MTH 309-4 Linear Algebra I/ Practice Exam I/Solutions F11 9/28/07

Show all your work

#1. Determine the solution set of

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & = & 4 \\ 2x_1 & + & 4x_2 & + & 7x_3 & = & 9 \\ -x_1 & + & x_2 & - & x_3 & = & 1 \end{array}$$

#2. Determine the solution set of

$$\begin{array}{rcrcrcrcrcrcr} x_1 & + & 2x_2 & & & + & 3x_4 & - & 2x_5 & & & = & 4 \\ & & & & x_3 & + & x_4 & + & x_5 & & & = & 9 \\ & & & & & & & & & & x_6 & = & -3 \end{array}$$

#3. For a, b in $\mathbb{R}$ define $a \oplus b = a + b$ and $a \odot b = ab^2$. Is $\mathbb{R}$ a vector space with these operations?

We will show that axiom 5 fails. Axiom 5 says:

$$a \odot (v \oplus w) = (a \odot v) \oplus (a \odot w) \tag{1}$$

for all $a \in \mathbb{R}$ and $v, w \in V$. Note that $V = \mathbb{R}$ in this exercise, so $a, v, w \in \mathbb{R}$. Using the definition of $\oplus$, (1) is equivalent to

$$a \odot (v + w) = (a \odot v) + (a \odot w) \tag{2}$$

Using the definition of $\odot$, (2) is equivalent to

$$a(v + w)^2 = av^2 + aw^2 \tag{3}$$

Computing in $\mathbb{R}$ we see that this is equivalent to

$$a(v^2 + 2vw + w^2) = av^2 + aw^2 \tag{4}$$

and so also equivalent to

$$2avw = 0 \tag{5}$$

Note that (5) is false for $a = 1, v = 1$ and $w = 1$ and so Axiom 5 fails. Thus $(\mathbb{R}, \oplus, \cdot)$ is not a vector space.

#4. Let a, b, c be vectors in a vector space V. Show that

$$2\big((4a+7c)+b\big) = (8a+2b)+14c.$$

(Show all your steps. In each step use at most one of the vector space axioms, and indicate which axiom your are using)

$$\begin{array}{rcll} 2\big((4a+7c)+b\big) & = & 2(4a+7c)+2b & \text{−Ax 5} \\ & = & \big(2(4a)+2(7c)\big)+2b & \text{−Ax 5} \\ & = & \big((2\cdot 4)a+(2\cdot 7)c\big)+2b & \text{−Ax 7, twice} \\ & = & (8a+14c)+2b & \text{−Properties of } \mathbb{R} \\ & = & 8a+(14c+2b) & \text{−Ax 2} \\ & = & 8a+(2b+14c) & \text{−Ax 1} \\ & = & (8a+2b)+14c & \text{−Ax 2} \end{array}$$

#5. Let I be a set, a a fixed element of I and put

$$W = \{f \in F(I) \mid f(a) = 0\}.$$

Show that W is a subspace of $F(I)$.

Recall first that $F(I)$ consists of all real valued function with domain I and that the vector addition and scalar multiplication on $F(I)$ are defined by

$$(f+g)(i) = f(i) + g(i) \tag{$*$}$$

and

$$(rf)(i) = rf(i) \tag{$**$}$$

for all $f, g \in F(I)$, $r \in \mathbb{R}$ and $i \in I$.

Next we determine when a given element f in $F(I)$ is an element of W. The definition of W says:

(***) Let $f \in F(I)$. Then $f \in W$ if and only if $f(a) = 0$.

We are now able to verify that the three conditions in the Subspace Theorem hold:

Condition (1): We need to show that $0_{F(I)} \in W$.

Recall that the additive identity in $F(I)$ is the real valued function 0^* defined by $0^*(i) = 0$ for all $i \in I$. In particular, $0^*(a) = 0$ and so (*) implies that $0^* \in W$. So Condition (1) is verified.

Condition (2): We need to show that $f + g \in W$ for all $f, g \in W$.

To check whether $f + g \in W$ we use (***):

$$\begin{array}{rcll} (f+g)(a) & = & f(a) + g(a) & -(*) \\ & = & 0 + 0 & \text{−follows from (***) since } f, g \in W \\ & = & 0 & \text{−Properties of real numbers} \end{array}$$

Thus $(f + g)(a) = 0$ and so $f + g \in W$ by (***). So Condition (2) is verified.

Condition (3): We need to show that $rf \in W$ for all $r \in \mathbb{R}$ and $f \in W$.

To check whether $rf \in W$ we use (***):

$$\begin{array}{rcll} (rf)(a) & = & rf(a) & -(**) \\ & = & r0 & \text{−follows from (*) since } f \in W \\ & = & 0 & \text{−Properties of real numbers} \end{array}$$

Thus $(rf)(a) = 0$ and so $rf \in W$ by (***). So Condition (3) is verified.

We verified all three conditions of the Subspace Theorem and so W is a subspace of $F(I)$.