MTH 309-4 **Linear Algebra I/ Practice Exam 2** **F11 10/26/11**

Solutions

#1. Use induction to prove that $2 \cdot 3^n \geq 7n - 1$ for all positive integers n.

Let S_n be the statement: $2 \cdot 3^n \geq 7n - 1$.
S_1 is equivalent to $2 \cdot 3 \geq 7 - 1$ and so to $6 \geq 6$. Hence S_1 holds.
Suppose now that S_n holds. Thus

$$(*) \qquad 2 \cdot 3^n \geq 7n - 1$$

We have

$$\begin{array}{rcll} 2 \cdot 3^{n+1} & = & 3(2 \cdot 3^n) & \\ & \geq & 3(7n-1) & \text{– (*) multiplied by 3} \\ & = & 7n + 14n - 3 & \\ & = & 7(n+1) - 1 + (14n - 9) & \\ & \geq & 7(n+1) - 1 & \text{– since } n \geq 1 \text{ and so } 14n - 9 \geq 0 \end{array}$$

Thus S_{n+1} holds. We proved that S_1 holds and that S_n implies S_{n+1}. So by the principal of induction, S_n holds for all positive integers n.

#2. Which of the following lists are linearly independent in the indicated vector space? (do not justify your answer)

(a) $\big((1,2,3),(1,1,1),(2,3,4)\big)$ in $\mathbb{R}^3$.

(b) $\left(\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}\right)$ in $\mathbb{M}(2,2)$.

(c) $(1, \sin^2 x, \cos^2 x)$ in $\mathbf{F}(\mathbb{R})$.

(a): The third vector is the sum of the last two. So the list is linearly dependent by Theorem 3.5.

(b): Let (A_1, A_2, A_3) be the list in (b). Let $(x_1, x_2, x_3) \in \mathbb{R}^3$ and put $B = x_1A_1 + x_2A_2 + x_3A_3$. Suppose that $B = \mathbf{0}$. Since x_1 is the $(1,1)$-entry of B we conclude that $x_1 = 0$. Hence x_2 is the $(1,2)$-entry of B and so $x_2 = 0$. Thus x_3 is the $(3,1)$-entry of B and so $x_3 = 0$. It follows that list is linearly independent.

(c) Since $\sin^2 x + \cos^2 x = 1$, the first vector is the sum of the first two. So the list is linearly dependent by Theorem 3.5.

#3. Let $A = \begin{bmatrix}1 & 0 & 1 & 0 & 1\\0 & 1 & 2 & 0 & 4\\0 & 1 & 2 & 1 & 5\\0 & 1 & 2 & 0 & 4\end{bmatrix}$. Find bases for $\mathrm{Col}A, \mathrm{Row}A$ and $\mathrm{Nul}A$.

Subtracting the second row of A from the third and fourth we obtain the matrix

$$B = \begin{bmatrix}1 & 0 & 1 & 0 & 1\\0 & 1 & 2 & 0 & 4\\0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

Note that B is in reduced echelon form.
By Theorem N3.7.5 the non-zero rows of B form a basis for $\mathrm{Row}A$. Thus

$$\big((1,0,1,0,1),(0,1,2,0,4),(0,0,0,1,0)\big)$$

is a basis for Row$A$.

By Theorem N3.7.5 the columns of A corresponding to the lead variables of B form a basis for Col$A$. The lead variables are x_1, x_2 and x_4 and so

$$\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right)$$

is a basis for Col$A$.

The free variables are x_3 and x_5. Solving the homogeneous system of equation corresponding to B gives:

$$\begin{array}{rcrcr} x_1 & = & -1x_3 & + & -1x_5 \\ x_2 & = & -2x_3 & + & -4x_5 \\ x_3 & = & 1x_3 & + & 0x_5 \\ x_4 & = & 0x_3 & + & 0x_5 \\ x_5 & = & 0x_3 & + & 1x_5 \end{array}$$

So by Theorem N3.7.5

$$\left(\begin{bmatrix}-1\\-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\-4\\0\\0\\1\end{bmatrix}\right)$$

is a basis for Nul$A$.

#4. Let $\mathbf{V}$ be a vector space, I a set and $x, y \in I$. Define the function $T : F(I, V) \to V$ by $T(f) = f(x) + f(y)$ for all $f \in F(I, V)$. Show that T is linear.

Let $f, g \in F(I, V)$ and $r \in \mathbb{R}$. Then

$$\begin{array}{cll} & T(f+g) & \\ = & (f+g)(x) + (f+g)(y) & \text{– definition of } T \\ = & \big(f(x) + g(x)\big) + \big(f(y) + g(y)\big) & \text{– definition of addition in } \mathbf{F}(I, V) \\ = & \big(f(x) + f(y)\big) + \big(g(x) + g(y)\big) & \text{– axioms of the vector space } V \\ = & T(f) + T(g) & \text{– definition of } T\text{, twice} \end{array}$$

and

$$\begin{array}{cll} & T(rf) & \\ = & (rf)(x) + (rf)(y) & \text{– definition of } T \\ = & r\big(f(x)\big) + r\big(f(y)\big) & \text{– definition of multiplication in } \mathbf{F}(I, V) \\ = & r\big(f(x) + f(y)\big) & \text{– Axiom 5 of the vector space } V \\ = & r\big(T(f)\big) & \text{– definition of } T \end{array}$$

So T is linear.

#5. True or false (do not justify your answer)

(a) Every vector space has a basis.

(b) Let $\mathbf{V}$ be a vector space with $\dim \mathbf{V} = 10$. Then any list of length 11 in V spans $\mathbf{V}$.

(c) If $\mathbf{W}$ is a 7-dimensional subspace of a 7-dimensional vector space $\mathbf{V}$, then $V = W$.

(d) If $\mathbf{U}$ and $\mathbf{W}$ are subspace of a vector space $\mathbf{V}$, then $\dim \mathbf{U} + \dim \mathbf{W} \leq \dim \mathbf{V}$.

(e) Let $(v_1, \dots, v_n)$ be a list in the vector space $\mathbf{V}$. Suppose that $(v_1, \dots, v_n, v)$ is linearly dependent for all $v \in V$. Then $(v_1, \dots, v_n)$ spans V.

(a) is false. For example $\mathbb{P}$ is infinite dimensional and so does not have a basis.
(b) is false. For example the list $(v_1, \dots, v_{11})$ with $v_i = \mathbf{0}$ for all $1 \leq i \leq 11$, does not span $\mathbf{V}$.
(c) is true by exercise 3.6.# 21.
(d) is false. For example if $U = V$ and $W = V$ and $\dim \mathbf{V} > 0$, then $\dim \mathbf{U} + \dim \mathbf{W} = 2 \dim \mathbf{V} > \dim \mathbf{V}$.
(e) is false. For example if $n = 1$ and $v_1 = \mathbf{0}$, then (v_1, v) is linearly dependent for all $v \in V$. But if $\dim V > 0$, $(\mathbf{0})$ does not span $\mathbf{V}$.

#6. Let $\mathbf{V}$ be a vector space, $(v_1, \dots, v_n)$ a list in V and $v \in V$. Prove that the following two statements are equivalent:

(a) $(v_1, \dots, v_n, v)$ is linearly dependent.

(b) $(v_1, \dots, v_n)$ is linearly dependent or $v \in \operatorname{span}(v_1, \dots, v_n)$

(a) $\Longrightarrow$ (b): Suppose that $(v_1, \dots, v_n, v)$ is linearly dependent. Then there exist $r_1, \dots, r_n, r \in \mathbb{R}$, not all zero with

$$r_1 v_1 + \ldots + r_n v_n + rv = \mathbf{0}.$$

Note that either $r \neq 0$ or $r = 0$.
If $r \neq 0$, then by Lemma N3.3.2, $v \in \operatorname{span}(v_1, \dots, v_n)$. So (b) holds in this case.
If $r = 0$, then $r_i \neq 0$ for some $1 \leq i \leq n$ and

$$r_1 v_1 + \ldots + r_n v_n = \mathbf{0}.$$

So $(v_1, \dots, v_n)$ is linearly dependent and again (b) holds.

(b) $\Longrightarrow$ (a): Suppose that $(v_1, \dots, v_n)$ is linearly dependent. Then there exist $r_1, \dots, r_n \mathbb{R}$, not all zero with

$$r_1 v_1 + \ldots + r_n v_n = \mathbf{0}$$

Then

$$r_1 v_1 + \ldots + r_n v_n + 0v = \mathbf{0}$$

and so $(v_1, \dots, v_n, v)$ is linearly dependent.

Suppose that $v \in \operatorname{span}(v_1, \dots, v_n)$. Then by Theorem 3.5, $(v_1, \dots, v_n, v)$ is linearly dependent.
So either of the two conditions in (b) imply (a) and so (b) implies (a).

#7. Define the function $T : \mathbb{P} \to \mathbb{P}$ by $T(a_0 + a_1 x + \ldots + a_n x^n) = a_1 + a_2 x + \ldots + a_n x^{n-1}$.

(a) Is T linear?

(b) Is T 1-1?

(c) Is T onto?

(Justify all you answers)

(a): Yes, T is linear. *Proof:* Let $p, q \in \mathbb{P}$ and let n be the larger of the degrees of p and q. Then $p = a_0 + a_1 x + \ldots a_n x^n$ and $q = b_0 + b_1 x + \ldots b_n x^n$ for some real numbers $a_0, \ldots, a_n, b_0, \ldots, b_n$. Then

$$p + q = (a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_n + b_n)x^n.$$

So using the definition of T three times

$$T(p+q) = (a_1+b_1)+(a_2+b_2)x+\ldots+(a_n+b_n)x^{n-1} = (a_1+a_2x+\ldots+a_nx^{n-1})+(b_1+b_2x+\ldots+b_nx^{n-1}) = T(p)+T(q)$$

Also for $r \in \mathbb{R}$, $rp = ra_0 + ra_1 \ldots + ra_n x^n$ and using the definition of T twice,

$$T(rp) = ra_1 + ra_2 x + \ldots + ra_n x^{n-1} = r(a_1 + a_2 x + \ldots a_n x^{n-1}) = r(T(p))$$

We proved that $T(p+q) = T(p) + T(q)$ and $T(rp) = r(T(p))$ and so T is linear.

(b) We have $T(0) = 0 = T(1)$ and so T is not 1-1.

(c) Let $p \in \mathbb{P}$. Then $p = a_0 + a_1 x + \ldots + a_n x^n$ for some $a_0, \ldots, a_n$ is $\mathbb{R}$. Put $q = a_0 x + a_1 x^2 + \ldots + a_n x^{n+1}$. Then $T(q) = p$ and so T is onto.