MTH 234-61 **Multivariable Calculus** **Quiz 10** **12/06/11**

Solutions

1. (3pts) Compute $\iint_S x d\sigma$, where S is the part of the plane $2x+2y+z=2$ in the first octant.

Solution: Note that $2x+2y+z=2$ if and only if $z=2(1-x-y)$. Thus (x,y,z) lies on S if and only if

$$z=2(1-x-y), z\geq 0, x\geq 0, y\geq 0$$

and so S can be parameterized via

$$S: \qquad \vec{r}=\langle x,y,2(1-x-y)\rangle \qquad x\geq 0, \quad y\geq 0 \quad x+y\leq 1$$

Put $f=2(1-x-y)$ and let R be the triangle $x\geq 0, y\geq 0, x+y\leq 1$. Then

$$S: \qquad \vec{r}=\langle x,y,f(x,y)\rangle \qquad (x,y) \text{ in } R$$

and

$$R: \qquad 0\leq x\leq 1, \quad 0\leq y\leq 1-x$$

We compute

$$f_x=-2, f_y=-2$$
$$d\sigma=\sqrt{f_x^2+f_y^2+1}dA=\sqrt{(-2)^2+(-2)^2+1}dA=\sqrt{9}dA=3dA$$
$$xd\sigma=3xdA$$
$$\iint_S xd\sigma=\int_R 3xdA \qquad =3\int_{x=0}^{x=1}\int_{y=0}^{y=1-x} dydx \qquad =3\int_{x=0}^{x=1}(1-x)dx$$
$$=3\left[x-\frac{1}{2}x^2\right]_{x=0}^{x=1} \qquad =3\left(\frac{1}{2}-(-\frac{1}{2})\right) \qquad =3$$

2. (4pts) Let $\vec{F}=\langle y-z,z-x,xy\rangle$. Use Stokes' Theorem to compute the outward flux of $\operatorname{curl}\vec{F}$ across the surface S with parametrization $\vec{r}=\langle r\cos\theta, r\sin\theta, 1-r^2\rangle, \quad 0\leq\theta\leq 2\pi, 0\leq r\leq 1$.

Solution: Recall that the flux of $\operatorname{curl}\vec{F}$ across S is defined as

$$\text{Flux}=\iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,d\sigma$$

Stokes Theorem says that

$$\oint_C \vec{F}\cdot d\vec{r}=\iint_S \operatorname{curl}\vec{F}\cdot\vec{n}d\sigma$$

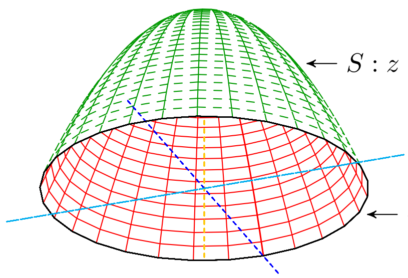
where C is the boundary of S. Note that the boundary of S consist of the points with $r = 1$ in the parameterization for S. So

$$C: \qquad \vec{r} = \langle \cos\theta, \sin\theta, 0\rangle, \quad 0 \le \theta \le 2\pi$$

is a parameterization for C.

Remark: It might be easier to see the boundary curve by using Cartesian coordinates to parameterize S:

$$S: \qquad \vec{r} = \langle x, y, 1 - x^2 - y^2\rangle \qquad x^2 + y^2 \le 1$$



We compute

$$\vec{r} = \langle \cos\theta, \sin\theta, 0\rangle$$
$$d\vec{t} = \vec{r}\,' d\theta = \langle -\sin\theta, \cos\theta\rangle d\theta$$
$$\vec{F} = \langle y - z, z - x, xy\rangle = \langle \sin\theta, -\cos\theta, \cos\theta\sin\theta\rangle$$
$$\vec{F} d\vec{r} = (-\sin^2\theta - \cos^2\theta - \cos\theta\sin\theta \cdot 0)d\theta = -1 d\theta$$
$$\text{Flux} = \iint_S \text{curl}\,\vec{F} \cdot \vec{n} d\sigma = \oint_C \vec{F} \cdot d\vec{r} = \int_{\theta=0}^{\theta=\pi} -1 d\theta = \boxed{-2\pi}$$

3. (3pts) Use the Divergence Theorem to compute the outward flux of the vector field $\vec{F} = \langle x^2, y + z, y^2\rangle$ across the boundary of the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$.

Solution: Let D be the cube $0 \le x \le 1$, $0 \le y \le 1$ and $0 \le z \le 1$. Let S be the boundary surface of D. Recall that the flux of $\vec{F}$ across S is defined as

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n}\, d\sigma$$

The Divergence Theorem says:

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iiint_D \text{div}\vec{F} dV$$

We compute

$$\vec{F} = \langle x^2, y+z, y^2 \rangle$$

$$\text{div}\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial y}(y+z) + \frac{\partial}{\partial z}y^2 = 2x+1$$

$$\text{Flux} = \int_S \vec{F} \cdot \vec{n} d\sigma = \iiint_D \text{div}\vec{F} dV \qquad = \int_{x=0}^{x=1} \int_{y=0}^{y=1} \int_{z=0}^{z=1} (2x+1) dz dy dx$$

$$= \int_{x=0}^{x=1} (2x+1) dx = \left[x^2 + x\right]_{x=0}^{x=1} = (1+1) - (0+0) = \boxed{2}$$