{R4-R3\to R4}\begin{bmatrix}1&1&3&0&3\\0&1&1&0&2\\0&0&0&1&1\\0&0&0&0&0\end{bmatrix}$$

The leading ones in the first four columns are in Columns 1, 3 and 4 and so Columns 1, 3 and 4 form basis for X. Hence

$$\left(\begin{pmatrix}1\\0\\2\\1\end{pmatrix},\begin{pmatrix}1\\2\\3\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\1\end{pmatrix}\right)$$

is a basis for X.

There is no leading 1 in Column 5, so the linear system of equation corresponding to B has a solution. So $(3,4,9,2)$ is a linear combination of the columns of B, that is $(3,4,9,2)\in X$.

14. Let $f:X\to Y$ and $g:Y\to Z$ be linear functions. Show that $g\circ f$ is a linear. (Do not quote any theorems).

Solution: Let $u,v\in X$. Then

$$\begin{aligned}(g\circ f)(u+v)&=g\Big(f(u+v)\Big)&&-\text{ definition of composition}\\&=g\Big(f(u)+f(v)\Big)&&-\text{ since } f \text{ is linear}\\&=g\Big(f(u)\Big)+g\Big(f(v)\Big)&&-\text{ since } g \text{ is linear}\\&=(g\circ f)(u)+(g\circ f)(v)&&-\text{ definition of composition,twice}\end{aligned}$$

Let $v \in V$ and $r \in \mathbb{R}$. Then

$$\begin{aligned}(g \circ f)(rv) &= g\Big(f(rv)\Big) && \text{– definition of composition}\\ &= g\Big(r\Big(f(v)\Big)\Big) && \text{– since } f \text{ is linear}\\ &= r\Big(g\Big(f(v)\Big)\Big) && \text{– since } g \text{ is linear}\\ &= r\Big((g \circ f)(v)\Big) && \text{– definition of composition}\end{aligned}$$

Thus $g \circ f$ is linear.

15. Let $B = (x, 1, x^2)$ and $B' = (1 + x, 1 - x, x^2)$ be ordered bases for $\mathbb{P}_3$.
   (a) Find the change-of-bases matrix from B' to B.
   (b) Find then change-of-bases matrix from B to B'.
   (c) Find the coordinate vector of $1 + 2x + x^2$ with respect to B'.

Solution: (a) $1 + x = 1x + 1 \cdot 1 + 0x^2$ and so $[1 + x]_B = \begin{pmatrix}1\\1\\0\end{pmatrix}$

$1 - x = (-1)x + 1 \cdot 1 + 0x^2$ and so $[1 - x]_B = \begin{pmatrix}-1\\1\\0\end{pmatrix}$.

$x^2 = -0x + 0 \cdot 1 + 1x^2$ and so $[x^2]_B = \begin{pmatrix}0\\0\\1\end{pmatrix}$. Thus the change-of-bases matrix from B' to B is

$$P = \begin{bmatrix}1 & -1 & 0\\1 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

(b) We use the Gauss Jordan Algorithm to compute the inverse of P:

$$\begin{bmatrix}1 & -1 & 0 & 1 & 0 & 0\\1 & 1 & 0 & 0 & 1 & 0\\0 & 0 & 1 & 0 & 0 & 1\end{bmatrix} \xrightarrow{R2 - R1 \to R2} \begin{bmatrix}1 & -1 & 0 & 1 & 0 & 0\\0 & 2 & 0 & -1 & 1 & 0\\0 & 0 & 1 & 0 & 0 & 1\end{bmatrix} \xrightarrow[\frac{1}{2}R2 \to R2]{R1 + \frac{1}{2}R2 \to R1} \begin{bmatrix}1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0\\0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0\\0 & 0 & 1 & 0 & 0 & 1\end{bmatrix}$$

So the change of basis matrix from B to B' is

$$P^{-1} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\-\frac{1}{2} & \frac{1}{2} & 0\\0 & 0 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1 & 0\\-1 & 1 & 0\\0 & 0 & 2\end{bmatrix}$$

(c) We have $1 + 2x + x^2 = 2x + 1 \cdot 1 + 1x^2$ and so $[1 + 2x + x^2]_B = \begin{pmatrix}2\\1\\1\end{pmatrix}$. Thus

$$[1 + 2x + x^2]_{B'} = P^{-1}[1 + 2x + x^2]_B = \frac{1}{2}\begin{bmatrix}1 & 1 & 0\\-1 & 1 & 0\\0 & 0 & 2\end{bmatrix}\begin{pmatrix}2\\1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}3\\-1\\2\end{pmatrix} = \begin{pmatrix}\frac{3}{2}\\-\frac{1}{1}\\1\end{pmatrix}$$

16. (a) State the definition of a basis for a vector space V.

(b) Show that the list

$$B = \left(x^3 + 3x^2, x^2 + 3x, x + 3, x^3 + 4x^2 + 4x + 4\right)$$

is a basis for $\mathbb{P}_3$.

Solution: (a) A basis for a vector space V is a linearly independent, spanning list in V.

(b) Let $V = \text{span}(B)$. Then $x^3 + 4x^2 + 4x + 3 = (x^3 + 3x^2) + (x^2 + 3x) + (x + 3) \in V$. Hence also

$$1 = (x^3 + 4x^2 + 4x + 4) - (x^3 + 4x^2 + 4x + 3) \in V$$

Hence $x = (x + 3) - 3 \cdot 1 \in V$, $x^2 = (x^3 + 3x) - 3x \in V$ and $x^3 = (x^3 + 4x^2) - 4x^2 \in V$. Since $(1, x, x^2, x^3)$ spans $\mathbb{P}_3$ we conclude that B spans $\mathbb{P}_3$. Since $\dim P_3 = 4$ any list of length 4 which spans $\mathbb{P}_3$ is a basis. So B is a basis for $\mathbb{P}_3$.

17. Let X and Y be subspaces of a vector space V. Put $X + Y := \{x + y \mid x \in X, y \in Y\}$.

(a) State the definition of a subspace of a vector space.

(b) State the Subspace Theorem

(c) Prove that $X + Y$ is a subspace of V.

(d) Prove that $\dim(X) \leq \dim(X + Y)$.

Solution: (a) Let $\mathbf{V} = (V, \oplus, \odot)$ be a vector space and W a subset of V. Put $\mathbf{W} = (W, \oplus, \odot)$. Then $\mathbf{W}$ is called a subspace of $\mathbf{V}$ provided that $\mathbf{W}$ is a vector space.

(b) Let $\mathbf{V} = (V, \oplus, \odot)$ be a vector space and W a subset of V. Put $\underset{\sim}{W} = (W, \oplus, \odot)$. Then $\mathbf{W}$ is a subspace of $\mathbf{V}$ if and only if the following three conditions hold:

(1) $\mathbf{0_V} \in W$,

(2) $a + b \in W$ for all $a, b \in W$, and

(3) $ra \in W$ for all $a \in W$ and $r \in \mathbb{R}$.

(c) Observe first that by definition of $X + Y$:

(*) Let v be an object. Then $v \in X + Y$ if and only if there exist $x \in X$ and $y \in Y$ with $v = x + y$.

We will now verify that the three Conditions of the Subspace Theorem are fulfilled for $X + Y$.

(1) Since X and Y are subspace of V the Subspace Theorem shows that $\mathbf{0} \in X$ and $\mathbf{0} \in Y$. Since $\mathbf{0} = \mathbf{0} + \mathbf{0}$ we conclude from (*) that $\mathbf{0} \in X + Y$.

(2) Let $a, b \in X + Y$. Then by (*) there exist $x, x' \in X$ and $y, y' \in Y$ with $a = x + y$ and $b = x' + y'$. Since X and Y are subspaces of V, the Subspace Theorem shows that $x + x' \in X$ and $y + y' \in Y$. Since $a + b = (x + y) + (x' + y') = (x + x') + (y + y')$ we conclude from (*) that $a + b \in X + Y$.

(3) Let $a \in X$ and $r \in \mathbb{R}$. Then by (*) there exist $x \in X$ and $y \in Y$ with $a = x + y$. Since X and Y are subspaces of V, the Subspace Theorem shows that $rx \in X$ and $ry \in Y$. Since

$ra = r(x+y) = rx + ry$ we conclude from (*) that $a + b \in X + Y$.We verified the three conditions of the Subspace Theorem and so $X + Y$ is a subspace of V.

(d) Let $x \in X$. Then $x = x + \mathbf{0}$ and so $x \in X + Y$. Let $n = \dim(X)$ and let $(x_1, \ldots, x_n)$ be a basis for X. Then $(x_1, \ldots, x_n)$ is linearly independent list in $X + Y$ and so by the Comparison Theorem $n \leq \dim(X + Y)$. So $\dim(X) \leq \dim(X + Y)$

18. Let V be a vector space and $\{v_1, \ldots, v_{n+1}\} \subseteq V$ a set of linearly independent vectors of V. Prove without quoting any theorem:
   (a) The set $\{v_1, \ldots, v_n\} \subseteq V$ is linearly independent.
   (b) $v_{n+1} \notin \operatorname{span}\{v_1, \ldots, v_n\}$

Solution: (a) Let $r_1, r_2 \ldots, r_n \in \mathbb{R}$ with $r_1v_1 + \ldots + r_nv_n = \mathbf{0}$. Then

$$r_1v_1 + \ldots + r_nv_n + 0v_{n+1} = r_1v_1 + \ldots + r_nv_n + \mathbf{0} = r_1v_1 + \ldots + r_nv_n = \mathbf{0}$$

and since $(v_1, \ldots, v_n, v_{n+1})$ is linearly independent we conclude that $r_1 = 0, r_2 = 0, \ldots, r_n = 0, 0 = 0$. Thus $(v_1, \ldots, v_n)$ is linearly independent.

(b) Suppose for a contradiction that $v_{n+1} \in \operatorname{span}\{v_1, \ldots, v_n\}$. Then there exist $r_1, \ldots, r_n \in \mathbb{R}$ with $v_{n+1} = r_1v_1 + \ldots + r_nv_n$. Thus

$$r_1v_1 + \ldots + r_nv_n + (-1)v_{n+1} = v_{n+1} - v_{n+1} = 0$$

a contradiction since $-1 \neq 0$ and $\{v_1, \ldots, v_{n+1}\}$ is linearly independent.

19. Let V be an infinite dimensional vector space. Use induction to show that for every $n \in \mathbb{N}$ there is a subspace $S_n \subseteq V$ with $\dim(S_n) = n$.

Solution: For a non-negative integer n let Q_n be the statement:

There exists a subspace S of V with $\dim S = n$.

Note that $\{\mathbf{0_V}\}$ is a 0-dimensional subspace of V and so S_0 is true.

Suppose that Q_n is true. Then there exists an n-dimensional subspace S of V. Since V is infinite dimensional, $S \neq V$ and so there exists $v \in V \setminus S$. Let $(v_1, \ldots, v_n)$ be a basis for S and put $T = \operatorname{span}(v_1, \ldots, v_n, v)$. Then T has a spanning list of length $n+1$ and a linear independent list of length n. So by the Comparison Theorem, $n \leq \dim T \leq n+1$. If $\dim T = n$, then $(v_1, \ldots, v_n)$ would be a basis for T, a contradiction since $v \in T$ but $v \notin S = \operatorname{span}(v_1, \ldots, v_n)$. Thus $\dim T = n + 1$. So Q_{n+1} is true.

The principal of induction now show that Q_n holds for all non-negative integers n.

20. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear function given by:

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+b \\ a-2b+2c \end{bmatrix}.$$

(a) Find the matrix of T with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

(b) Is T one-to-one? Justify your answer!

(c) Is T onto? Justify your answer!

Solution: (a) We have

$$\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+b \\ a-2b+2c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

and so the matrix of T is $\begin{bmatrix} 1 & 1 & 0 \\ 1 & -2 & 2 \end{bmatrix}$.

(b) Note that

$$T\left(\begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right).$$

and so T is not 1-1.

(c) Let $\begin{bmatrix} c \\ d \end{bmatrix} \in \mathbb{R}^2$. Then

$$T\left(\begin{bmatrix} c \\ 0 \\ \frac{d-c}{2} \end{bmatrix}\right) = \begin{bmatrix} c \\ d \end{bmatrix}$$

and so T is onto.

Solution: We will give a second, more systematic, solution.

(a)

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

and so the matrix of A with respect to the standard bases is

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -2 & 2 \end{bmatrix}$$

(b) and (c) We use the Gauss-Jordan algorithm to determine $\dim \mathrm{Col}A = \dim \mathrm{Im}T$ and $\dim \mathrm{Nul}A = \dim \ker T$:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -2 & 2 \end{bmatrix} \xrightarrow{R2 - R1 \to R1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -3 & 2 \end{bmatrix} \xrightarrow[-\frac{1}{3}R2 \to R2]{R1 + \frac{1}{3}R2 \to R1} \begin{bmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & -\frac{2}{3} \end{bmatrix}$$

Since A has one free variable, $\dim \mathrm{Nul}A = 1$. Thus $\ker T = \mathrm{Nul}A \neq \{\mathbf{0}\}$ and T is not 1-1. Since A has two lead variables, $\dim \mathrm{Col}A = 2$. Thus $\mathrm{Im}T = \mathrm{Col}A = \mathbb{R}^2$ and T is onto.

21. Let $T : \mathbb{R}^6 \to \mathbb{M}(3, 2)$ be a linear function. Show:

(a) If T is one-to-one, then T is onto.

(b) If T is onto, then T is one-to-one.

Solution: Note that $\dim \mathbb{R}^6 = 6 = \dim \mathbb{M}(3,2)$. Also $\dim \ker T + \dim \mathrm{Im} T = \dim \mathbb{M}(3,2)$ by the dimension formula. Thus

$$\begin{array}{cc} & T \text{ is 1-1} \\ \iff & \ker T = \{\mathbf{0}\} \\ \iff & \dim \ker T = 0 \\ \iff & \dim \mathbb{M}(3,2) = \dim \mathrm{Im} T \\ \iff & \mathbb{M}(3,2) = \mathrm{Im} T \\ \iff & T \text{ is onto} \end{array}$$

22. Let A be an $m \times n$ matrix and $\mu_A : \mathbb{R}^n \to \mathbb{R}^m$ the linear function defined by $\mu_A(v) = Av$. Let $b \in \mathbb{R}^m$. Show that b lies in the image of μ_A if and only if the linear system $Ax = b$ has a solution.

Solution: We have

$$\begin{array}{ccl} & b \in \mathrm{Im}(\mu_A) & \\ \iff & b = \{\mu_A(x) \mid x \in \mathbb{R}^n\} & - \text{ definition of } \mathrm{Im}(\mu_A) \\ \iff & b = \mu_A(x) \text{ for some } x \in \mathbb{R}^n & - \text{ definition of } \{\mu_A(x) \mid x \in \mathbb{R}^n\} \\ \iff & b = Ax \text{ for some } x \in \mathbb{R}^n & - \text{ definition of } \mu_A \\ \iff & Ax = b \text{ has a solution} & - \text{ definition of solution} \end{array}$$

23. Let A be an $n \times n$ matrix and $v, w \in \mathbb{R}^n \setminus \{0\}$ with $Av = v$ and $Aw = 3w$. Show directly that the set $\{v, w\}$ is linearly independent. (Don't just quote a theorem!)

Solution: Let $r, s \in \mathbb{R}$ with

$$(*) \qquad rv + sw = \mathbf{0}$$

Then $\mathbf{0} = A\mathbf{0} = A(rv + sw) = rAv + sAw = r(-v) + s(3w) = -rv + 3sw$. So

$$-rv + 3sw = \mathbf{0}$$

Adding this equation to (*) we get $4sw = \mathbf{0}$. Since $w \neq \mathbf{0}$ this gives $4s = 0$ and so $s = 0$. Thus (*) implies $rv = \mathbf{0}$ and since $v \neq \mathbf{0}$, $r = 0$. So (v, w) is linearly independent.

24. Consider the ordered basis

$$B = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

of $\mathbb{R}^3$ and the linear function $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by:

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+c \\ 2a+b+c \\ b+c \end{bmatrix}.$$

Find the matrix of T with respect to the bases B and B.

Solution: Let $E = (e_1, e_2, e_3)$ be the standard basis for $\mathbb{R}^3$. Since

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+c \\ 2a+b+c \\ b+c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

the matrix of T with respect to E and E is

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Since $[x]_E = x$ for all $x \in \mathbb{R}^3$, the change-of-basis matrix from B to E is

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Thus the matrix A' for T with respect to B and B is $P^{-1}AP$. We use the Gauss-Jordan algorithm to compute that inverse of P:

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R3} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \xrightarrow[R2 - R3 \to R2]{R1 - R2 \to R1} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

So

$$P^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad P^{-1}A = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and

$$A' = (P^{-1}A)P = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

25. Let V, W be finite dimensional vector spaces and X a subspace of V. Prove that the following two statements are equivalent

    (a) $\dim X + \dim W \geq \dim V$.

    (b) There exists a linear function $T : V \to W$ with $\ker T = X$.

Solution:

(a) $\implies$ (b): Suppose that $\dim X + \dim W \geq \dim V$. Let $l = \dim X$, $n = \dim V$ and $m = \dim W$. Then $l + m \geq n$ and so $n - l \leq m$. Put $k = n - l$. Then $n = l + k = \dim V$ and $k \leq m$. Let $(x_1, \ldots, x_l)$ be a basis for X. By the Expansion Theorem, $(x_1, \ldots, x_l)$ can be expanded to a basis $(x_1, \ldots, x_l, v_1, \ldots, v_k)$ of V. Let $(w_1, \ldots, w_m)$ be basis for W. By Theorem 6.9 there exists unique linear function $T : V \to W$ such that $T(x_i) = \mathbf{0}$ for $1 \leq i \leq l$ and $T(v_i) = w_i$ for $1 \leq i \leq k$. We claim that $\ker T = X$. Let $v \in V$. Since $(x_1, \ldots, x_l, v_1, \ldots, v_k)$ is a basis for V, $v = r_1x_1 + \ldots + r_lx_l + s_1v_1 + \ldots + s_kv_k$ for some $r_1, \ldots r_l, s_1, \ldots, s_k$ in $\mathbb{R}$. Then since T is linear, Theorem 6.2 shows that

$$\begin{aligned} T(v) &= r_1T(x_1) + \ldots + r_lT(x_l) + s_1T(v_1) + \ldots + s_kT(v_k) \\ &= r_1\mathbf{0} + \ldots r_l\mathbf{0} + s_1w_1 + \ldots + s_kw_k \\ &= s_1w_1 + \ldots + s_kw_k \end{aligned}$$

Thus $v \in \ker T$ if and only if $T(v) = \mathbf{0}$ and if and only if $s_1w_1 + \ldots + s_kw_k = \mathbf{0}$. Since $(w_1, \ldots, w_m)$ is linearly independent, this holds if and only of $s_1 = s_2 = \ldots = s_k = 0$.

Thus $v \in \ker T$ if and only if $v = r_1x_1 + \ldots + r_lx_l$ for some $r_1, \ldots r_k \in \mathbb{R}$ and if and only if $v \in X$. Thus $\ker T = X$. Hence (b) holds.

(b) $\implies$ (a): Suppose $T : V \to W$ is a linear function with $\ker T = X$. By the dimension formula, $\dim \ker T + \dim \mathrm{Im}T = \dim V$. Since $\ker T = X$, $\dim \ker T = \dim X$ and since $\mathrm{Im}T \leq W$, $\dim \mathrm{Im}T \leq W$. Thus

$$\dim X + \dim W \geq \dim \ker T + \dim \mathrm{Im}T = \dim V$$

So (a) holds.

26. Compute the inverse and determinant of $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 2 & 5 & 8 \end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 4 & 5 & 0 & 1 & 0 \\ 2 & 5 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R3-2R1 \to R3]{R2-R1 \to R2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 \\ 0 & 1 & 2 & -2 & 0 & 1 \end{array}\right] \xrightarrow[R2 \leftrightarrow R3]{R1-R2 \to R1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -2 & 0 & 1 \\ 0 & 2 & 2 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R3-2R2 \to R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -2 & 0 & 1 \\ 0 & 0 & -2 & 3 & 1 & -2 \end{array}\right] \xrightarrow[\substack{R2+R3 \to R2 \\ -\frac{1}{2}R3 \to R3}]{R1+\frac{1}{2}R3 \to R1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -\frac{3}{2} & -\frac{1}{2} & 1 \end{array}\right]$$

So the inverse is

$$\begin{bmatrix} \frac{7}{2} & -\frac{1}{2} & -1 \\ 1 & 1 & -1 \\ -\frac{3}{2} & -\frac{1}{2} & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 7 & -1 & -2 \\ 2 & 2 & -2 \\ -3 & -1 & 2 \end{bmatrix}$$

All but two of the row operations were adding one row to another and so did not change the determinant. One of the operation was interchanging two rows and one was multiplication of a row by $-\frac{1}{2}$. So the determinant of the original matrix is

$$(-1)\frac{1}{-\frac{1}{2}} = 2.$$