Review Problems for MTH 234

1. Let $\mathbf{A} = -5\mathbf{i} + \mathbf{k}, \mathbf{B} = 2\mathbf{i} + 10\mathbf{j} + \sqrt{17}\mathbf{k}$. Find $\mathbf{A} \cdot \mathbf{B}$, $\mathbf{A} \times \mathbf{B}$, $|\mathbf{A}|$ and $\text{proj}_{\mathbf{A}}\mathbf{B}$.

Solution:

$$\mathbf{A} \cdot \mathbf{B} = \langle -5, 0, 1\rangle \cdot \langle 2, 10, \sqrt{17}\rangle = -5 \cdot 2 + 0 \cdot 10 + 1 \cdot \sqrt{17} = \boxed{\sqrt{17} - 10}.$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & 0 & 1 \\ 2 & 10 & \sqrt{17} \end{vmatrix} = \langle -10, -(-5\sqrt{17} - 2), -5 \cdot 10\rangle = \boxed{\langle -10, 5\sqrt{17} + 2, -50\rangle}$$

$$|\mathbf{A}| = |\langle -5, 0, 1\rangle| = \sqrt{(-5)^2 + 1} = \boxed{\sqrt{26}}$$

$$\text{proj}_{\mathbf{A}}\mathbf{B} = \frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{A}}\mathbf{A} = \boxed{\frac{\sqrt{17} - 10}{26}\langle -5, 0, 1\rangle}$$

2. Find the distance between the point $P(1, 2, 3)$ and the plane $5x - 3y + z = 5$.

Solution: Let $\mathbf{n} = \langle 5, -3, 1\rangle$. Note that $Q(1, 0, 0)$ is a point on the plane and so the distant d of P from plane $\mathbf{r} \cdot \mathbf{n} = 5$ is the absolute value of the scalar projection of $\overrightarrow{QP}$ onto $\mathbf{n}$:

$$d = \frac{|\overrightarrow{QP} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{|\langle 0, 2, 3\rangle \cdot \langle 5, -3, 1\rangle|}{|\langle 5, -3, 1\rangle|} = \frac{|0 - 6 + 3|}{\sqrt{25 + 9 + 1}} = \boxed{\frac{3}{\sqrt{35}}}$$

3. Find the equation of the line perpendicular to $5x - 3y + z = 5$ and through the point $Q(1, 2, 3)$. Compute the point of intersection between this line and the plane.

Solution: Let $\mathbf{n} = \langle 5, -3, 1\rangle$. The vector equation for the line is

$$\boxed{\mathbf{r} = \overrightarrow{OP} + \mathbf{n}t = \langle 1, 2, 3\rangle + \langle 5, -3, 1\rangle t, \qquad -\infty < t < \infty}$$

$\mathbf{r}$ is on the plane if $\mathbf{r} \cdot \mathbf{n} = 5$. So $(\overrightarrow{OP} + \mathbf{n}t) \cdot \mathbf{n} = 5$, $(\mathbf{n} \cdot \mathbf{n})t = 5 - \overrightarrow{OP} \cdot \mathbf{n}$ and

$$t = \frac{5 - \overrightarrow{OP} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} = \frac{5 - \langle 1, 2, 3\rangle \cdot \langle 5, -3, 1\rangle}{\langle 5, -3, 1\rangle \cdot \langle 5, -3, 1\rangle} = \frac{5 - (5 - 6 + 3)}{25 + 9 + 1} = \frac{3}{35}$$

So the position vector of the intersection point is

$$\langle 1, 2, 3\rangle + \langle 5, -3, 1\rangle \frac{3}{35} = \frac{1}{35}\langle 35 + 15, 70 - 9, 105 + 3\rangle = \frac{1}{35}\langle 50, 61, 108\rangle$$

Thus the intersection point is $\boxed{\left(\frac{50}{35}, \frac{61}{35}, \frac{108}{35}\right)}$

4. Find the parametric equations for the line that is tangent to the curve $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + \sin(2t)\mathbf{k}$ at $t_0 = \frac{\pi}{2}$. Find the unit tangent vector $\mathbf{T}(t)$.

Solution:

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 2\cos(2t)\rangle$$

and so

$$\mathbf{r}'(\frac{\pi}{2}) = \langle -1, 0, 2\rangle$$

is parallel to the line.

Also the $\mathbf{r}(\frac{\pi}{2}) = \langle 0, 1, 0\rangle$ is the position vector of a point on the line. So the vector equation is

$$\mathbf{r} = \langle 0, 1, 0\rangle + \langle -1, 0, 2\rangle t, \qquad -\infty \le t \le \infty$$

and the parametric equations are

$$\boxed{x = -t, y = 1, z = 2t, \qquad -\infty \le t \le \infty}$$

The unit tangent vector is

$$\mathbf{T}(t) = \frac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t) = \frac{1}{\sqrt{\sin^2 t + \cos^2 t + 4\cos^2(2t)}}\langle -\sin t, \cos t, 2\cos(2t)\rangle = \boxed{\frac{1}{\sqrt{1 + 4\cos^2(2t)}}\langle -\sin t, \cos t, 2\cos(2t)\rangle}$$

5. Let $w = w(x, y)$, $x = x(u, v)$, $y = y(u, v)$. Write out the formula for $\frac{\partial w}{\partial u}$. For $w = x^2 + yx^{-1}$, $x = 3u - 5v + 1$ and $y = 5u - v + 3$, find $\frac{\partial w}{\partial u}$ in terms of u and v only.

Solution:

$$\boxed{\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}}$$

For $w = x^2 + yx^{-1}$, $x = 3u - 5v + 1$ and $y = 5u - v + 3$ we have

$$\frac{\partial w}{\partial x} = 2x - yx^{-2} = \frac{2x^3 - y}{x^2} = \frac{2(3u - 5v + 1)^3 - (5u - v + 3)}{(3u - 5v + 1)^2}$$

$$\frac{\partial w}{\partial y} = x^{-1} = \frac{1}{3u - 5v + 1}$$

$$\frac{\partial x}{\partial u} = 3 \text{ and } \frac{\partial y}{\partial u} = 5$$

and so

$$\begin{aligned}
\frac{\partial w}{\partial u} &= \frac{2(3u - 5v + 1)^3 - (5u - v + 3)}{(3u - 5v + 1)^2}3 + \frac{1}{3u - 5v + 1}5 \\
&= \frac{6(3u - 5v + 1)^3 - 15u + 3v - 9 + (15u - 25v + 5)}{(3u - 5v + 1)^2} \\
&= \frac{6(3u - 5v + 1)^3 - 22v - 4}{(3u - 5v + 1)^2}
\end{aligned}$$

6. Consider $x^6yz^3 + 4x + 5y + 10z - 20 = 0$. Find $\frac{\partial x}{\partial y}$. Evaluate it at $P_0(1, 1, 1)$.

Solution:
We view x as a function of y and z. Differentiating both sides of $x^6yz^3 + 4x + 5y + 10z - 20 = 0$ with respect to y (and viewing z as a constant) we get:

$$6x^5yz^3\frac{\partial x}{\partial y} + x^6z^3 + 4\frac{\partial x}{\partial y} + 5 = 0$$

and so

$$\boxed{\frac{\partial x}{\partial y} = -\frac{5 + x^6z^3}{6x^5yz^3 + 4}}$$

At $(1, 1, 1)$ this evaluates to

$$\left.\frac{\partial x}{\partial y}\right|_{(1,1,1)} = -\frac{5+1}{6+4} = -\frac{6}{10} = \boxed{\frac{3}{5}}$$

7. Let $f(x, y) = xy + x^2 + 5y^2$. Find $L(x, y)$ at $P_0(1, 1)$. Find an estimate of the error $E(x, y)$ on $R : |x-1| \leq \frac{2}{10}, |y-1| \leq \frac{2}{10}$.

Solution: Part 1: We have $f(1, 1) = 1+1+5 = 7$, $\nabla f = \langle y+2x, x+10y\rangle$ and $\nabla f(1, 1) = \langle 1+2, 1+10\rangle = \langle 3, 11\rangle$. Thus

$$L(x, y) = f(1, 1) + \nabla f(1, 1) \cdot \langle x - 1, y - 1\rangle = 7 + \langle 3, 11\rangle \cdot \langle x - 1, y - 1\rangle = 7 + 3x - 3 + 11y - 11 = \boxed{3x + 11y - 7}$$

Part 2 Recall that

$$E(x, y) \leq \frac{1}{2}M(|x - x_0| + |y - y_0|)^2$$

where R is an upper bound for f_{xx}, f_{xy} and f_{yy} on R. Since $f_x = y + 2x$ and $f_y = x + 10y$ we have $f_{xx} = 2$, $f_{xy} = 1$ and $f_{yy} = 10$. So we can use $M = 10$. On R we have $|x - 1| \leq \frac{2}{10} = \frac{1}{5}$ and $|y - 1| \leq \frac{2}{10} = \frac{1}{5}$. Thus

$$E(x, y) \leq \frac{1}{2}10(\frac{1}{5} + \frac{1}{5})^2 = 5(\frac{2}{5})^2 = \frac{4}{5} = \frac{8}{10}$$

8. Find the derivative of $f(x, y, z) = \ln(xy) + yz + zx$ at $P_0(1, -1, 2)$ in the direction of $\mathbf{v} = \sqrt{8}\mathbf{i} + 3\mathbf{j} - \sqrt{8}\mathbf{k}$.

Solution:

$$\nabla f = \langle \frac{y}{x} + z, \frac{x}{y} + z, y + x\rangle$$

and so

$$\nabla f(P_0) = \langle \frac{-1}{1} + 2, \frac{1}{-1} + 2, -1 + 1\rangle = \langle 1, 1, 0\rangle,$$

$$|\mathbf{v}| = \sqrt{8 + 9 + 8} = \sqrt{25} = 5$$

and so the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v} = \frac{1}{5}\langle\sqrt{8}, 3, -\sqrt{8}\rangle$$

Thus

$$f_{\mathbf{u}} = \nabla f(P_0) \cdot \mathbf{u} = \langle 1, -1, 0\rangle \cdot \frac{1}{5}\langle\sqrt{8}, 3, -\sqrt{8}\rangle = \boxed{\frac{\sqrt{8}-3}{5}}$$

9. Find the equation of the tangent plane to the level surface $\ln(xy) + yz + xz + 1 = \ln(2) + 4$ at the point $P_0(2, 1, 1)$.

Solution: Let $f = \ln(xy) + yz + xz + 1$. Then

$$\nabla f = \langle \frac{y}{x} + z, \frac{x}{y} + z, y + x\rangle$$

and so

$$\nabla f(P_0) = \langle \frac{1}{2} + 1, \frac{2}{1} + 1, 1 + 2\rangle = \langle \frac{3}{2}, 3, 3\rangle = \frac{3}{2}\langle 1, 2, 2\rangle$$

So $\langle 1, 2, 2\rangle$ is a normal vector to the tangent plane. $P_0(2, 1, 1)$ is on the plane and $\langle 1, 2, 2\rangle \cdot \langle 2, 1, 1\rangle = 2 + 2 + 2 = 6$. Thus the equation of the tangent plane is

$$\boxed{x + 2y + 2z = 6}$$

10. Find all saddle points, all local maxima and local minima for $f(x, y) = x^3 + 3xy + y^3$.

Solution:

To find the critical points we solve the equation $\nabla f = \vec{0}$. We have

$$\nabla f = \langle 3x^2 + 3y, 3x + 3y^2\rangle$$

$$\begin{array}{rcl} \nabla f & = & \vec{0} \\ 3x^2 + 3y = 0 & \text{and} & 3x + 3y^2 = 0 \\ y = -x^2 & \text{and} & x = -y^2 \\ y = -x^2 & \text{and} & x = -(-x^2)^2 = -x^4 \\ y = -x^2 & \text{and} & (x = 0 \text{ or } 1 = -x^3) \\ y = -x^2 & \text{and} & (x = 0 \text{ or } x = -1) \\ (x = 0 \text{ and } y = -0^2 = 0) & \text{or} & (x = -1 \text{ and } y = -(-1)^2 = -1) \end{array}$$

$\boxed{\text{So the critical points are } (0, 0) \text{ and } (-1, -1).}$

Put $J(x, y) = f_{xx}f_{yy} - f_{xy}^2$. We compute

$$f_{xx} = (3x^2 + 3y)_x = 6x, f_{xy} = (3x^2 + 3y)_y = 3 \text{ and } f_{yy} = (3x + 3y^2)_y = 6y.$$

So $J(x, y) = (6x)(6y) - 3^2 = 36xy - 9 = 9(4xy - 1)$.

Hence $J(0, 0) = 9(4 \cdot 0 - 1) = -9 < 0$ and $J(-1, -1) = 9\Big(4 \cdot (-1) \cdot (-1) - 1\Big) = 9 \cdot 3 = 27 > 0$.

Since $J(0,0) < 0$, the Second Order Derivative Test shows that $\boxed{(0,0) \text{ is a saddle point}}$.

Since $J(-1,-1) > 0$ and $f_{xx}(-1,-1) = 6 \cdot (-1) = -6 < 0$, the Second Order Derivative Test shows that $\boxed{(1,1) \text{ is a local maximum.}}$

11. Find the area inside the cardioid $r = 1 + \cos(\theta)$ and outside the circle $r = 1$.

Solution: The area is described in polar coordinates by $1 \le r \le 1 + \cos\theta$. This implies $\cos\theta \ge 0$ and so $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Thus

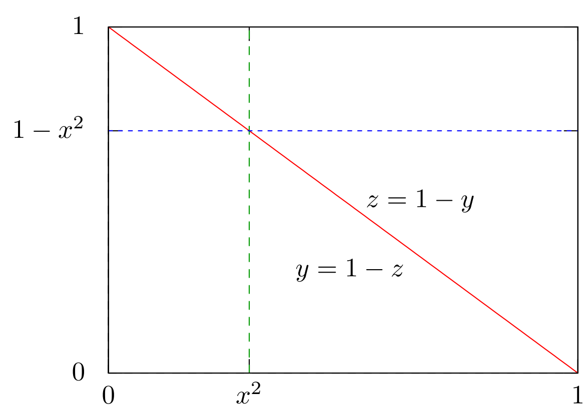
$$\begin{aligned}
\text{Area} &= \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{r=1}^{1+\cos\theta} r\,dr\,d\theta = \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \left[\frac{r^2}{2}\right]_{r=1}^{1+\cos\theta} d\theta = \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \frac{1}{2}(2\cos\theta + \cos^2\theta)d\theta \\
&= \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \left(\cos\theta + \frac{1}{4}(1+\cos 2\theta)\right) d\theta = \left[\sin\theta + \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta\right]_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} = \left(1 + \frac{\pi}{8} + 0\right) - \left(-1 - \frac{\pi}{8} + 0\right) = 2 + \frac{\pi}{4}
\end{aligned}$$

12. Evaluate $\int_0^1 \int_0^1 \int_0^{\sqrt{y}} 2xze^{zy^2}\,dx\,dy\,dz$. Change the order of integration of $\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} dz\,dy\,dx$ into $dy\,dz\,dx$. Do not evaluate the integral.

Solution: First Part:

$$\begin{aligned}
\int_0^1 \int_0^1 \int_0^{\sqrt{y}} 2xze^{zy^2}\,dx\,dy\,dz &= \int_0^1 \int_0^1 \left[x^2\right]_{x=0}^{x=\sqrt{y}} ze^{zy^2}\,dy\,dz = \int_0^1 \int_0^1 yze^{zy^2}\,dy\,dz = \int_0^1 \left[\frac{1}{2}e^{zy^2}\right]_{y=0}^{y=1} dz \\
&= \int_0^1 \frac{1}{2}(e^z - 1)\,dz = \frac{1}{2}\left[e^z - z\right]_{z=0}^{z=1} = \frac{1}{2}\left((e^1 - 1) - (1 - 0)\right) = \frac{e}{2} - 1
\end{aligned}$$

Second Part: For a fixed x in $[0,1]$, the region $x^2 \le y \le 1, 0 \le z \le 1 - y$ in the yz-plane is the same as the region $0 \le z \le 1 - x^2$, $x^2 \le y \le 1 - z$.



So

$$\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} dz\,dy\,dx = \int_{-1}^1 \int_0^{1-x^2} \int_{x^2}^{1-z} dy\,dz\,dx$$

13. Set up, do not evaluate, an integral in cylindrical coordinates for the volume of the solid D. The base of D is $z = 0$, the top is in the plane $z = 4 - y$, the sides are given by $r = 2\sin(\theta)$.

Solution: D is described by $0 \leq z \leq 4 - y$ and $0 \leq r \leq 2\sin\theta$. This implies $\sin\theta \geq 0$ and so $0 \leq \theta \leq \pi$.

Since $y = r\sin\theta$, $0 \leq z \leq 4 - y$ is equivalent to $0 \leq z \leq 4 - r\sin\theta$. (Note that this demands $r\sin\theta \leq 4$, but since $rsin\theta \leq 2\sin\theta \leq 2$, this is already fulfilled.) So the limits of integrations are

$$0 \leq \theta \leq \pi, \qquad 0 \leq r \leq 2\sin\theta, \qquad 0 \leq z \leq 4 - r\sin\theta$$

and the integral is

$$\int_{\theta=0}^{\pi} \int_{r=0}^{2\sin\theta} \int_{z=0}^{4-r\sin\theta} r dz dr d\theta$$

14. Use spherical coordinates to set up an integral for the volume of the solid bounded above by $z = 1$ and bounded below by the cone $z = \sqrt{x^2 + y^2}$.

Solution: The cone $z = \sqrt{x^2 + y^2}$ is in spherical coordinates described by $\phi = \frac{\pi}{4}$ and the region above this cone by $\phi \leq \frac{\pi}{4}$. $z \leq 1$ is described by $\rho\cos\phi \leq 1$ and so $\rho \leq \frac{1}{\cos\phi}$. Thus the limits of integrations are

$$0 \leq \theta \leq 2\pi, \qquad 0 \leq \phi \leq \frac{\pi}{4}, \qquad 0 \leq r \leq \frac{1}{\cos\phi}$$

and the integral is

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\frac{\pi}{4}} \int_{\rho=0}^{\frac{1}{\cos\phi}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

15. Evaluate $\int_C f(x, y, z) ds$ when C is given by $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + 6t\mathbf{k}$, $(0 \leq t \leq 2\pi)$ and $f(x, y, z) = x + y + z$.

Solution:

$$\begin{aligned} \mathbf{r}' &= \langle -\sin t, \cos t, 6 \rangle \\ |\mathbf{r}'| &= \sqrt{\sin^2 t + \cos^2 t + 36} = \sqrt{37} \\ ds &= |\mathbf{r}'| dt = \sqrt{37} dt \\ f(\mathbf{r}(t)) &= \cos t + \sin t + 6t \end{aligned}$$

$$\int_C f ds = \int_{t=0}^{2\pi} \left(\cos t + \sin t + 6t\right) \sqrt{37} dt = \sqrt{37} \left[\sin t - \cos t + 3t^2\right]_{t=0}^{2\pi} = \sqrt{37} 3(2\pi)^2 = 12\sqrt{37}\pi^2$$

16. Find the work done by $\mathbf{F} = -3y\mathbf{i} + 3x\mathbf{j} + (x + y)\mathbf{k}$ over the curve $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + 4t\mathbf{k}$, $(0 \leq t \leq 2\pi)$ in the direction of increasing t.

Solution:

$$\begin{aligned}
\mathbf{r} &= \langle \cos t, \sin t, 4t\rangle \\
\mathbf{r}' &= \langle -\sin t, \cos t, 4\rangle \\
d\mathbf{r} &= \mathbf{r}' dt = \langle -\sin t, \cos t, 4\rangle dt \\
\mathbf{F} &= \langle -3y, 3x, x+y\rangle = \langle -3\sin t, 3\cos t, \cos t + \sin t\rangle
\end{aligned}$$

$$\begin{aligned}
\mathbf{F}\cdot d\mathbf{r} =& \langle -3\sin t, 3\cos t, \cos t + \sin t\rangle \cdot \langle -\sin t, \cos t, 4\rangle dt \\
=& \left(3\sin^2 t + 3\cos^t + 4\cos t + 4\sin t\right) dt \\
=& (3 + 4\cos t + 4\sin t) dt
\end{aligned}$$

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_{t=0}^{2\pi} \left(3 + 4\cos t + 4\sin t\right) dt = [3t + 4\sin t - 4\cos t]_{t=0}^{2\pi} = 6\pi$$

17. Consider $\mathbf{F} = (2xy^3z^4 + y)\mathbf{i} + (3x^2y^2z^4 + 2yz + x)\mathbf{j} + (4x^2y^3z^3 + y^2)\mathbf{k}$. Show that $\mathbf{F}$ is conservative. Find f so that $\nabla f = \mathbf{F}$. Evaluate $\int_{(0,0,0)}^{(1,1,1)} \mathbf{F}\cdot d\mathbf{r}$.

Solution: $f_x = 2xy^3z^4 + y$ so $f = x^2y^3z^4 + yx + g$ where g is a function of y and z. Hence $f_y = 3x^2y^2z^4 + x + g_y$ and since $f_y = 3x^2y^2z^4 + 2yz + x$ we get $g_y = 2yz$. So $g = y^2z + h$, h a function of z. Thus $f = x^2y^3z^4 + yx + y^2z + h$ and $f_z = 4x^2y^3z^3 + y^2 + h_z$. Since $f_z = 4x^2y^3z^3 + y^2$ this shows that h is a constant. We choose $h = 0$ and so

$$\boxed{f = 4x^2y^3z^3 + yx + y^2z}$$

So $\mathbf{F}$ has a potential and thus is conservative.

We could also have used the Component test to show that F is conservative:

$$\begin{aligned}
\frac{\partial}{\partial y}(4x^2y^3z^3 + y^2) =& 12x^2y^2z^3 + 2y = \frac{\partial}{\partial z}(3x^2y^2z^4 + 2yz + x) \\
\frac{\partial}{\partial x}(4x^2y^3z^3 + y^2) =& \qquad 8xy^3z^3 = \frac{\partial}{\partial z}(2xy^3z^4 + y) \\
\frac{\partial}{\partial x}(3x^2y^2z^4 + 2yz + x) =& \quad 6xy^2z^4 + 1 = \frac{\partial}{\partial y}(2xy^3z^4 + y)
\end{aligned}$$

Since f is a potential for $\mathbf{F}$, the Fundamental Theorem for Line Integrals tells us

$$\int_{(0,0,0)}^{(1,1,1)} \mathbf{F}\cdot d\mathbf{r} = f(1,1,1) - f(0,0,0) = 4 + 1 + 1 - 0 = \boxed{6}$$

18. Quote the circulation form of Green's Theorem. Use it to find $\oint_C (2x + y^2 + 2y)dx + (2xy + 3y + 5x)dy$ where C is the boundary of the triangle bounded by $y = 0$, $x = 0$ and $x + y = 4$.

Solution:

Part 1: **Green's Theorem (Circulation-Curl-Form)** *Let R be a region in the plane whose boundary is a piecewise smooth simple closed, curve C. Let $\mathbf{F}$ be a vector field with continuous first order partial derivatives on an open region containing C. Then*

$$\oint_C \mathbf{F} d\mathbf{r} = \iint_R \operatorname{curl} \mathbf{F} dA$$

Part 2:

$$\mathbf{F} = \langle 2x + y^2 + 2y, 2xy + 3y + 5x \rangle$$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2x + y^2 + 2y & 2xy + 3y + 5x \end{vmatrix} = (2y + 5) - (2y + 2) = 3$$

and so

$$\oint_C \mathbf{F} d\mathbf{r} = \iint_R \operatorname{curl} \mathbf{F} dA = \iint_R 3 dA = 3 \iint_R dA = 3 \operatorname{Area}(R) = 3 \cdot \frac{1}{2} \cdot 4 \cdot 4 = 24$$

19. Find the area of the surface cut from $z = 10 + 4xy$ by the cylinder $x^2 + y^2 = 4$.

Solution:

Let $f = 4xy - z$ and $\mathbf{p} = \mathbf{k} = \langle 0, 0, 1 \rangle$. The surface S in question is the part of surface $f = 10$ above the circle $R : x^2 + y^2 \leq 4$. We compute

$$\nabla f = \langle 4y, 4x, -1 \rangle$$
$$|\nabla f| = \sqrt{16x^2 + 16y^2 + 1}$$
$$|\nabla f \cdot \mathbf{p}| = |\langle 4y, 4x, -1 \rangle \cdot \langle 0, 0, 1 \rangle| = |-1| = 1$$
$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot p|} dA = \sqrt{16x^2 + 16y^2 + 1}$$

In polar coordinates R can be described as $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$. Also $dA = r dr d\theta$ and so

$$\begin{aligned} \operatorname{Area}(S) = \iint_S d\sigma = \iint_R \sqrt{16x^2 + 16y^2 + 1} dA &= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \sqrt{16r^2 + 1} r dr d\theta \\ &= 2\pi \left[\frac{2}{3} \frac{1}{16} \frac{1}{2} (16r^2 + 1)^{\frac{3}{2}} \right]_0^2 = \frac{\pi}{24} \left(65^{\frac{3}{2}} - 1 \right). \end{aligned}$$

20. Use the divergence theorem to evaluate $\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma$, where $\mathbf{F} = x^2 \mathbf{i} + y^2 \mathbf{j} + 3z \mathbf{k}$ and S is the surface of the cube bounded by $x = \pm 1, y = \pm 1, z = \pm 1$.

Solution:

$$\operatorname{div} \mathbf{F} = 2x + 2y + 3$$

and so by the Divergence Theorem

$$
\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma &= \int_{x=-1}^{x=1}\int_{y=-1}^{y=1}\int_{z=-1}^{z=1}(2x+2y+3)dzdydx \\
&= \Big((1-(-1))(1-(-1))[x^2]_{x=-1}^{x=1}\Big) + \Big((1-(-1))(1-(-1))[y^2]_{y=-1}^{y=1}\Big) + \Big((1-(-1))(1-(-1))[3z]_{z=-1}^{z=1}\Big) \\
&= 2\cdot 2\cdot 3(1-(-1)) = 24
\end{aligned}
$$

21. Integrate the function $f(x,y) = x^3/y$ along the plane curve C given by $y = x^2/2$ for $x \in [0,2]$, from the point $(0,0)$ to $(2,2)$.

Solution: Using $t = x$, C can be parameterized by $\mathbf{r} = \langle t, \frac{t^2}{2}\rangle$, $0 \le t \le 2$. So

$$
\begin{aligned}
\mathbf{r}' &= \langle 1, t\rangle \\
ds &= |\mathbf{r}'|dt = \sqrt{1+t^2}dt \\
f &= \frac{x^3}{y} = \frac{t^3}{\frac{t^2}{2}} = 2t \\
\int_C f ds &= \int_0^2 2t\sqrt{1+t^2}dt = \left[\frac{2}{3}(1+t^2)^{\frac{3}{2}}\right]_0^2 = \frac{2}{3}(5^{\frac{3}{2}}-1)
\end{aligned}
$$

22. Find the work done by the force $\mathbf{F} = \langle yz, zx, -xy\rangle$ in a moving particle along the curve $\mathbf{r}(t) = \langle t^3, t^2, t\rangle$ for $t \in [0,2]$.

Solution: Recall that Work $= \int_C \mathbf{F}\cdot d\mathbf{r}$. We compute

$$
\begin{aligned}
\mathbf{r} &= \langle t^3, t^2, t\rangle \\
d\mathbf{r} &= \mathbf{r}'dt = \langle 3t^2, 2t, 1\rangle dt \\
\mathbf{F} &= \langle yz, zx, -xy\rangle = \langle t^2 t, tt^3, -t^3t^2\rangle = \langle t^3, t^4, -t^5\rangle \\
\mathbf{F}\cdot d\mathbf{r} &= \langle t^3, t^4, -t^5\rangle\cdot\langle 3t^2, 2t, 1\rangle dt = (3t^5 + 2t^5 - t^5)dt = 4t^5 dt \\
\text{Work} &= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^2 4t^5 dt = \left[\frac{4}{6}t^6\right]_0^2 = \frac{2}{3}2^6 = \frac{2^7}{3}.
\end{aligned}
$$

23. Find the flow of the velocity field $\mathbf{F} = \langle xy, y^2, -yz\rangle$ from the point $(0,0,0)$ to the point $(1,1,1)$ along the curve of intersection of the cylinder $y = x^2$ with the plane $z = x$.

Solution: Recall that Flow $= \int_C \mathbf{F}\cdot d\mathbf{r}$. Using $t = x$, the curve of intersection from $(0,0,0)$ to $(1,1,1)$ can be parameterized by

$$
\mathbf{r} = \langle t, t^2, t\rangle, \qquad 0 \le t \le 1
$$

So

$$\begin{aligned}
d\mathbf{r} &= \mathbf{r}'dt = \langle 1, 2t, 1\rangle dt\\
\mathbf{F} &= \langle xy, y^2, -yz\rangle = \langle tt^2, (t^2)^2, -t^2t\rangle = \langle t^3, t^4, -t^3\rangle\\
\mathbf{F}\cdot d\mathbf{r} &= \langle t^3, t^4, -t^3\rangle \cdot \langle 1, 2t, 1\rangle dt = (t^3 + 2t^5 - t^3)dt = 2t^5\\
\text{Flow} &= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 2t^5 dt = \left[\frac{2}{6}t^6\right]_0^1 = \frac{1}{3}
\end{aligned}$$

24. Find the flux of the field $\mathbf{F} = \langle -x, x-y\rangle$ across the loop C given by the circle $\mathbf{r}(t) = \langle a\cos(t), a\sin(t)\rangle$ for $t \in [0, 2\pi]$.

Solution: Recall that Flux $= \int_C \mathbf{F}\cdot\mathbf{n}ds = \int_C M dy - N dx$.

$$\begin{aligned}
x &= a\cos t & y &= a\sin t\\
dx &= -a\sin t\,dt & dy &= a\cos t\,dt\\
M &= -x = -a\cos t & N &= x - y = a\cos t - a\sin t
\end{aligned}$$

and so

$$\begin{aligned}
M dy - N dx &= (-a\cos t)(a\cos t)\,dt + (a\cos t - a\sin t)(-a\cos t)\,dt\\
&= -a^2(\sin^2 t + \cos^2 t - \sin t\cos t)\,dt\\
&= -a^2(1 - \sin t\cos t)\,dt
\end{aligned}$$

Thus

$$\text{Flux} = \int_C M dy - N dx = -a^2\int_0^{2\pi}((1-\sin t\cos t)dt = -a^2\left[t - \frac{1}{2}\sin^2 t\right]_0^{2\pi} = -2\pi a^2$$

25. (a) Is the field $\mathbf{F} = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ conservative?
   (b) If yes, then find the potential function.
   (c) Compute $I = \int_C y\sin(z)dx + x\sin(z)dy + xy\cos(z)dz$, where C is given by $\mathbf{r}(t) = \left\langle \cos(2\pi t), 1 + t^5, \cos^2(2\pi t)\pi/2\right\rangle$ for $t \in [0, 1]$.

Solution: (a)

$$\begin{aligned}
\frac{\partial}{\partial y}(xy\cos z) =& x\cos z = \frac{\partial}{\partial z}(x\sin z)\\
\frac{\partial}{\partial x}(xy\cos z) =& y\cos z = \frac{\partial}{\partial z}(y\sin z)\\
\frac{\partial}{\partial x}(x\sin z) =& \sin z = \frac{\partial}{\partial y}(y\sin z)
\end{aligned}$$

and so $\mathbf{F}$ is conservative by the component test.

(b) Let f be a potential for $\mathbf{F}$. Then $\nabla f = \mathbf{F}$ and so

$$f_x = y\sin z, \quad f_y = x\sin z, \quad f_z = xy\cos z$$

From $f_x = y\sin z$ we get $f = xy\sin z + g$, g a function of y and z. Thus $f_y = x\sin z + g_y$ and since $f_y = x\sin z$, $g_y = 0$ and so $g = h$, h a function of z. Hence $f = xy\sin z + h$ and $f_z = xy\cos z + h_z$. Since $f_z = xy\cos z$ we get $h_z = 0$ and so h is a constant. We may choose $h = 0$ and so $f = xy\cos z$ is a potential for $\mathbf{F}$.

(c) Note that C is a curve from $\mathbf{r}(0) = \langle 1, 1, \frac{\pi}{2}\rangle$ to $\mathbf{r}(1) = \langle 1, 2, \frac{\pi}{2}\rangle$. By the Fundamental Theorem of Line Integrals,

$$I = \int_C y\sin(z)dx + x\sin(z)dy + xy\cos(z)dz = f\left(1, 2, \frac{\pi}{2}\right) - f\left(1, 2, \frac{\pi}{2}\right) = 1\cdot 1\cdot \sin\frac{\pi}{2} = 1\cdot 1\cdot \sin\frac{\pi}{2} = 2 - 1 = 1$$

26. Show that the differential form in the integral below is exact,

$$\int_C \left[3x^2dx + \frac{z^2}{y}dy + 2z\ln y dz\right], y > 0$$

Solution: Let $\mathbf{F} = \langle 3x^2, \frac{z^2}{y}, 2z\ln y\rangle$. Recall that the differential form is exact if and only if the vector field $\mathbf{F}$ is a conservative. So we can use the component test:

$$\begin{aligned}\frac{\partial}{\partial y}(2z\ln y) =& \frac{2z}{y} = \frac{\partial}{\partial z}(\frac{z^2}{y})\\ \frac{\partial}{\partial x}(2z\ln y) =& \; 0 = \frac{\partial}{\partial z}3x^2\\ \frac{\partial}{\partial x}(\frac{z^2}{y}) =& \; 0 = \frac{\partial}{\partial y}(3x^2)\end{aligned}$$

So $\mathbf{F}$ is conservative and corresponding differential form is exact.

27. Compute

$$\int_{(0,0,0)}^{(1,-1,0)} 2x\cos(z)dx + zdy + (y - x^2\sin(z))dz$$

Solution: The notion $\int_{(0,0,0)}^{(1,-1,0)} 2x\cos(z)dx + zdy + (y - x^2\sin(z))dx$ is only defined if the vector field

$$\mathbf{F} = \langle 2x\cos z, y - x^2\sin z\rangle$$

is conservative (since otherwise the integral would depended on actual path from $(0,0,0)$ to $(1,-1,0)$.) So we will compute a potential f for $\mathbf{F}$ and then use the Fundamental Theorem to evaluate the integral.
We have $\nabla f = \mathbf{F}$ and so

$$f_x = 2x\cos z, \quad f_y = z, \quad f_z = y - x^2\sin z$$

From $f_x = 2x\cos z$ we get $f = x^2\cos z + g$, g a function of y and z. Thus $f_y = g_y$ and since $f_y = z$, $g_y = z$ and so $g = zy + h$, h a function of z. Hence $f = x^2\cos z + zy + h$ and $f_z = -x^2\sin z + y + h_z$. Since $f_z = y - x^2\sin z$ we get $h_z = 0$ and so h is a constant. We may choose $h = 0$ and so $f = x^2\cos z + zy$ is a potential for $\mathbf{F}$. Therefore

$$\int_{(0,0,0)}^{(1,-1,0)} 2x\cos(z)dx+zdy+(y-x^2\sin(z))dz = f(1,-1,0)-f(0,0,0) = ((-1)^2\cdot\cos 0+0\cdot-1)-(0\cdot\cos 0+0\cdot 0) = 1\cdot 1-0 = 1$$

28. Use Green's Theorem in the plane to evaluate the line integral given by $\oint_C (6y+x)dx+(y+2x)dy$ on the circle C defined by $(x-1)^2+(y-3)^2=4$.

Solution: C is the boundary of the disk R defined by $(x-1)^2+(y-3)^2 \leq 4$. Let $\mathbf{F} = \langle 6y+x, y+2x\rangle$. By the circulation-curl form of Green's Theorem

$$\oint_C \mathbf{F} d\mathbf{r} = \iint_R \operatorname{curl} \mathbf{F} dA$$

We compute

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 6y+x & y+2x \end{vmatrix} = 2-6=-4$$

$$\iint_R \operatorname{curl} \mathbf{F}\, dA = \iint_R -4dA = -4\iint_R 1dA = -4\operatorname{Area}(R) = -4\pi 2^2 = -16\pi$$

So

$$\boxed{\oint_C (6y+x)dx+(y+2x)dy = -12\pi}$$

29. Use Green's Theorem in the plane to find the flux of $\mathbf{F} = (x-y^2)\mathbf{i} + (x^2+y)\mathbf{j}$ through the ellipse $9x^2+4y^2=36$.

Solution: The ellipse C is the boundary of the region R described by $9x^2+4y^2 \leq 36$. So by the flux-divergence form of Green's Theorem

$$\oint_C \mathbf{F}\cdot\mathbf{n}\, ds = \iint_R \operatorname{div} \mathbf{F}\, dA$$

We compute

$$\operatorname{div} \mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(x-y^2) + \frac{\partial}{\partial y}(x^2+y) = 1+1=2$$

$$\iint_R \operatorname{div} \mathbf{F}\, dA = \iint_R 2dA = 2\iint_R 1dA = 2\operatorname{Area}(R) = 2(3\cdot 2\cdot \pi) = 12\pi$$

Remark: Let $a>0$ and $b>0$. The area of an ellipse $R: a^2x^2+b^2y^2 \leq a^2b^2$ is $ab\pi$. Indeed, R can be described as $(\frac{x}{b})^2 + (\frac{y}{a})^2 \leq 1$ and using $u=\frac{x}{b}$ and $v=\frac{y}{a}$ we can parameterize R via $\mathbf{r} = \langle bu, av, 0\rangle$, (u,v) in the region $S: u^2+v^2 \leq 1$. Then $\mathbf{r}_u = \langle b,0,0\rangle$, $\mathbf{r}_v = \langle 0,a,0\rangle$, $\mathbf{r}_u\times\mathbf{r}_v = \langle 0,0,ab\rangle$ and so $|\mathbf{r}_u\times\mathbf{r}_v| = ab$. Thus $d\sigma = |\mathbf{r}_u\times\mathbf{r}_v|dA = ab\,dA$ and $\operatorname{Area}(R) = \int_R d\sigma = \int_S ab dA = ab\int_S dA = ab\operatorname{Area}(S) = ab\pi$.

30. Set up the integral for the area of the surface cut from the parabolic cylinder $z=4-y^2/4$ by the planes $x=0, x=1, z=0$.

Solution: (x,y,z) is on the surface S, if and only if $z=4-\frac{y^2}{4}$, $0\leq x\leq 1$ and $z\geq 0$. From $z\geq 0$ and $z=4-\frac{y^2}{4}$ we get $y^2\leq 16$ and so $-4\leq y\leq 4$. Thus the projection of S onto the xy-plane is the rectangle $R: 0\leq x\leq 1, -4\leq y\leq 4$. Put $f=4-\frac{y^2}{4}$. Then S can be parameterized by

$$\mathbf{r} = \langle x, y, f(x,y)\rangle \quad 0 \le x \le 1, -4 \le y \le 4$$

Thus

$$\text{Area}, S = \iint_S 1 d\sigma = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dA = \iint_R \sqrt{0^2 + (\frac{-2y}{4})^2 + 1}\, dA = \int_{x=0}^{x=1} \int_{y=-4}^{y=4} \sqrt{1 + \frac{y^2}{4}}\, dydx$$

31. Integrate the function $g(x,y,z) = x\sqrt{4+y^2}$ over the surface cut from the parabolic cylinder $z = 4 - y^2/4$ by the planes $x = 0, x = 1$ and $z = 0$.

Solution: In the previous problem we saw that

$$\mathbf{r} = \langle x, y, f(x,y)\rangle \quad 0 \le x \le 1, -4 \le y \le 4$$

is a parametrization of the surface S and $d\sigma = \sqrt{1 + \frac{y^2}{4}} dydx$. We have

$$g(\mathbf{r}) = g(x, y, f(x,y)) = x\sqrt{4+y^2}$$

So

$$\begin{aligned}
\int_S g d\sigma &= \iint_R x\sqrt{4+y^2}\sqrt{1+\frac{y^2}{4}} dydx & &= \frac{1}{2}\iint_R x\sqrt{4+y^2}\sqrt{4+y^2} dydx \\
&= \frac{1}{2}\int_{x=0}^{x=1}\int_{y=-4}^{y=4} x(4+y^2) dydx & &= \frac{1}{2}\int_{x=0}^{x=1} x\left[4y + \frac{1}{3}y^3\right]_{y=-4}^{y=4} dx \\
&= \frac{1}{2}\int_{x=0}^{x=1} x\left((16+\frac{64}{3}) - (-16-\frac{64}{3})\right) dx & &= \frac{1}{2}\left[\frac{1}{2}x^2\left(32+\frac{128}{3}\right)\right]_{x=0}^{x=1} \\
&= 8 + \frac{32}{3} & &= \frac{56}{3}
\end{aligned}$$

32. Use Stokes' Theorem to find the flux of $\nabla \times \mathbf{F}$ outward through the surface S, where $\mathbf{F} = \langle -y, x, x^2\rangle$ and

$$S = \{x^2 + y^2 = a^2, z \in [0,h]\} \cup \{x^2 + y^2 \le a^2, z = h\}.$$

Solution: Stokes' Theorem says that

$$\int_C \mathbf{F} d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

where C is the boundary of S. S consists of the side

$$x^2 + y^2 = a^2, \qquad 0 \le z \le h$$

and the top

$$x^2 + y^2 \le a^2, \qquad z = h$$

of the cylinder

$$x^2 + y^2 \le a^2, \qquad 0 \le z \le h.$$

But the bottom

$$x^2 + y^2 \le a^2, \qquad z = 0$$

is missing. So C is the circle

$$x^2 + y^2 = a^2, \qquad z = 0$$

Note that C can be parameterization by

$$\mathbf{r} = \langle a\cos t, a\sin t, 0\rangle, \qquad 0 \le t \le 2\pi$$

We have

$$d\mathbf{r} = \mathbf{r}'dt = \langle -a\sin t, a\cos t, 0\rangle dt$$
$$\mathbf{F} = \langle -y, x, x^2\rangle = \langle -a\sin t, a\cos t, a^2\cos^2 t\rangle$$
$$\mathbf{F}\cdot d\mathbf{r} = a^2\sin^2 t + a^2\cos^2 t + 0 = a^2$$

and so

$$\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_C \mathbf{F}d\mathbf{r} = \int_0^{2\pi} a^2 dt = a^2\pi$$

33. Use the Divergence Theorem to find the outward flux of the field $\mathbf{F} = \langle x^2, -2xy, 3xz\rangle$ across the boundary of the region

$$D = \{x^2 + y^2 + z^2 \le 4, x \ge 0, y \ge 0, z \ge 0\}.$$

Solution: The Divergence Theorem says

$$\iint_S \mathbf{F}\cdot\mathbf{n}d\sigma = \iiint_D \mathrm{div}\mathbf{F}\,dV$$

where S is the boundary of D.

$$\mathrm{div}\mathbf{F} = \nabla\cdot F = \frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(3xz) = 2x - 2x + 3x = 3x$$

In spherical coordinates D is described by

$$0 \le \rho \le 2, \qquad 0 \le \phi \le \frac{\pi}{2}, \qquad 0 \le \theta \le \frac{\pi}{2}$$

Also

$$dV = \rho^2\sin\phi d\theta d\phi d\rho$$
$$\mathrm{div}\mathbf{F} = 3x = 3\rho\sin\phi\cos\theta$$
$$\mathrm{div}\mathbf{F}\,dV = 3\rho^3\sin^2\phi\cos\theta d\theta d\phi d\rho$$

$$
\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_D \mathrm{div}\mathbf{F}\,dV &= \int_{\rho=0}^{\rho=2}\int_{\phi=0}^{\phi=\frac{\pi}{2}}\int_{\theta=0}^{\theta=\frac{\pi}{2}} 3\rho^3\sin^2\phi\cos\theta\, d\theta d\phi d\rho &&= 3\int_{\rho=0}^{\rho=2}\int_{\phi=0}^{\phi=\frac{\pi}{2}} \rho^3\sin^2\phi\,[\sin\theta]_{\theta=0}^{\theta=\frac{\pi}{2}}\ d\phi d\rho \\
&= 3\int_{\rho=0}^{\rho=2}\int_{\phi=0}^{\phi=\frac{\pi}{2}} \rho^3\sin^2\phi\cdot 1 d\phi d\rho &&= 3\int_{\rho=0}^{\rho=2}\int_{\phi=0}^{\phi=\frac{\pi}{2}} \rho^3\frac{1}{2}(1-\cos(2\phi))d\phi d\rho \\
&= 3\int_{\rho=0}^{\rho=2}\rho^3\frac{1}{2}\left[\phi-\frac{1}{2}\sin(2\phi)\right]_{\phi=0}^{\phi=\frac{\pi}{2}}\ d\rho &&= 3\int_{\rho=0}^{\rho=2}\rho^3\frac{1}{2}\frac{\pi}{2}\,d\rho \\
&= \frac{3\pi}{4}\left[\frac{1}{4}\rho^4\right]_{\rho=0}^{\rho=2} &&= \frac{3\pi}{4}\frac{2^4}{4} = 3\pi
\end{aligned}
$$