Solutions

1. (4 points) Find a potential for $\vec{F} = \langle 2xyz, x^2z, x^2y\rangle$ and evaluate $\int_C \vec{F} d\vec{r}$, where C is a smooth curve from $(0,0,0)$ to $(1,2,3)$.

Solution: We need to find a function f with $\nabla f = \vec{F}$, that is a function f with

$$f_x = 2xyz, \quad f_y = x^2z, \quad f_z = x^2y$$

From $f_x = 2xyz$ we get $f = x^2yz + g$, where g a function of y and z. So $f_y = x^2z + g_y$ and since $f_y = x^2z$ we get $g_y = 0$. Therefore $g = h$, where h is a function of z. Hence $f = x^2yz + h$ and $f_z = x^2y + h_z$. Since $f_z = x^2y$ we get $h_z = 0$. Thus $h = c$, c a constant. We are only asked to find find one potential, so we can choose $c = 0$. Then $\boxed{f = x^2yz}$

By the Fundamental Theorem of line integrals

$$\int_C \vec{F} d\vec{r} = f(1,2,3) - f(0,0,0) = 1^2 \cdot 2 \cdot 3 - 0 \cdot 0 \cdot 0 = 6 - 0 = \boxed{6}.$$

2. (4 points) Use the Flux-Divergence form of Green's Theorem to evaluate $\oint_C \langle 2x - 3y, x + y\rangle \cdot \vec{n}\, ds$, where C is the unit circle $\vec{r} = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$.

Solution: Let R be the unit circle $x^2 + y^2 \le 1$. Then C is the counter clockwise boundary of R. By Green's Theorem

$$\oint_C \langle 2x - 3y, x + y\rangle \cdot \vec{n}\, ds = \iint_R \operatorname{div}\langle 2x - 3y, x + y\rangle dA.$$

We have $\operatorname{div}\langle 2x - 3y, x + y\rangle = \frac{d}{dx}(2x - 3y) + \frac{d}{dy}(x + y) = 2 + 1 = 3$ and so

$$\iint_R \operatorname{div}\langle 2x - 3y, x + y\rangle dA = \iint_R 3 dA = 3 \iint_R dA = 3\,\text{area}(R) = \boxed{3\pi}.$$

3. (3 points) Find a parametrization for the portion of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant between the xy-plane and the cone $z = \sqrt{x^2 + y^2}$.

I will present three different solutions:

Solution 1: In terms of spherical coordinates the sphere $x^2 + y^2 + z^2 = 4$ can be described as $\rho = 2$. So

$$\vec{r} = \langle 2\sin\phi\cos\theta, 2\sin\phi\cos\theta, 2\cos\phi\rangle, \qquad 0 \le \phi \le \pi, 0 \le \theta \le 2\pi$$

is a parametrization of the whole sphere $x^2 + y^2 + z^2 = 4$.

The cone $z = \sqrt{x^2 + y^2}$ can be described as $\phi = \frac{\pi}{4}$, and the xy-plane as $\phi = \frac{\pi}{2}$. So the region between the xy plane and the cone $z = \sqrt{x^2 + y^2}$ can be described by $\frac{\pi}{4} \le \phi \le \frac{\pi}{2}$

The first octant is described by $0 \le \theta \le \frac{\pi}{2}$ and $0 \le \phi \le \frac{\pi}{2}$. So we obtain the following parametrization:

$$\boxed{\vec{r} = \langle 2\sin\phi\cos\theta, 2\sin\phi\cos\theta, 2\cos\phi\rangle, \qquad \frac{\pi}{4} \le \phi \le \frac{\pi}{2}, 0 \le \theta \le \frac{\pi}{2}}$$

Solution 2: In Cartesian coordinates the semisphere $x^2 + y^2 + z^2 = 4, z \ge 0$ can be described as $z = \sqrt{4 - x^2 - y^2}$ and so

$$\vec{r} = \langle x, y, \sqrt{4 - x^2 - y^2}\rangle, \qquad x^2 + y^2 \le 4$$

is a parametrization of the semisphere $x^2 + y^2 + z^2 = 4,\ z \ge 0$.

The region between the xy-plane and the cone $z = \sqrt{x^2 + y^2}$ can be described by $0 \le z \le \sqrt{x^2 + y^2}$. So for $z = \sqrt{4 - x^2 - y^2}$ this means $\sqrt{4 - x^2 - y^2} \le \sqrt{x^2 + y^2}$ and so $4 - x^2 - y^2 \le x^2 + y^2, 4 \le 2(x^2 + y^2)$ and $2 \le x^2 + y^2$. The first octant is $x \ge 0, y \ge 0, z \ge 0$ and we obtain the following parametrization:

$$\boxed{\vec{r} = \langle x, y, \sqrt{4 - x^2 - y^2}\rangle, \qquad 2 \le x^2 + y^2 \le 4, x \ge 0, y \ge 0}$$

Solution 3: In cylindrical coordinates the semisphere $x^2 + y^2 + z^2 = 4, z \ge 0$ can be described as $z = \sqrt{4 - r^2}$ and so

$$\vec{r} = \langle r\cos\theta, r\sin\theta, \sqrt{4 - r^2}\rangle, \qquad 0 \le r \le 2, 0 \le \theta \le 2\pi$$

is a parametrization of the semisphere $x^2 + y^2 + z^2 = 4,\ z \ge 0$.

The cone $z = \sqrt{x^2 + y^2}$ can be described by $z = r$, the xy-plane by $z = 0$ and so the region between the xy-plane and the cone $z = \sqrt{x^2 + y^2}$ can be described by $0 \le z \le r$. For $z = \sqrt{4 - r^2}$ this means $\sqrt{4 - r^2} \le r$ and so $4 - r^2 \le r^2, 4 \le 2r^2,\ 2 \le r^2$ and $\sqrt{2} \le r$. The first octant is $0 \le \theta \le \frac{\pi}{2}, z \ge 0$ and we obtain the following parametrization:

$$\boxed{\vec{r} = \langle r\cos\theta, r\sin\theta, \sqrt{4 - r^2}\rangle \qquad \sqrt{2} \le r \le 2, 0 \le \theta \le \frac{\pi}{2}}$$