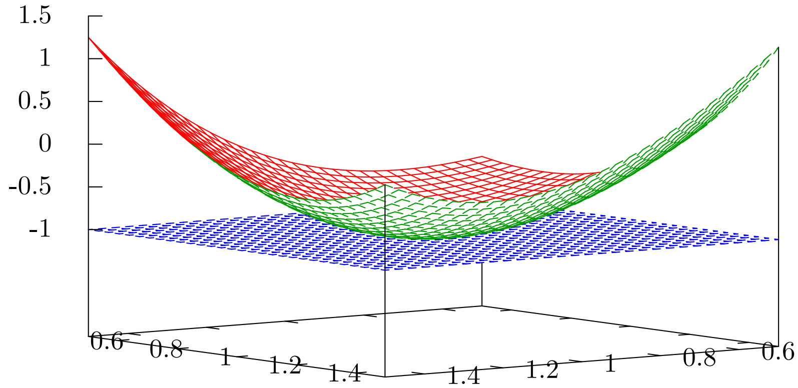
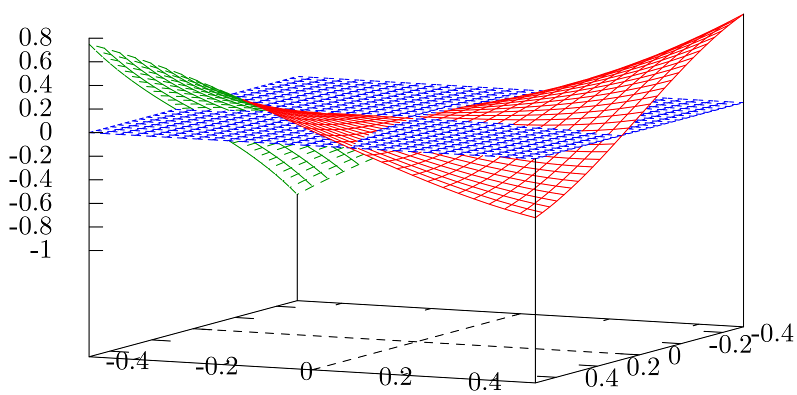
MTH 234-61 **Multivariable Calculus** **11/08/11**

Exam 3/Solutions

1. Find all local maxima, local minima and saddle points of the function

$$f(x,y) = x^3 - 3xy + y^3.$$





To find the critical points we solve the equation $\nabla f = \vec{0}$. We have

$$\nabla f = \langle 3x^2 - 3, -3x + 3y^2 \rangle$$

$$\begin{array}{rcl} \nabla f & = & \vec{0} \\ 3x^2 - 3y = 0 & \text{and} & -3x + y^2 = 0 \\ y = x^2 & \text{and} & x = y^2 \\ y = x^2 & \text{and} & x = (x^2)^2 = x^4 \\ y = x^2 & \text{and} & (x = 0 \text{ or } 1 = x^3) \\ y = x^2 & \text{and} & (x = 0 \text{ or } x = 1) \\ (x = 0 \text{ and } y = 0^2 = 0) & \text{or} & (x = 1 \text{ and } y = 1^2 = 1) \end{array}$$

$\boxed{\text{So the critical points are } (0,0) \text{ and } (1,1).}$

Put $J(x,y) = f_{xx}f_{yy} - f_{xy}^2$. We compute

$$f_{xx} = (3x^2 - 3y)_x = 6x, f_{xy} = (3x^2 - 3y)_y = -3 \text{ and } f_{yy} = (-3x + 3y^2) + y = 6y$$

So $J(x,y) = (6x)(6y) - (-3)^2 = 36xy - 9 = 9(4xy - 1)$.

Hence $J(0,0) = 9(4 \cdot 0 - 1) = -9 < 0$ and $J(1,1) = 9(4 \cdot 1 - 1) = 9 \cdot 3 = 27 > 0$.

Since $J(0,0) < 0$, the Second Order Derivative Test shows that $\boxed{(0,0) \text{ is a saddle point}}$.

Since $J(1,1) > 0$ and $f_{xx}(1,1) = 6 \cdot 1 = 6 > 0$, the Second Order Derivative Test shows that $\boxed{(1,1) \text{ is a local minimum.}}$

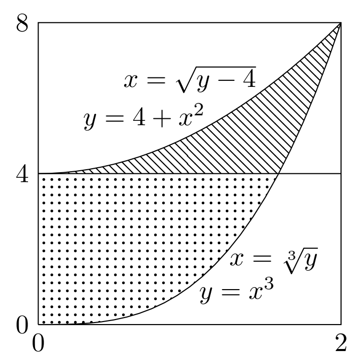
2. Consider the double integral

$$I = \int_{x=0}^{x=2} \left(\int_{y=x^3}^{y=4+x^2} e^{x+y} dy \right) dx.$$

Sketch the region of integration and write I as a sum of two double integrals with the order of integration reversed. Do not evaluate any of the integrals.

From the limits of integration we see that region of integration is

$$0 \le x \le 2, x^3 \le y \le 4 + x^2$$

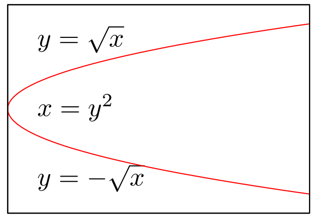


For $0 \le y \le 4$, the region of integration is bounded on the left by $x = 0$ and on the right by $x = \sqrt[3]{y}$. For $4 \le y \le 8$, the region of integration is bounded on the left by $x = \sqrt{y-4}$ and on right by $x = \sqrt[3]{y}$.

$$\boxed{I = \int_{y=0}^{y=4} \left(\int_{x=0}^{x=\sqrt[3]{y}} e^{x+y} dx \right) dy + \int_{y=4}^{y=8} \left(\int_{x=\sqrt{y-4}}^{x=\sqrt[3]{y}} e^{x+y} dx \right) dy}$$

3. Compute

$$\int_{x=0}^{x=1} \left(\int_{y=-\sqrt{x}}^{y=\sqrt{x}} xy^2 dy \right) dx.$$



$$\begin{aligned}
\int_{x=0}^{x=1}\left(\int_{y=-\sqrt{x}}^{y=\sqrt{x}} xy^2dy\right)dx &= \int_{y=-1}^{y=1}\left(\int_{x=y^2}^{x=1} xy^2dx\right)dy\\
=\int_{y=-1}^{y=1}\frac{1}{2}\left[x^2y^2\right]_{x=y^2}^{x=1}dy &= \int_{y=-1}^{y=1}\frac{1}{2}\left(y^2-y^6\right)\\
=\frac{1}{2}\left[\frac{1}{3}y^3-\frac{1}{7}y^7\right]_{y=-1}^{y=1} &= \frac{1}{3}-\frac{1}{7}=\frac{7-3}{21}=\boxed{\frac{4}{21}}
\end{aligned}$$

4. Let $f(x)$ be a continuous function on interval $[0,2]$ and $g(y)$ a continuous function on the interval $[0,3]$. Let h be the function defined by $h(x,y)=f(x)+g(y)$. Let

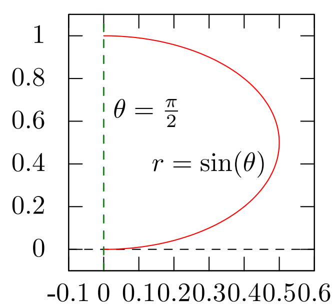
$$K=\int_0^2 f(x)dx,\quad L=\int_0^3 g(y)dy,\ \text{and } M=\iint_R h(x,y)dA,$$

where R is the rectangle $0\le x\le 2, 0\le y\le 3$. Show that $M=3K+2L$.

$$\begin{aligned}
M &= \iint_R h(x,y)dA &&= \int_0^2\left(\int_0^3(f(x)+g(y))dy\right)dx\\
&= \int_0^2\left(\int_0^3 f(x)dy\right)dx+\int_0^2\left(\int_0^3 g(y)dy\right)dx &&= \int_0^2 3f(x)dx+\int_0^2 Ldx\\
&= 3\int_0^2 f(x)dx+2L &&= 3K+2L
\end{aligned}$$

5. Compute $\iint_R xdA$, where R is the region in the plane described in polar coordinates via

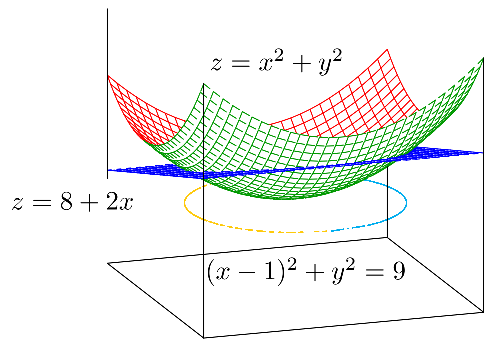
$$0\le r\le \sin\theta \text{ and } 0\le\theta\le\frac{\pi}{2}.$$



$x=r\cos\theta$ so

$$\begin{aligned}
\iint_R x\,dA &= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left(\int_{r=0}^{r=\sin\theta} r\cos\theta r dr \right) d\theta &&= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \cos\theta \left[\frac{1}{3} r^3 \right]_{r=0}^{r=\sin\theta} d\theta \\
&= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{1}{3} \sin^3\theta \cos\theta d\theta &&= \frac{1}{3} \left[\frac{1}{4} \cos^4\theta \right]_{\theta=0}^{\theta=\frac{\pi}{2}} \\
&= \frac{1}{12}(1^4 - 0^4) &&= \boxed{\frac{1}{12}}
\end{aligned}$$

6. Set up a triple integral in cartesian coordinates which gives the volume of the region bounded below by the paraboloid $z = x^2 + y^2$ and above by plane $z = 8 + 2x$. Do not evaluate the integral.



The limits of integration for z are

$$x^2 + y^2 \le z \le 8 + 2x$$

The x and y coordinates have to satisfy the inequality

$$\begin{aligned}
x^2 + y^2 &\le 8 + 2x \\
(x^2 - 2x + 1) + y^2 &\le 8 + 1 \\
(x-1)^2 + y^2 &\le 9 \\
y^2 &\le 9 - (x-1)^2
\end{aligned}$$

Thus the limits of integration for y are

$$-\sqrt{9-(x-1)^2} \le y \le -\sqrt{9-(x-1)^2}$$

For $\sqrt{9-(x-1)^2}$ to be defined we need $(x-1)^2 \le 9$, that is $-3 \le x - 1 \le 3$. So the limits of integration for x are

$$-2 \le x \le 4$$

Hence the volume of the region is

$$\boxed{\int_{x=-2}^{x=4} \left(\int_{y=-\sqrt{9-(x-1)^2}}^{y=\sqrt{9-(x-1)^2}} \left(\int_{z=x^2+y^2}^{z=8+2x} 1dz \right) dy \right) dx}$$

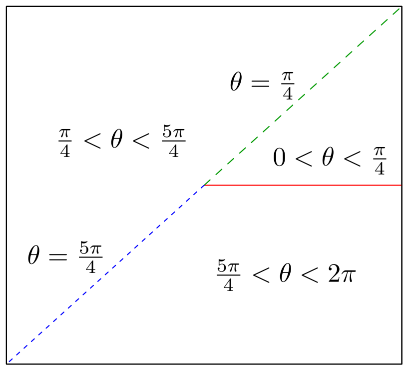
7. Set up a triple integral (in cylindrical coordinates) for evaluating the volume of the region D described by the equations:

$$4 \le x^2 + y^2 \le 9, \quad x \le y \quad \text{and} \quad 0 \le z \le 4 - y.$$

(Do not evaluate the integral).

The equation $4 \le x^2 + y^2 \le 9$ converts to $4 \le r^2 \le 9$ and so $2 \le r \le 3$ in polar coordinates.

$x \le y$ converts to $\frac{\pi}{4} \le \theta \le \frac{5\pi}{4}$, see the following graph:



and $0 \le z \le 4 - y$ converts to $z - 4r \sin\theta$. So we can use the following triple integral to compute the volume:

$$\boxed{\int_{r=2}^{r=3} \left(\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{5\pi}{4}} \left(\int_{z=0}^{z=4-\sin\theta} r\,dz \right) d\theta \right) dr}$$