MTH 309-4 **Linear Algebra I/ Exam I/Solutions** **F11 9/30/11**

#1. Determine the solution set of

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & - & 2x_3 & = & 1 \\ x_1 & + & 3x_2 & + & x_3 & = & 6 \\ 2x_1 & + & 4x_2 & + & x_3 & = & 3 \end{array}$$

We use the Gauss-Jordan algorithm applied the augmented matrix of linear system of equations:

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 1 & 3 & 1 & 6 \\ 2 & 4 & 1 & 3 \end{bmatrix} \begin{matrix} \scriptstyle -R1+R2\to R2 \\ \scriptstyle -2R1+R3\to R3 \end{matrix} \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 5 & 1 \end{bmatrix} \begin{matrix} \scriptstyle \frac{1}{5}R3\to R3 \\ \scriptstyle -2R2+R1\to R1 \end{matrix} \begin{bmatrix} 1 & 0 & -8 & -9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & \frac{1}{5} \end{bmatrix} \begin{matrix} \scriptstyle -3R3+R2\to R2 \\ \scriptstyle 8R3+R1\to R1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{-37}{5} \\ 0 & 1 & 0 & \frac{12}{5} \\ 0 & 0 & 1 & -\frac{1}{5} \end{bmatrix}$$

So the solution is $x_1 = \frac{-37}{5}, x_2 = \frac{12}{5}, x_3 = -\frac{1}{5}$ and the solution set is

$$S = \left\{ \left(\frac{-37}{5}, \frac{12}{5}, -\frac{1}{5} \right) \right\}$$

#2. Determine the solution set of

$$\begin{array}{rcrcrcrcrcrcl} x_1 & + & 2x_2 & - & 4x_3 & & & + & 7x_5 & & & = & -2 \\ & & & & & & x_4 & - & x_5 & & & = & 4 \\ & & & & & & & & & & x_6 & = & 12 \end{array}$$

Note that this system of linear equation is already in reduced echelon form. The leading variables are x_1, x_4 and x_6 and so the free variables are x_2, x_3 and x_5. Moving the free variables to the right we obtain

$$\begin{array}{rcrcrcrcr} x_1 & = & -2x_2 & + & 4x_3 & - & 7x_5 & - & 2 \\ x_2 & = & x_2 & & & & & & \\ x_3 & = & & & x_3 & & & & \\ x_4 & = & & & & & x_5 & + & 4 \\ x_5 & = & & & & & x_5 & & \\ x_6 & = & & & & & & & 12 \end{array}$$

So the solution set is

$$S = \left\{ x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 4 \\ 0 \\ 12 \end{bmatrix} \middle| \; x_2, x_3, x_5 \in \mathbb{R} \right\}$$

#3. For a, b in $\mathbb{R}$ define $a \oplus b = a + b$ and $a \odot b = a^2 b$. Is $\mathbb{R}$ a vector space with these operations?

We will show that Axiom 6 fails. Axiom 6 says:

$$(a+b) \odot v = (a \odot v) \oplus (b \odot v) \tag{1}$$

for all $a, b \in \mathbb{R}$ and $v \in V$. Note that $V = \mathbb{R}$ in this exercise, so $a, b, v \in \mathbb{R}$. Using the definition of $\oplus$, (1) is equivalent to

$$(a+b) \odot v = (a \odot v) + (b \odot v) \tag{2}$$

Using the definition of $\odot$, (2) is equivalent to

$$(a+b)^2 v = a^2 v + b^2 v \tag{3}$$

Computing in $\mathbb{R}$ we see that this is equivalent to

$$a^2 v + 2abv + bv^2 = a^2 v + b^2 v \tag{4}$$

and so also equivalent to

$$2abv = 0 \tag{5}$$

Note that (5) is false for $a = 1, b = 1$ and $v = 1$ and so Axiom 6 fails. Thus $(\mathbb{R}, \oplus, \cdot)$ is not a vector space.

Remark: All the other Axioms hold.

#4. Let V be a vector space, $r, s \in \mathbb{R}$ and $v, w \in V$. Prove that

$$(r - s)(v - w) = (rv - sv) + (sw - rw).$$

At each step use only one Definition, Theorem or Axiom and indicated exactly which Definition, Theorem or Axiom you are using.

$$\begin{array}{rcll}
(r-s)(v-w) & = & (r-s)v - (r-s)w & \text{–Theorem 1.7m} \\
 & = & (rv - sv) - (rw - sw) & \text{–Theorem 1.7n, twice} \\
 & = & (rv - sv) + (-(rw - sw)) & \text{–Definition of } - \\
 & = & (rv - sv) + (-(rw + (-sw))) & \text{–Definition of } - \\
 & = & (rv - sv) + (-(-(sw) - rw) & \text{–Theorem 1.7d} \\
 & = & (rv - sv) + (sw - rw) & \text{–Theorem 1.5f}
\end{array}$$

#5. Let V be the vector space and U and W subspaces of V. Prove that $U \cap W$ is a subspace of V.

Observe first that by definition of $U \cap W$:

(*) Let $v \in V$. Then $v \in U \cap W$ if and only if $v \in U$ and $v \in W$.

We will now verify the three conditions in the Subspace Theorem.

Condition (1): We need to show that $0_V \in U \cap W$.

Since U and W are subspaces of V we have $0_V \in U$ and $0_V \in W$ by the Subspace Theorem. Thus $0_V \in U \cap W$ by (*). So Condition (1) holds.

Condition (2): We need to show that $x + y \in U \cap W$ for all $x, y \in U \cap W$.

Let $x, y \in U \cap W$. Then $x, y \in U$ and $x, y \in W$ by (*). Since U and W are subspaces of V we have $x + y \in U$ and $x + y \in W$ by the Subspace Theorem. Hence $x + y \in U \cap W$ by (*).

Condition (3): We need to show that $rx \in U \cap W$ for all $r \in \mathbb{R}$ and $x \in U \cap W$.

Let $r \in \mathbb{R}$ and $x \in U \cap W$. By (*), $x \in U$ and $x \in W$.
Since U and W are subspaces of V we have $rx \in U$ and $rx \in W$ by the Subspace Theorem. Hence $rx \in U \cap W$ by (*).

We verified all three conditions of the Subspace Theorem and so $U \cap W$ is a subspace by the Subspace Theorem.