Solutions

#1. Use induction to prove that

$$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

for all positive integers n.

For a positive integer n, let S_n be the statement:

$$(S_n) \qquad 1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

S_1 is the statement $1^2 = \frac{1(2\cdot 1-1)}{3}$, which is equivalent $1 = 1$. Hence S_1 holds.
Suppose now that S_n holds. We compute

$$\begin{aligned}
& 1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 + (2(n+1)-1)^2 \\
= \ & \big(1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2\big) + (2n+1)^2 \\
= \ & \tfrac{n(2n-1)(2n+1)}{3} + (2n+1)^2 \qquad -(S_n) \\
= \ & \tfrac{n(2n-1)(2n+1)+3(2n+1)(2n+1)}{3} \\
= \ & \tfrac{(2n+1)\big(n(2n-1)+3(2n+1)\big)}{3} \\
= \ & \tfrac{(2n+1)(2n^2-n+6n+3)}{3} \\
= \ & \tfrac{(2n+1)(2n^2+5n+3)}{3} \\
= \ & \tfrac{(2n+1)(n+1)(2n+3)}{3} \\
= \ & \tfrac{(n+1)(2(n+1)-1)(2(n+1)+1)}{3}
\end{aligned}$$

Thus S_{n+1} holds. We proved that S_1 holds and that S_n implies S_{n+1}. So by the principal of induction, S_n holds for all positive integers n.

#2. Which of the following lists are linearly independent in the indicated vector space? (do not justify your answer)

(a) $\big((1,2,3,4), (1,0,1,0), (1,0,3,1)\big)$ in $\mathbb{R}^4$.

Let $a, b, c \in \mathbb{R}$ with

$$a(1,2,3,4) + b(1,0,1,0) + c(1,0,3,1) = (0,0,0,0)$$

Considering the second entry we see that $a = 0$. Then considering the last entry gives $c = 0$ and finally the first entry shows that $b = 0$. So the list is linearly independent.

(b) $\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$ in $\mathbb{M}(2,2)$.

The second matrix is the sum of the first and the last. So the list is linearly dependent by Theorem 3.5.

(c) $(x^2 + x + 1, x^2 - 1, 3x^2 + x - 1)$ in $\mathbb{P}_2$.

The third polynomial is the sum of the first and two times the second. So the list is linearly dependent by Theorem 3.5.

#3. Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 2 & 2 & 2 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$. Find bases for Col$A$, Row$A$ and Nul$A$.

We use the Gauss Jordan Algorithm to compute the reduced row echelon form of A.

$$A \xrightarrow[-R_1+R_3\to R_3]{-2R_1+R_2\to R_2} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_3\leftrightarrow R_4]{R_2\leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

Note that B is in reduced echelon form.
By Theorem N3.7.5 the non-zero rows of B form a basis for Row$A$. Thus

$$\big((1,1,1,0,0,1),(0,0,0,1,0,2),(0,0,0,0,1,2)\big)$$

is a basis for Row$A$.

By Theorem N3.7.5 the columns of A corresponding to the lead variables of B form a basis for Col$A$. The lead variables are x_1, x_4 and x_5 and so

$$\left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

is a basis for Col$A$.

The free variables are x_2, x_3 and x_6. Solving the homogeneous system of equation corresponding to B gives:

$$\begin{array}{rcrcrcr} x_1 & = & -1x_2 & + & -1x_3 & + & -1x_6 \\ x_2 & = & 1x_2 & + & 0x_3 & + & 0x_6 \\ x_3 & = & 0x_2 & + & 1x_3 & + & 0x_6 \\ x_4 & = & 0x_2 & + & 0x_3 & + & -2x_6 \\ x_5 & = & 0x_2 & + & 0x_3 & + & -2x_6 \\ x_6 & = & 0x_2 & + & 0x_3 & + & -1x_6 \end{array}$$

So by Theorem N3.7.5

$$\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -2 \\ -2 \\ 1 \end{bmatrix}\right)$$

is a basis for Nul$A$.

#4. Let $\mathbf{V}$ be a vector space, I a set and $x, y \in I$. Define the function $T : F(I,V) \to V$ by $T(f) = f(x) - f(y)$ for all $f \in F(I,V)$.

(a) Show that T is linear.

Let $f, g \in F(I,V)$ and $r \in \mathbb{R}$. Then

$$\begin{aligned} & T(f+g) & \\ = \;& (f+g)(x) - (f+g)(y) & \text{– definition of } T \\ = \;& \big(f(x)+g(x)\big) - \big(f(y)+g(y)\big) & \text{– definition of addition in } \mathbf{F}(I,V) \\ = \;& \big(f(x)-f(y)\big) + \big(g(x)-g(y)\big) & \text{– axioms of the vector space V, definition of } - \\ = \;& T(f) + T(g) & \text{– definition of } T, \text{twice} \end{aligned}$$

and

$$\begin{aligned} & T(rf) & \\ = \;& (rf)(x) - (rf)(y) & \text{– definition of } T \\ = \;& r\big(f(x)\big) - r\big(f(y)\big) & \text{– definition of multiplication in } \mathbf{F}(I,V) \\ = \;& r\big(f(x) - f(y)\big) & \text{– Thm 1.7(n) for the vector space } V \\ = \;& r\big(T(f)\big) & \text{– definition of} \end{aligned}$$

Thus T is linear.

(b) If $x \neq y$, show that T is onto.

Let $v \in V$. Define the function $f : I \to V$ by $f(x) = v$ and $f(z) = \mathbf{0}_V$ for all $z \in I$ with $z \neq x$. Since $x \neq y$ we have $f(y) = \mathbf{0}$. Thus

$$T(f) = f(x) - f(y) = v - \mathbf{0} = v$$

and so T is onto.

#5. True or false (do not justify your answer)

(a) Every finite dimensional vector space has a basis.

True It follows from the definition of a finite dimensional vector space in the book, or by Theorem N3.4.4 in my notes.

(b) Let $\mathbf{V}$ be a 12-dimensional vector space. Then $\mathbf{V}$ has a unique 6-dimensional subspace.

False. Let $(v_1, \dots, v_{12})$ be a basis for $\mathbf{V}$. Then both span$(v_1, \dots, v_6)$ and span$(v_7, \dots, v_{12})$ are 6-dimensional subspaces of $\mathbf{V}$. So $\mathbf{V}$ has a 6-dimensional subspace, but it is not unique.

(c) Let $\mathbf{V}$ be a vector space, $(v_1, \dots, v_n)$ a linearly independent list in $\mathbf{V}$ and $(w_1, \dots, w_m)$ a spanning list of $\mathbf{V}$. Then there exists a sublist $(u_1, \dots, u_l)$ of $(w_1, \dots, w_m)$ such that $(v_1, \dots, v_n, u_1, \dots, u_l)$ is a basis of $\mathbf{V}$.

True. Let $(u_1, \dots, u_l)$ be sublist of $(w_1, \dots, w_m)$ maximal such that $(v_1, \dots, v_n, u_1, \dots, u_l)$ is linearly independent. Then by N3.4.1 $(v_1, \dots, v_n, u_1, \dots, u_l)$ is a basis for V.

(d) Let $\mathbf{V}$ be a vector space with $\dim \mathbf{V} = 4$. Then there exists a spanning list of $\mathbf{V}$ which has length 7.

True Let $(v_1, \dots, v_4)$ be a basis for $\mathbf{V}$. Then for example $(v_1, v_2, v_3, v_4, v_1 + v_2, v_2 + v_3, v_3 + v_4)$ is a spanning list of length 7 for $\mathbf{V}$.

(e) If $(v_1, \ldots, v_n)$ is a linearly dependent list in the vector space $\mathbf{V}$, then v_n is a linear combination of $(v_1, \ldots, v_{n-1})$ in $\mathbf{V}$.

False For example $(0, 1)$ is a linearly dependent list in $\mathbb{R}$, but 1 is not a linear combination of (0).