Locally Finite, Simple Groups

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Contents

Chapter 1

Local Properties

local

Definition 1.1 *Let $\mathcal{C}$ be a class of groups and G a group. Then G is called a* locally $\mathcal{C}$-group*, provided that for each finite subset X of G there exists $Y \leq G$ with $X \subseteq Y$ and $Y \in \mathcal{C}$.* locally

For example a locally abelian group is abelian as any two elements of G have to lie in an abelian subgroup of G and so commute. A less trivial observation is:

Lemma 1.2 *Let G be a locally simple group. Then G is simple.* ls=s

Proof: Let N be a non-trivial normal subgroup of G and $1 \neq x \in N$. We need to show that $G = N$. For this let $y \in G$. Since G is locally simple there exists a simple subgroup S of G so that $x, y \in S$. As $x \in S \cap N$, $S \cap N$ is a non-trivial normal subgroup of S and so $S = S \cap N \leq N$. Thus $y \in N$. Now y was arbitrary and so $G = N$ □

Another explanation why "simple" is a local property is as follows. A group G is simple if and only if $G = \langle x^G \rangle$ for all $1 \neq x \in G$. This is true if and only if $y \in \langle x^G \rangle$ for all $y \in G$. This in turn just means that y can be written as a product of finitely many conjugates of x and x^{-1}. But the last statement just involves finitely many elements of G and so is a local property.

The next lemma introduces a fairly general method to construct locally $\mathcal{C}$-groups.

Lemma 1.3 *Let $\mathcal{C}$ be a class of groups and let* clcg

$$G_1 \leq G_2 \leq G_3 \leq \ldots \leq G_i \leq G_{i+1} \leq \ldots$$

be an ascending chain of $\mathcal{C}$-groups . Put $G = \bigcup_{i=1}^{\infty} G_i$. Then G is a locally $\mathcal{C}$-group.

Proof: Let X be a finite subset of G. As G is the union of the G_i, for each $x \in X$ there exists $i_x \geq 1$ with $x \in G_{i_x}$. Put $i = \max\{i_x \mid x \in X\}$. Then $x \in G_{i_x} \leq G_i$ and so $X \subseteq G_i$. Since G_i is a $\mathcal{C}$ group, this proves that G is locally a $\mathcal{C}$-group. □

For example the infinite cyclic group $(\mathbb{Z}, +)$ has lots of subgroups isomorphic to itself and we easily obtain a chain of groups as in 1.3 with each of the G_i's isomorphic to $\mathbb{Z}$ and properly contained in G_{i+1}. The resulting group G is locally cyclic, but (as any possible element of G lies in one of the G_i's and so generates a proper subgroup), G is not cyclic. A more concrete example of a locally cyclic but not cyclic group is $(\mathcal{Q}, +)$. The reader might also convince herself that our first example of such a locally cyclic, but not cyclic group is isomorphic to a subgroup of $\mathcal{Q}$. For example if $|G_{i+1}/G_i| = 2$ for all i, then G is isomorphic to $\{\frac{n}{2^k} \mid n \in \mathbb{Z}, k \in \mathcal{N}\}$.

To obtain an example of an infinite, locally finite, locally cyclic, groups, let p be prime and choose $G_i \cong C_{p^i} = \mathbb{Z}/p^i\mathbb{Z}$. The resulting group G is called the Prüfer p-group and will be denoted by C_{p^∞} or $\mathbb{Z}_{p^\infty}$.[1]

Next let $\Omega_i = \{1, 2, \ldots, i\}$. Then we can identify $\mathrm{Sym}(\Omega_i)$ with the subgroup of $\mathrm{Sym}(\Omega_{i+1})$ fixing $i+1$. Application of 1.3 yields a locally finite group $G = \bigcup_{i=1}^{\infty} \mathrm{Sym}(\Omega_i)$. Note that we can view G as a subgroup of $\mathrm{Sym}(\Omega)$, where $\Omega = \{1, 2, 3, \ldots\}$. But G is not the full symmetric group, indeed

$$\ldots \to 8 \to 6 \to 4 \to 2 \to 1 \to 3 \to 5 \to 7 \to 9 \to \ldots$$

is an element of infinite order in $\mathrm{Sym}(\Omega)$ and so cannot be contained in G. Note also that the elements in G fix all but finitely many elements of Ω. Conversely, if $\pi \in \mathrm{Sym}(\Omega)$ fixes all but finitely many elements of Ω, then $\pi \in G_i$, where i is the largest positive integer not fixed by π. For a set Δ define the finitary symmetric group on Δ by

$$\mathrm{FSym}(\Delta) = \{\pi \in \mathrm{Sym}(\Delta) \mid |\mathrm{Supp}(\pi)| < \infty\}.$$

With this notation G is $\mathrm{FSym}(\Omega)$. The groups $\mathrm{FSym}(\Delta)$ are always locally finite and we have found locally finite groups of arbitrary cardinality. $\mathrm{FSym}(\Delta)$ is a normal subgroup of $\mathrm{Sym}(\Delta)$ and has as a normal subgroup the alternating group $\mathrm{Alt}(\Omega)$, which consists of those permutations which are product of an even number of 2-cycles. Also it might be interesting to notice that for infinite Δ, $|\mathrm{FSym}(\Delta)| = |\Delta|$ while $|\mathrm{Sym}(\Delta)| = 2^{|\Delta|}$.

Next we use 1.3 one more time to construct one of the most fascinating locally finite groups: P.Hall's universal locally finite group U.

For this let $n_1 = 3$ and put $G_1 = \mathrm{Sym}(3)$. Let $n_2 = |G_1| = n_1!$ and $G_2 = \mathrm{Sym}(n_2) = \mathrm{Sym}(G_1)$. View G_1 as a subgroup of G_2 via the regular permutation action. Inductively put $n_{k+1} = |G_k| = n_k!$, $G_{k+1} = \mathrm{Sym}(n_{k+1}) = \mathrm{Sym}(G_k)$ and view G_k as a subgroup of G_{k+1}

[1] The reader may check that the Prüfer p-groups are subgroups of the multiplicative group of complex numbers with norm equal 1

via the regular permutation. Put $U = \bigcup_{i=1}^{\infty} G_i$. Then U is a locally finite group. We will study U in more details later on but for now let us notice that as U contains symmetric groups of arbitrary large finite groups, U contains an isomorphic copy of any finite group.

Chapter 2

P.Hall's universal locally finite group

PHall

Definition 2.1 *A locally finite group G is called universal provided that G fulfills the following property:* universal

(Uni) For all finite groups F and E and embeddings $\alpha : F \to G$ and $\beta : F \to E$ there exists an embedding $\gamma : E \to G$ with $\alpha = \gamma\beta$.

The goal of this chapter is to show that P.Hall's group U is up to isomorphism the unique countable, universal, locally finite group.

Definition 2.2 *Let G be a group acting on a set Ω. Then we say that G acts semi-regularly on Ω provided that $C_G(\omega) = \{g \in G \mid \omega^g = \omega\} = 1$ for all $\omega \in \Omega$. G acts regularly on Ω if G acts transitively and semi-regularly on Ω.* semiregular

The proofs of the next three lemmas are left to the reader.

Lemma 2.3 *Let G be acting on a set Ω. Then the following three statements are equivalent:* esr

(a) *G acts semi-regularly on Ω.*

(b) *G acts regularly on each of its orbits on Ω.*

(c) *Each orbit for G on Ω is isomorphic to the action of G on G by right multiplication.* $\square$

Lemma 2.4 *Let G be a group acting semi-regularly on a set Ω and $H \leq G$. Then also H acts semi-regularly on Ω.* $\square$ ssr

Lemma 2.5 *Let G be acting semi-regularly on the sets Ω_1 and Ω_2. Then Ω_1 and Ω_2 are isomorphic as G-sets if and only if $|\Omega_1| = |\Omega_2|$.* $\square$ isr

Proposition 2.6 *P.Hall's group U is universal.* phu

Proof: Let F and E be finite groups and $\alpha : F \to U$ and $\beta : F \to E$ be embeddings. Also let $G_1 < G_2 < \ldots < G_i < \ldots$ be the ascending chain of finite symmetric groups used to construct U. Since $\alpha(F)$ is finite, there exists j with $\alpha(F) \leq G_j$. Moreover, pick k, $n_k \geq |E|$ and $k \geq j$. Then $\alpha(F) \leq G_k$ and there exists an embedding $\gamma^* : E \to G_k$. Let $i = k+1$ and $\Omega = G_k$. Then $G_i = \mathrm{Sym}(\Omega)$. By construction G_k acts regularly on Ω and so by 2.4 we see that both α and $\gamma^*\beta$ define semi-regular actions of F on Ω. By 2.5 these actions are isomorphic. But that just means that there exists $g \in \mathrm{Sym}(\Omega) = G_i \leq U$ so that $\alpha = \mathrm{Inn}(g)\gamma^*\beta$. Putting $\gamma = \mathrm{Inn}(g)\gamma^*$ we see that (Uni) in Definition 2.1 is fulfilled. □

uulg **Proposition 2.7** *Let X and Y be countable, universal, locally finite groups. Then X and Y are isomorphic.*

Proof: Since X and Y are countable, $X = \{x_1, x_2, x_3, \ldots\}$ and $Y = \{y_1, y_2, y_3, \ldots\}$. Put $X_i = \langle x_1, x_2, \ldots, x_i\rangle$ and $Y_i = \langle y_1, y_2, \ldots, y_i\rangle$ and note that X_i and Y_i are finite subgroups of X and Y, respectively. Put $X_0^* = 1 = Y_0^*$. By induction we will define finite subgroups $X_i^* \leq X$ and $Y_i \leq Y_i^*$ and embeddings $\phi_i : X_i^* \to Y$ and $\psi_i : Y_i^* \to X$ so that for all $i > 0$

1. $X_i^* = \langle \psi_{i-1}(Y_{i-1}^*), X_i\rangle$
2. $\phi_i \circ \psi_{i-1}$ is the identity on Y_{i-1}^*.
3. ϕ_i equals ϕ_{i-1}, when restricted to X_{i-1}^*
4. $Y_i^* = \langle \phi_i(X_i^*), Y_i\rangle$
5. $\psi_i \circ \phi_i$ is the identity on X_i^*.
6. ψ_i equals ψ_{i-1}, when restricted to Y_{i-1}^*

For $i = 0$ there is nothing to do. So assume such groups and maps exist for $i-1$. Then use 1. as the definition of X_i^*. Let $F = \psi_{i-1}(Y_{i-1}^*)$ and $\alpha : F \to Y_{i-1}^* \leq Y$ the inverse map of ψ_{i-1}. Let β be the inclusion map from F to X_i^*.. As Y is universal there exist $\gamma : X_i^* \to Y$ with $\gamma\beta = \alpha$. But this just means that γ restricted to F is α and so 2. holds with $\phi_i = \gamma$. In particular, ϕ_i is the inverse of ψ_{i-1} on X_{i-1}^*. By 5. the same is true for ϕ_{i-1} and so 3. holds.

4., 5. and 6. are done in a similar way.

Define $\phi : X \to Y$ by $\phi(x_i) = \phi_i(x_i)$ and $\psi : Y \to X$ by $\psi(y_i) = \psi_i(y_i)$. Then it is easy to check that ψ and ϕ are group homomorphisms which are inverse to each other. □

Chapter 3

Kegel-covers of locally finite, simple groups

¿From now on G will always be a locally finite group and $\mathcal{F}$ the set of finite subgroups of G, i.e the set of finitely generated subgroups of G. Kegel

Definition 3.1 kegelcover

(a) *A set of pairs* $\{(H_i, M_i)|i \in I\}$ *is called a* sectional cover *for* G *if, for all* i *in* I, $H_i \in \mathcal{F}$ *and* M_i *is a normal subgroup of* H_i*, and if, for each* $H \in \mathcal{F}$ *there exists* i *in* I *with* $H \leq H_i$ *and* $H \cap M_i = 1$*. The groups* H_i/M_i, $i \in I$*, are called the* factors *of the sectional cover.*

(b) *A* Kegel-cover *for* G *is a sectional cover all of whose factors are simple groups.*

Definition 3.2 G^∞ *is subgroup of* G *generated by the all the perfect finite subgroups of* G*.* G *is called* absolutely perfect *if* $G = G^\infty$*.* gi

Note that G^∞ is the smallest normal subgroup N of G so that G/N is locally solvable. Also for finite G, G^∞ is just the last term of the derived series of G and G is perfect if and only if its is absolutely perfect.

Lemma 3.3 *Let* $\{(G_i, N_i)|i \in I\}$ *a sectional cover for* G*. Suppose that* $I = \bigcup_{t=1}^{k} I_t$*. Then for at least one* $1 \leq t \leq k$, $\{(G_i, N_i)|i \in I_t\}$ *is a sectional cover for* G*.* psc

Proof: Suppose not. Then for each t there exists a finite subgroup $F_t \leq G$ so that for all $i \in I_t$ with $F_t \leq G_i$ we have $F_t \cap N_i \neq 1$. Define $F = \langle F_t \mid 1 \leq t \leq k \rangle$. By the definition of a sectional cover there exists $i \in I$ with $F \leq G_i$ and $F \cap N_i = 1$. But then $F_t \leq G_i$ and $F_t \cap M_i = 1$. Hence $i \notin I_t$, a contradiction to $I = \bigcup_{t=1}^{k} I_t$. □

Lemma 3.4 *Let* G *be a locally finite, simple group and* $\{(G_i, N_i)|i \in I\}$ *a sectional cover for* G*.* elementarykege

(a) *There exists a Kegel cover* $\{(H_j, M_j)|j \in J\}$ *such that for all* $j \in J$ *there exists* $i \in I$ *with* $N_i \leq M_j \trianglelefteq H_j \trianglelefteq G_i$.

(b) *For* $i \in I$ *let* M_i *be a normal subgroup of* G_i. *Then at least one of* $\{(G_i, M_i)|i \in I\}$ *and* $\{(M_i, M_i \cap N_i)|i \in I\}$ *is a sectional cover for* G.

(c) *Suppose* G *is not cyclic of prime order.Then* $\{(G_i^\infty, G_i^\infty \cap N_i)|i \in I\}$ *is a sectional cover for* G.

(d) *Suppose* G *is not cyclic of prime order. Let* $\mathcal{E}$ *be a class of groups such for each* $i \in I$ *and each non-abelian composition factor* K *of* G_i/N_i *one has* $K \in \mathcal{E}$. *Then there exists a Kegel cover for* G *all of whose factors are in* $\mathcal{E}$.

Proof: (a) Let E be a non trivial finite subgroup of G and $1 \neq e \in E$.

As a first step we show there exists a finite subgroup T of G with $E \leq T$ and $E \leq \langle e^{\langle e^T \rangle} \rangle$ for all $1 \neq e \in E$. Indeed since G is simple, $E \leq \langle e^G \rangle$ and since G is locally finite, $E \leq \langle e^{F_e} \rangle$ for some finite subgroup F_e of G. Similarly $F_e \leq \langle e^{T_e} \rangle$ for some finite subgroup T_e of G. Then $E \leq \langle e^{\langle e^{T_e} \rangle} \rangle$. Now just let T be the finite subgroup of G generated by E and all the T_e, $1 \neq e \in E$ and the first step is completed.

Pick $i \in I$ with $T \leq G_i$ and $T \cap N_i = 1$. Put $H_E = \langle E^{G_i} \rangle N_i$.

Our second step is to find a maximal normal subgroup M_E of H_E with $N_i \leq M_E$ and $E \cap M_E = 1$. For this let $\mathcal{R}$ be the set of the maximal normal subgroups of H_E containing N_i and $R = \bigcap \mathcal{R}$. Then G_i leaves $\mathcal{R}$ invariant and so R is a normal subgroup of G_i with $R < H_E$. Suppose that $E \leq R$. Then $H_E = \langle E^{G_i} \rangle N_i \leq R$, a contradiction. Thus $E \not\leq R$. Thus there exists $M_E \in \mathcal{R}$ with $E \not\leq M_E$. To complete the second step it remains to show that $E \cap M_E = 1$. Otherwise pick $1 \neq e \in E \cap M_E$. Then

$$E \leq \langle e^{\langle e^T \rangle} \rangle \leq \langle e^{H_E} \rangle \leq M_E,$$

a contradiction.

It follows now immediately from the second step that $\{(H_E, M_E) \mid E \in \mathcal{F}\}$ is a Kegel cover that fulfills (a).

(b) Assume that $\{(G_i, M_i)|i \in I\}$ is not a sectional cover for G. Then there exists a $H \in \mathcal{F}$ such that $H \cap M_i \neq 1$ for all $i \in I$ with $H \leq G_i$. Without loss $H \leq G_i$ for all $i \in I$. For $1 \neq h \in H$ let $I_h = \{i \in I | h \in M_i\}$. Then I is the finite union of these I_h and so by 3.3 there exists $1 \neq h \in H$ such that $\{(G_i, N_i)|i \in I_h\}$ is a sectional cover for G. Hence we may assume that $I = I_h$. Let E be any finite subgroup of G. Since G is LFS, there exists a finite subgroup T in G with $E \leq T$ and $E \leq \langle h^T \rangle$. Pick $i \in I$ with $T \leq G_i$ and $T \cap N_i = 1$. Then $E \leq \langle h^T \rangle \leq M_i$ and $E \cap (N_i \cap M_i) \leq T \cap N_i = 1$. Thus $\{(M_i, M_i \cap N_i)|i \in I\}$ is a sectional cover for G.

(c) Otherwise we conclude from (b) that $\{(G_i, G_i^\infty)|i \in I\}$ is a sectional cover for G. Hence by (a) G has a Kegel cover all of whose factor are of prime order. Hence every finite subgroup of G is embedded into a group of prime order and so G itself has prime order.

(d) By (a) there exists a Kegel cover all of whose factors have prime order or lie in $\mathcal{E}$. Thus by 3.3 there exists a Kegel cover all of whose factors have prime order or there exists a Kegel cover all of factors are in $\mathcal{E}$. In the first case G itself has prime order and (d) is proved. □

We remark that there exists examples of locally finite groups which do have a Kegel cover, but are not simple.

Corollary 3.5 *Non-cyclic, locally finite simple groups are absolutely perfect. In particular,* lss
a locally solvable, locally finite, simple group is cyclic of prime order.

Proof: Observe that every locally finite simple groups has a sectional cover, for example $\{(F, 1) \mid F \in \mathcal{F}\}$. Hence the corollary follows from 3.4c. □

Chapter 4

Serial Subgroups

The main goal of this section is to show that a subgroup of G is serial if and only if its locally subnormal. Serial

For the next definition recall that a directed set is a partially ordered set so that each two elements have an upper bound. For example $\mathcal{F}$ is a directed set under inclusion.

Definition 4.1 *Let I be a directed set and for $i \in I$ let L_i be a subgroup of G. Then $\{L_i \mid i \in I\}$ is called a* local system *for G with respect to I provided:* localsystem

(a) $L_i \leq L_j$ *for all* $i \leq j \in I$.

(b) $\bigcup_{i \in I} L_i = G$

By (b) each of the elements of G lie in one of the L_i's and by (a) and since I is directed each finite subgroup of G lies in one of L_i's.

Definition 4.2 *Let R be a group and H a group acting on R. Let $\mathcal{S}$ a set of H-invariant subgroups of R.* series

(a) *$\mathcal{S}$ is called an H-*series *on R provided that:*

  (a) $1 \in \mathcal{S}$ *and* $R \in \mathcal{S}$.

  (b) *$\mathcal{S}$ is totally ordered with respect to inclusion. That is, if $D, E \in \mathcal{S}$ then $D \leq E$ or $E \leq D$.*

  (c) *$\mathcal{S}$ is closed with respect to arbitrary intersections and unions. That is, if $\mathcal{D} \subseteq \mathcal{S}$, then $\bigcap \mathcal{D} \in \mathcal{S}$ and $\bigcup \mathcal{D} \in \mathcal{S}$.*

  (d) *For $D \in \mathcal{S}$ put $D^- = \bigcup\{E \in \mathcal{S} \mid E < D\}$. Then $D^- \trianglelefteq D$ for all $D \in \mathcal{S}$.*

(b) *If $D \neq D^-$ for some $D \in \mathcal{S}$, then (D^-, D) is called a* jump *of $\mathcal{S}$ and D/D^- a* factor *of $\mathcal{S}$.*

(c) *An* H-composition series *is an* H*-series so that none of the factors has a proper* H*-invariant normal subgroup.*

(d) *A* normal H-series *on* R *is a series all of its members are normal in* R *and a* H-chief series *is a normal composition series.*

(e) *A* series, normal series, composition series *and* chief series *on* R *is a* 1*-series, normal* 1*-series,* 1*-composition series and* 1*-chief series on* R.

(f) *A subgroup* N *of* R *is called* H-serial *in* R *provided that* N *is the member of some* H*-series on* R.

locallysubnormal **Definition 4.3** *Let* N *be a subgroup of* G. *Then* N *is called* locally subnormal *provided that for each* $F \in \mathcal{F}$, $N \cap F$ *is subnormal in* F.

fsn **Lemma 4.4** *Let* G *be finite,* H *a subgroup of* G, N *a subnormal subgroup of* G, M *a normal subgroup of* G *and* $\mathcal{N}$ *a set of subnormal subgroups of* G. *Then*

(a) $H \cap N$ *is subnormal in* H. *In particular,* N *is locally subnormal in* G.

(b) NM/M *is subnormal in* G/M.

(c) $\bigcap \mathcal{N}$ *and* $\langle \mathcal{N} \rangle$ *are subnormal in* G.

(d) *If* $H \leq \langle N^H \rangle$, *then* $H \leq N$.

(e) *If* H *is subnormal in* N, *then* H *is subnormal in* G.

Proof: (a) Just intersect a subnormal series from N to G with H to obtain a subnormal series from $H \cap N$ to H.

(b) Just take the images in G/M of a subnormal series from N to G to obtain a subnormal series from NM/M to G/M.

(e) follows immediately from the definition of subnormal.

(c) By induction on $|\mathcal{N}|$ we may assume that $\mathcal{N} = \{N_1, N_2\}$. Then by (a) $N_1 \cap N_2 \trianglelefteq \trianglelefteq N_1 \trianglelefteq \trianglelefteq G$ and so $\bigcap \mathcal{N}$ is subnormal in G.

For the proof that $\langle \mathcal{N} \rangle$ is subnormal in G we will use induction on $|G|$ and then on $|G/N_1|$. If $|G| = 1$ or $|G/N_1| = 1$, there is nothing to prove. If $N_1 \trianglelefteq G$ then by (b) $N_1 N_2/N_1$ is subnormal in G/N_1 and so $N_1 N_2$ is subnormal in G. So we may assume that N_1 is not normal in G. Thus there exist subnormal subgroups N_3, N_4 with

$$N_1 \triangleleft N_3 \trianglelefteq \trianglelefteq N_4 \triangleleft G$$

Let $n \in N_2$. Since n normalizes N_4, N_1^n is a subnormal subgroup of N_4. Since $|N_4| < |G|$, induction on $|G|$ implies that $\langle N_1^{N_2} \rangle = \langle N_1^n \mid n \in N_2 \rangle$ is a subnormal subgroup of N_4 and so also a subnormal subgroup of G.

If N_2 does not normalize N_1, then $|\langle N_1^{N_2}\rangle| > |N_1|$ and so by induction on $|G/N_1|$, $\langle N_1, N_2\rangle = \langle\langle N_1^{N_2}\rangle, N_2\rangle$ is a subnormal subgroup of G.

So we may assume that N_2 does normalize N_1. Thus $\langle N_3, N_2\rangle \leq N_G(N_1) < G$. Thus by induction on $|G|$, $\langle N_1, N_2\rangle$ is subnormal in $\langle N_3, N_2\rangle$. Moreover by induction on $|G/N_1|$, $\langle N_3, N_2\rangle$ is subnormal in G. Thus $\langle N_1, N_2\rangle$ is subnormal in G by (e).

(d) By induction on $|G|$. If $N = G$ we are done. So suppose $N \trianglelefteq \trianglelefteq M \lhd G$. Then $H \leq \langle N^H\rangle \leq \langle M^H\rangle = M$. As $|M| < |G|$ and N and H are contained in M, we are done by induction. □

Lemma 4.5 *Let $\{L_i \mid i \in I\}$ be a* local system *for G and for $i \in I$ let N_i be a subgroup of* clsn
L_i. Suppose that

(a) *For all $i \in I$, N_i is locally subnormal in L_i.*

(b) *For all $i \leq j \in I$, $N_i \leq N_j$.*

Then $N = \bigcup_{i\in I} N_i$ is a locally subnormal subgroup of G.

Proof: First notice that by (b) and as I is directed, N is a subgroup of G. Let $F \in \mathcal{F}$. Then since F and $F \cap N$ are finite there exists $i \in I$ so that $F \leq L_i$ and $F \cap N \leq N_i$. Since N_i is locally subnormal in L_i, $F \cap N_i$ is subnormal in F. But

$$F \cap N_i \leq F \cap N \leq F \cap N_i.$$

So $F \cap N_i = F \cap N$ and $F \cap N$ is subnormal in F, as desired. □

We are now able to generalize 4.4 to infinite locally finite groups.

Proposition 4.6 *Let H a subgroup of G, N a locally subnormal subgroup of G, M a* lfsn
normal subgroup of G and $\mathcal{N}$ a set of locally subnormal subgroups of G. Then

(a) *$H \cap N$ is locally subnormal in H.*

(b) *NM/M is locally subnormal in G/M.*

(c) *$\bigcap \mathcal{N}$ and $\langle \mathcal{N}\rangle$ are locally subnormal in G.*

(d) *If $H \leq \langle N^H\rangle$ and H is finite, then $H \leq N$.*

(e) *If H is locally subnormal in N, then H is locally subnormal in G.*

Proof: We will prove (d) and the second part of (c). The easy proofs for the remaining parts are left to the reader.

(c) part 2: For $F \in \mathcal{F}$ put $N_F = \langle E \cap F \mid E \in \mathcal{N}\rangle$. As each $E \in \mathcal{N}$ is locally subnormal $E \cap F$ is subnormal in F and so by 4.4c, N_F is subnormal in F. Also $F_1 \leq F_2 \in \mathcal{F}$ clearly implies $N_{F_1} \leq N_{F_2}$. Application of 4.5 (with $I = \mathcal{F}$ and $L_F = F$) yields that $N = \bigcup_{F\in\mathcal{F}} N_F$ is locally subnormal in G. To complete the proof of (c) it now suffice to show that $N = \langle \mathcal{N}\rangle$.

By definition of N_F, $N_F \leq \langle \mathcal{N} \rangle$ and so also $N \leq \langle \mathcal{N} \rangle$. Conversely, if $g \in E \in \mathcal{N}$, then $g \in N_{\langle g \rangle} \leq N$ and so $\langle \mathcal{N} \rangle \leq N$.

(d) Since H is finite and $H \leq \langle N^H \rangle$, there exists a finite subset A of N with $H \leq \langle A^H \rangle$. Put $F = \langle A, H \rangle$. Then $A \subseteq F \cap N$ and so $H \leq \langle (F \cap N)^H \rangle$. As N is locally subnormal $F \cap N$ is subnormal in F. So by 4.4d $H \leq F \cap N \leq N$. □

mlsnen **Lemma 4.7** *Let N subgroup of G that is maximal with respect to being locally subnormal. Then N is normal in G.*

Proof: Suppose N is not normal in G. Then there exists $g \in G$ with $N \neq N^g$. By 4.6c, $\langle N, N^g \rangle$ is locally subnormal in G and thus by maximality of $\langle N, N^g \rangle = G$. In particular, $\langle g \rangle \leq \langle N^{\langle g \rangle} \rangle$ and so by 4.6d, $g \in N$ a contradiction to $N \neq N^g$. □

selsn **Theorem 4.8** *Let H be a subgroup of G. Then H is serial in G if and only if it is locally subnormal in G.*

Proof: Suppose first that $\mathcal{S}$ is a series on G containing H. Let $F \in \mathcal{F}$. Then $\mathcal{S} \cap F = \{E \cap F \mid E \in \mathcal{S}\}$ is a subnormal series of F containing $F \cap H$. So H is locally subnormal.

Suppose next that H is locally subnormal in G. Let $\mathcal{S}$ be a maximal chain of locally subnormal subgroups of G with $H \in \mathcal{S}$. That such a maximal chain exists follows from Zorn's lemma. To complete the proof of the theorem it suffices to show that $\mathcal{S}$ is a series. Clearly $1 \in \mathcal{S}$ and $G \in \mathcal{S}$.

Let E be a locally subnormal subgroup of G so that for each $S \in \mathcal{S}$, $E \leq S$ or $S \leq E$. Then $\mathcal{S} \cup \{E\}$ is still a chain of locally subnormal subgroups and so the maximality of $\mathcal{S}$ implies $E \in \mathcal{S}$.

Let $\mathcal{D} \subseteq \mathcal{S}$. By 4.6c, $\bigcup \mathcal{D}$ is locally subnormal in G. Let $E \in \mathcal{S}$. If $E \leq D$ for some $D \in \mathcal{D}$, then $E \leq \bigcup \mathcal{D}$. Otherwise $D \leq E$ for all $D \in \mathcal{D}$ and so $\bigcup \mathcal{D} \leq E$. Hence by the previous paragraph, $\bigcup \mathcal{D} \in \mathcal{S}$. With a similar argument $\bigcap \mathcal{D} \in \mathcal{S}$.

Finally we need to show that $E^- \trianglelefteq E$ for all $E \leq \mathcal{S}$. As $E^- \in \mathcal{S}$, it is locally subnormal. Now let $E^- \leq R \leq E$ so that R is locally subnormal in E. Then R is also locally subnormal in G and so by maximality of $\mathcal{S}$ we get $R \in \mathcal{S}$. But this implies that $R = E^-$ or $R = E$. So E^- is a maximal locally subnormal subgroup of G. Thus by 4.7, $E^- \trianglelefteq E$.

So $\mathcal{S}$ is indeed a series, H is serial in G and the theorem is proved. □

Chapter 5

On the Jordan Hölder Theorem for locally finite groups

sJordan

Definition 5.1 *Let H be a groups acting on the groups X and Y.* djoho

(a) *X and Y are called* H-isomorphic *provided that there exists an isomorphism $\alpha : X \to Y$ which commutes with the action of H.*

(b) *Let $\mathcal{C}$ and $\mathcal{D}$ be sets of groups acted upon by H. Then $\mathcal{C}$ and $\mathcal{D}$ are H* isomorphic *provided that there exists a bijection $\beta : \mathcal{C} \to \mathcal{D}$ so that C and $\beta(C)$ are H-isomorphic for all $C \in \mathcal{C}$.*

(c) *We say that the* Jordan-Hölder theorem holds for H on X *provided that the sets of factors for any two composition series for H on X are H-isomorphic.*

(d) *Let calE be a class of groups acted upon by H. We say that the* Jordan-Hölder theorem holds for H on X with respect to $\mathcal{E}$ *provided that the sets of calE-factors for any two composition series for H on X are H-isomorphic.*

The following lemma will produces lots of groups for which the Jordan Hölder theorem fails.

Lemma 5.2 *Let $\mathcal{I}$ be a set of groups and $\oplus\mathcal{I} \leq H \leq \prod\mathcal{I}$. Then H has a normal series whose set of factors is isomorphic to $\mathcal{I}$.* sersub

Proof: Let " $\preceq$ " be any reversed well ordering (that is any non empty subset has a maximal element) on $\mathcal{I}$ such that I has a minimal element. For $I \in \mathcal{I}$ let $H_I^+ = \{f \in H \mid f_J = 1 \text{ for all } I \prec J \in \mathcal{I}\}$ and $H_I^- = \{f \in H \mid f_J = 1 \text{ for all } I \preceq J \in \mathcal{I}\}$. Let $\mathcal{S} = \{H_I^+, H_I^- \mid I \in \mathcal{I}\}$. We claim that $\mathcal{S}$ is a series for H with jumps (H_I^-, H_I^+), $I \in \mathcal{H}$. As $\oplus\mathcal{I} \leq H$, it is clear that $H_I^+/H_I^- \cong I$, so the lemma is proved once the claim is established.

If m is the minimal element of $\mathcal{I}$, then $1 = H_m^- \in \mathcal{H}$ and if M is the maximal element of I, then $H = H^+M \in \mathcal{I}$.

If $J \prec I \in \mathcal{I}$, then $H_J^+ \leq H_I^- \leq H_I^+$ and so $\mathcal{S}$ is totally ordered.

Let $\mathcal{D}$ be a subset of $\mathcal{S}$. Let $\mathcal{R}$ be the set of all $I \in \mathcal{I}$ with $H_I^\epsilon \in \mathcal{D}$ for some $\epsilon \in \{\pm\}$.

As $\mathcal{R}$ has a minimimal element, there exist a largest lower bound r for for $\mathcal{R}$. If $H_r^- \in \mathcal{R}$, then $\bigcap \mathcal{D} = H_r^-$. If $H_r^- \notin \mathcal{R}$ we will show that $\bigcap \mathcal{D} = H_r^+$. Clearly $H_r^+ \leq \bigcap \mathcal{D}$. Let $f \in \bigcap \mathcal{D}$ and let $I \in \mathcal{I}$ with $r \prec I$. As r is the largest lower bound of $\mathcal{R}$, there exists $J \in \mathcal{R}$ with $J \prec I$. Then $H_J^\epsilon \in \mathcal{D}$ for some ϵ and so $f \in H^\epsilon \leq H_J^+$. Hence $f_I = 1$ and $f \in H_r^+$. So indeed $\bigcap \mathcal{D} = H_r^+$.

Let R be the maximal element of $\mathcal{R}$. If $H_R^+ \in \mathcal{R}$, then $\bigcup \mathcal{D} = H_R^+$ and if $H_R^+ \notin \mathcal{R}$ then $H_R^- \in \mathcal{R}$ and $\bigcup \mathcal{D} = H_R^-$.

So $\mathcal{S}$ is closet under union and intersection. Next let (H_J^ϵ, H_I^μ). be the a jump of $\mathcal{S}$. We need to show that $(H_J^\epsilon, H_I^\mu) = (H_K^-, H_K^+)$ for some $K \in \mathcal{I}$. If $\mu = +$, then $H_J^\epsilon \leq H_I^- < H_I^+$ and so $(H_J^\epsilon, H_I^\mu) = (H_I^-, H^+I)$. If $\epsilon = -$, then $H_J^- < H_J^+ \leq H_I^\mu$ and so $(H_J^\epsilon, H_I^\mu) = (H_J^-, H^+J)$. So suppose $\epsilon = +$ and $\mu = -$. Then $J \prec I$. Suppose there exists $K \in \mathcal{I}$ with $J < K < I$. Then $H_J^\epsilon \leq H_K^- < H_K^+ \leq H_I^-$ and so $(H_J^\epsilon, H_I^\mu) = (H_K^-, H^+K)$. So we may assume that no such K exist. But then $H_J^+ = H^-I$, a contradiction. □

countjor **Example 5.3** *There exists a elementary abelian p-groups which has a composition series countable many factors and another with uncountable many factors.*

Proof: Let $A = \mathbb{F}_p^{\mathbb{N}}$. Then by 5.2 A has a composition series with countable many factors. On the otherhand, A is a vector space over $\mathbb{F}_p$ and so A is also a direct sum of copies of $\mathbb{F}_p$. As A is countable this has to be a uncountable sum and so again by 5.2, A has an uncountable series. □

incs **Proposition 5.4** *Let H be acting an the group X and let $\mathcal{C}$ be a H-series on X. Let M be an H-invariant subgroup of X.*

(a) *$M \cap \mathcal{C} = \{M \cap C \mid C \in \mathcal{C}$ is a H series on X. If M is normal in X and $\mathcal{C}$ is an composition series, then $M \cap \mathcal{C}$ is an H-series whose factors are H-isomorphic to a factors of jumps (B^*, T^*) of $\mathcal{C}$ with $T^* \cap M \neq B^* \cap M$.*

(b) *Suppose that M is normal in X and M fulfills the descending chain condition on H-serial subgroups. Then $\mathcal{C}M/M = \{CM/M \mid C \in \mathcal{C}\}$ is an H-series on X/M. If $\mathcal{D}$ is a composition series, then $\mathcal{D}\mathcal{M}/M$ is a composition series whose factors H-isomorphic to a factors of jumps (B^*, T^*) of $\mathcal{C}$ with $T^* \cap M = B^* \cap M$*

Proof: (a) It is readily verified that $M \cap \mathcal{D}$ contains $1, M$ and is closed under intersections and unions. Let (B, T) be a jump of $M \cap \mathcal{D}$. Put $B^* = \bigcup \{C \in \mathcal{C} \mid M \cap C = B\}$ and $T_* = \bigcap \{C \in \mathcal{C} \mid M \cap C = T\}$. Then clearly $M \cap B^* = B$, $M \cap T_* = T$ and (B^*, T^*) is a jump of $\mathcal{C}$. Thus $B^* \trianglelefteq T_*$ and so $B \trianglelefteq T$. Moreover, $M \cap B^* = (M \cap T_*) \cap B^*$, $T/B = M \cap T_*/M \cap B^* \cong (M \cap T_*)B^*/B^* \leq T_*/B^*$. If $\mathcal{C}$ is a composition series the last "$\leq$" actually is an equality and the lemma holds.

(b) It is readily verified that $\mathcal{D}M/M$ contains $1, X/M$ and is closed under unions. Let $\mathcal{D} \subseteq \mathcal{C}$. Let $E = \bigcap\{DM \mid D \in \mathcal{D}$ and $R = \bigcap \mathcal{D}$.

Claim $E = RD$.

By the descending chain conditions, $\mathcal{D} \cap M$ has a minimal element. Clearly this element is $R \cap M$. So we may assume without loss that $R \cap M = D \cap M$ for all $D \in \mathcal{D}$. Let $D \in \mathcal{D}$. Then $M \leq E \leq DM$ and so $E = (D \cap E)M$. If $D^* \leq \mathcal{D}$ with $D^* \leq D$, then $D^* \cap E \leq D \cap E \leq R = (D^* \cap E)M$ and so $D \cap E = (D \cap E^*)(D \cap M)$. But $D \cap M = R \cap M \leq (D^* \cap M)$ and so $D \cap E \leq D^* \cap E) \leq D^*$. Since this is true for all D^*, $D \cap E \leq R$. Clearly $R \leq E$ and so $E = (D \cap E)M = RM$.

By the claim the intersection property holds for $\mathcal{C}M/M$. Next let $(B/M, T/M)$ be a jump of $\mathcal{C}M/M$. Let $B^* = \bigcup\{C \in \mathcal{C} \mid CM = B\}$ and $T^* = \bigcup\{C \in \mathcal{C} \mid CM = T\}$. Then $B = B^*M$ and by the claim $T = T^*M$. In particular, $B/M \trianglelefteq T/M$ and $\mathcal{C}M/M$ is an H-series. Moreover,

$$T/B = T^*M/B^*M \cong T^*/(B^*M \cap T^*) = T^*/B^*(M \cap T^*).$$

Thus T/B is isomorphic to a quotient of T^*/B^*. If $\mathcal{D}$ is a composition series then $T/B \cong T^*/B^*$. Conversely (B^*, M^*) is a jump with $B^* \cap M \neq T^* \cap M$, then clearly $(B^*M/M, T^*M/M)$ is jump of $\mathcal{D}M/M$ with factor $T^*/B^*(M \cap T^*)$. □

Lemma 5.5 *Let $\mathcal{D}$ be a subnormal H-series on X and that all factors of H have the minimal conditions on H-serial subgroups. Then also X has the minimal condition on H-serial subgroups.* mibymi

Proof: By induction on $|\mathcal{D}|$ we may assume that $\mathcal{D} = \{1, M, X\}$. Let $\mathcal{S}$ be a set of serial subgroups of X. By 5.4 $\mathcal{S} \cap M$ and $\mathcal{S}M/M$ are sets of serial subgroups of M and X/M, respectively. Hence these sets of minimal elements say A and B/M. Without loss $S \cap M = A$ for all $S \in \mathcal{S}$ and then also $SM = B$ for all $S \in \mathcal{S}$. We claim that then all elements of $\mathcal{S}$ are minimal. Indeed, let $R, S \in \mathcal{S}$ with $R \leq S$. Then $S \leq RM = B = RM$ and so $S = R(S \cap M) = RA = R(R \cap M) = R$. □

Using 5.5 the follwing corollary follows easily from 5.4 and induction:

Corollary 5.6 *Suppose H acts on X and $\mathcal{D}$ is subnormal H series all of whose factors have the minimum conditions an H-serial subgroups. Then every H-composition series on X has isomorphic factors as some composition series containing $\mathcal{D}$. In particular, if the Jordan Hölder theorem holds for all factors of $\mathcal{D}$ it holds for X.* □ jh

Proposition 5.7 *Let $\mathcal{E}$ be a class of finite groups, which is closed with respect to subgroup, quotients and extensions. Call G to $\mathcal{E}$-perfect if no non-trivial quotient of G is locally $\mathcal{E}$.* bacae

(a) *Let $O_{\mathcal{E}}(G)$ be the subgroup of G generated by all the serial locally $\mathcal{E}(G)$ subgroup. Then $O_{\mathcal{E}}(G)$ is locally $\mathcal{E}$*

(b) *Let $O^{\mathcal{E}}(G)$ be the intersection of all normal subgroups of G whose quotient is locally $\mathcal{E}$. Then $G/O^{\mathcal{E}}(G)$ is locally $\mathcal{E}$ and $G = \langle O^{\mathcal{E}}(F) | F \in \mathcal{F}$.*

(c) *$O^{\mathcal{E}}(G)$ is $\mathcal{E}$-perfect.*

(d) *Let $\mathcal{C}$ be series on G and $H \in \mathcal{C}$. If all the factors of $\mathcal{C}$ above H are locally $\mathcal{E}$, then $O^{\mathcal{E}}(G) = O^{\mathcal{E}}(H)$.*

(e) *If $M \trianglelefteq G$, then $O^{\mathcal{E}}(G/M) = O^{\mathcal{E}}(G)M/M$*

(f) *If $H \leq G$, then $O^{\mathcal{E}}(H) \leq O^{\mathcal{E}}(G)$*

Proof: If G is finite, all statements are readily verified. The general case follows from the finite one via the local system $\mathcal{F}$. □

dces **Definition 5.8** *Let $\mathcal{E}$ be a class of finite groups, which is closed with respect to subgroup, quotients and extensions.*

(a) *A $\mathcal{E}$-series on G is a set of subgroups $calS$ such that*

  (a) *$1 \in \mathcal{C}$ and $O^{\mathcal{E}}(G) \in \mathcal{C}$.*
  (b) *All members of $\mathcal{S}$ are $\mathcal{E}$-perfect.*
  (c) *If $\mathcal{S}$ is a chain.*
  (d) *If $\mathcal{D} \subseteq \mathcal{S}$, then $\bigcup \mathcal{D} \in \mathcal{S}$ and $O^{\mathcal{E}}(\bigcup \mathcal{D}) \in \mathcal{S}$.*
  (e) *For $D \in \mathcal{S}$ put $D_i = \bigcup\{E \in \mathcal{S} \mid E < D\}$. Then $D^- \trianglelefteq D$ for all $D \in \mathcal{S}$.*

(b) *If $D \neq D^-$ for some $D \in \mathcal{S}$, then (D^-, D) is called a* jump *of $\mathcal{S}$. Let $D_-/D^- = O_{\mathcal{E}}(D/D^-)$. Then D/D^- a* factor *of $\mathcal{S}$.*

(c) *An $calE$-* composition series *is an $\mathcal{E}$-series so that none of the factors has a proper H-invariant normal subgroup.*

isces **Lemma 5.9** *Let $\mathcal{E}$ be a class of finite groups, which is closed with respect to subgroups, quotients and extensions. Let $\mathcal{S}$ be a series on G. Then $O^{\mathcal{E}}(\mathcal{S}) = \{O^{\mathcal{E}}(S) \mid S \in \mathcal{S}\}$ is a $\mathcal{E}$ series whose factors are $O^{\mathcal{E}}(Q)/O^{\mathcal{E}}(Q) \cap O_{\mathcal{E}}(Q)$, Q a factor of $\mathcal{S}$. If $\mathcal{S}$ is a composition series, then the factors of $O^{\mathcal{E}}(\mathcal{S})$ are precisely the $\neq \mathcal{E}$ factors of $\mathcal{S}$.*

Proof: Clearly part (a),(b) and (c) of the definition of a $\mathcal{E}$-series holds for $O^{\mathcal{E}}(\mathcal{S})hold$. Let $\mathcal{D} \subseteq \mathcal{S}$.

Claim $O^{\mathcal{E}}(\bigcup \mathcal{D}) = \bigcup\{O^{\mathcal{E}}(D) \mid D \in \mathcal{D}\}$ and $O^{\mathcal{E}}(\bigcap \mathcal{D}) = O^{\mathcal{E}}(\bigcap\{O^{\mathcal{E}}(D) \mid D \in \mathcal{D}\})$.

Indeed the first part is immediate from 5.7b. For the second let $T = \bigcap \mathcal{D}$ and $B = \bigcap\{O^{\mathcal{E}}(D) \mid D \in \mathcal{D}\}$. Then clearly B is a normal subgroups of T and $O^{\mathcal{E}}(T) \leq B$. Thus $O^{\mathcal{E}}(T) = O^{\mathcal{E}}(T)$. So also the second statement holds.

By the claim $O^{\mathcal{E}}(\mathcal{S})$ is closed under union and "$\mathcal{E}$"- intersections. Also the jumps of $O^{\mathcal{E}}(\mathcal{S})$ are clearly of the form $(O^{\mathcal{E}}(B), O^{\mathcal{E}}(T))$ for a jump (B,T) of $\mathcal{S}$. The claim about the factors is now readily verified.

jhce **Proposition 5.10** *Let $\mathcal{E}$ be a class of finite groups, which is closed with respect to subgroups, quotients and extensions. Then 5.4 and 5.6 hold for locally finite groups if H-series, H-serial, H-composition series is replaced by $\mathcal{E}$-series, $\mathcal{E}$-serial, $\mathcal{E}$-composition series.*

With minor modification the proof for 5.4 goes through. □

Corollary 5.11 *Let $\mathcal{E}$ be a class of finite groups, which is closed with respect to subgroups, quotients and extensions. Suppose that G has a absolute composition series $\mathcal{C}$ with only finitely many factors which are $\neg\mathcal{E}$. Then the Jordan-Hölder Theorem holds for the $\neg\mathcal{E}$-factors on G.* fmccf

By 5.9 has a finite $\mathcal{E}$-composition series. Thus the claim follows from the E version of 5.6. □

Lemma 5.12 *Let G be a locally finite group with a composition series with only finitely many non-abelian factors. Then* $\mathrm{LSol}(G)E(G) \neq 1$. fmnac

Proof:

By 5.9 G has a finite "solvable" series. The lowest term of the series is either locally solvable or a component.

Lemma 5.13 *Let G be countable such that G has some absolute composition series with only finite many non-abelian factors. Then the Jordan-Hölder theorem holds for G.* jhls

Proof: Each composition factor is non abelian or C_p for a prime p. By 5.11 applied to the class of finite solvable groups, the Jordan Hölder theorem holds for the non-abelian factors. Suppose that G has some composition series with only finitely many factors C_p. The all but finitely many composition factors are C_q for $q \neq p$. Hence G has only finitely many $\neg p'$-factors. Hence the Jordan Hölder theorem holds for the $\neq p'$ factors and in particular for the C_p-factors. If each composition series has infinitely many factors C_p then as G is countable, this number is countable infinite and again the Jordan Hölder theorem holds for the C_p-factors. □.

Chapter 6

Absolute Simplicity

Absolute

Definition 6.1 *A group R is called* absolutely simple *if* $\{1, R\}$ *is the only series on* R.

The main result in this section is due to R.E Phillips. It gives a sufficient conditions for a locally finite group to be absolutely simple. The importance of this conditions is that it can be verified locally.

Definition 6.2 ddefect

(a) *Let N be subnormal in G. Then the defect $d_G(N)$ of N in G is smallest length of a subnormal series from N to G.*

(b) *Let $H \leq G$. Then $\mathrm{NC}^0_G(H) = G$ and inductively define $\mathrm{NC}^{i+1}_G(H) = \langle H^{\mathrm{NC}^i_G(H)} \rangle$.*

(c) *Let $H \leq G$. Then $\mathrm{LSC}_G(H)$ is the intersection of the locally subnormal subgroups of G containing H. In other words, $\mathrm{LSC}_G(H)$ is the smallest locally subnormal subgroup of G containing H.*

Lemma 6.3 *Let N be subnormal in G and $d = d_G(N)$. Then* dgn

$$N = \mathrm{NC}^d_G(N) \lhd \mathrm{NC}^{d-1}_G(N) \lhd \ldots \lhd \mathrm{NC}^1_G(N) \lhd \mathrm{NC}^0_G(N) = G$$

is a subnormal series of length d from N to G.

Proof: Let $N = N_d \lhd N_{d-1} \lhd \ldots \lhd N_1 \lhd N_0 = G$ be a subnormal series of length d from N to G. If $d = 0$ (that is $G = N$) there is nothing to prove. So suppose $d \geq 1$. The $\mathrm{NC}^1_G(N) = \langle N^G \rangle \leq \langle N_1^G \rangle = N_1$ and with the same argument and using induction we get that $\mathrm{NC}^i_G(N) \leq N_i$ for all $i \geq 0$. In particular, $N \leq \mathrm{NC}^d_G(N) \leq N_d = N$ and so $NC^d_G(N) = N$. □

Lemma 6.4 *Let $N \leq G$. Then* $\mathrm{LSC}_G(N) = \bigcup_{F \in \mathcal{F}} \mathrm{LSC}_F(F \cap N)$ scgn

Proof: Let R be the group on the right hand. Let $F \in \mathcal{F}$ and $F \leq E \leq G$. Note that $F \cap N \leq E \cap N \leq \mathrm{LSC}_E(E \cap N)$ and that the latter group is locally subnormal in E. Thus $F \cap N \leq F \cap \mathrm{LSC}_E(E \cap N)$ and $F \cap \mathrm{LSC}_E(E \cap N)$ is subnormal in F. Thus by definition of $\mathrm{LSC}_F(F \cap N)$ we get $\mathrm{LSC}_F(F \cap N) \leq F \cap \mathrm{LSC}_E(E \cap N) \leq \mathrm{LSC}_E(E \cap N)$.

Choosing $E = G$ we conclude that $R \leq \mathrm{LSC}_G(N)$.

Choosing E to be finite we see that we can apply 4.5. Thus R is locally subnormal in G. Clearly $N \leq R$ and so by definition of $\mathrm{LSC}_G(N)$, $\mathrm{LSC}_G(N) \leq R$. □

clsnd **Lemma 6.5** *Let $\{L_i \mid i \in I\}$ be a local system for G, and for $i \in I$ let N_i be a subgroup of L_i. Suppose that there exists a non-negative integer d so that*

(a) *For all $i \in I$, N_i is subnormal of defect at most d in L_i.*

(b) *For all $i \leq j \in I$, $N_i \leq N_j$.*

Then $N = \bigcup_{i\in I} N_i$ is a subnormal subgroup of defect at most d in G.

Proof: An easy induction proof shows that for all $t \geq 0$, $\mathrm{NC}^t_G(N) = \bigcup_{i\in I} \mathrm{NC}^t_{L_i}(N_i)$. By 6.3 $\mathrm{NC}^d_{L_i}(N_i) = N_i$ and so $\mathrm{NC}^d_G(N) = N$. Thus N is subnormal in G and has defect at most d. □

lsr **Lemma 6.6** *Let $\mathrm{LSol}(G)$ be the subgroup of G generated by all the locally solvable, locally subnormal subgroups of G. Then $\mathrm{LSol}(G)$ itself is locally solvable and locally subnormal.*

In particular $\mathrm{LSol}(G)$ is the unique maximal locally solvable, normal subgroup of G.

Proof: Suppose first that G is finite. Let A and B solvable normal subgroups of G. We claim that AB is a solvable normal subgroup of G as well. Indeed let i and j be the derived length of A and B respectively. Then $(AB)^{(j)} \leq AB^{(j)} \leq A$ and so $(AB)^{(j+i)} \leq A^{(i)} = 1$. Thus AB is solvable and clearly also normal in G. Hence a finite group G has a unique largest solvable normal subgroup $\mathrm{Sol}(G)$. We claim that $\mathrm{Sol}(G) = \mathrm{LSol}(G)$. For this let A be any solvable, subnormal subgroup of G and $A \trianglelefteq N_1 \triangleleft N_2 \triangleleft \ldots \triangleleft N_d = G$ a subnormal series from A to G. Then $A \leq \mathrm{Sol}(N_1)$. Since $\mathrm{Sol}(N_1)$ is a characteristic subgroup of N_1, $\mathrm{Sol}(N_1)$ is normal in N_2 and

$$A \leq \mathrm{Sol}(N_1) \leq \mathrm{Sol}(N_2) \leq \mathrm{Sol}(N_3) \leq \ldots \leq \mathrm{Sol}(N_d) = \mathrm{Sol}(G).$$

In the general case let $\mathcal{N}$ be the set of locally solvable, locally subnormal subgroups of G. Then $\mathrm{LSol}(G) = \langle \mathcal{N} \rangle$. Let $F \in \mathcal{F}$. Then $\mathcal{N} \cap F$ is a set of solvable, subnormal subgroups of F and so by the finite case $\langle \mathcal{N} \cap F \rangle$ is a solvable subnormal subgroup of F. Since $\{\langle \mathcal{N} \cap F \rangle | F \in \mathcal{F}\}$ is a local system for $\mathrm{LSol}(G)$, $\mathrm{LSol}(G)$ is indeed a locally solvable, locally subnormal subgroup of G. □

bdas **Theorem 6.7** *Let G be a locally finite, simple group and $\mathcal{L}$ a local system for G. Suppose that there exists a positive integer d with the following property:*

If $L \in \mathcal{L}$ and M is an absolutely perfect, locally subnormal subgroup of L, then M is subnormal in L with defect at most d.

Then G is absolutely simple.

Proof: Let N be a non-trivial serial subgroup of G. We need to show that $N = G$. By 4.8 N is locally subnormal in G.

Suppose first that N is locally solvable. Then $1 \neq N \leq \mathrm{LSol}(G)$. Since G is simple we conclude that $G = \mathrm{LSol}(G)$ and so by 6.6 G is locally solvable. But then by 3.4c $G \cong C_p$. Clearly $N = G$ in this case.

Next suppose that N is not locally solvable. Then $N^\infty \neq 1$. Moreover, it is easy to see that $\{(N \cap L)^\infty \mid L \in \mathcal{L}\}$ is a local system for N^∞. By assumption $(N \cap L)^\infty$ is subnormal of defect at most d in L and so by 6.5, $1 \neq N^\infty$ is subnormal of defect at most d in G. Since G is simple, G has no proper subnormal subgroups. Thus $G = N^\infty = N$. $\square$

Chapter 7

Simple locally finite groups which are not absolutely simple

Simple

In this section we will construct the only known examples of simple locally finite groups which are not absolutely simple.

Lemma 7.1 *Let H be a group, S a group acting faithfully and transitively on a set I,* nsowp
$K = H \wr_I S$, where $\wr$ stands for the restricted wreath product. Then any normal subgroup of K either is contained in the base group $B(K)$ or it contains $B(K)'$.

Proof: Let N be a normal subgroup of K not contained in $B(K) = \prod_{i \in I} H_i$. Pick $n \in N \setminus B(K)$. Since $K/B(K) \cong S$, K acts on I with kernel $B(K)$ and $H_i^n = H_{i^n}$. Pick $i \in I$ with $i \neq i^n$. Let $a, b \in H_i$. Then $aa^{-n} = [a^{-1}, n] \leq [K, N] \leq N$. Since $a^{-n} \leq H_i^n$ we have $[a^{-n}, b] = 1$ and so $[a, b] = [aa^{-n}, b] \leq [N, K] \leq N$.

Since a and b are arbitrary we conclude $H_i' \leq N$. And since K is transitive on I we get $H_j' \leq N$ for all $j \in I$. Thus $B(K)' \leq N$. □

Lemma 7.2 *Let H be a perfect finite group. Then there exists a perfect finite group H^* containing H and function X which associates to each subgroup A of H a subgroup $X(A)$ of H^* such that*

(a) $H \leq \langle h^{H^*} \rangle$ for all $1 \neq h \in H$.

(b) $X(A) \cap H = A$ for all $A \leq H$.

(c) If $A \leq B \leq H$, then $A \trianglelefteq B$ if and only if $X(A) \trianglelefteq X(B)$.

(d) $X(H) \trianglelefteq \langle X(H)^{H^*} \rangle$.

Proof: Let S be any finite simple group such that there exists a monomorphism $\alpha : H \to S$ and let T be any non trivial finite perfect group. Furthermore, let S and T act transitively and non-trivially on the sets I and J, respectively. We assume that $0 \in I$ and $\{0, 1\} \subseteq J$. Let $K = H \wr_I S$. For $i \in I$ let $\beta_i : H \to K$ be the canonical isomorphism between H and the i'th component of the base group of K and let β be the canonical monomorphism from S to K. Let $H^* = K \wr_J T$ and for $j \in J$ let $\gamma_j : K \to H^*$ be the canonical isomorphism between K and the j'th component of the base group of H^*. Note that as H, S and T are perfect also K and H^* are perfect. Define $\rho : H \to H^*$ by $\rho(h) = \gamma_0(\beta_0(h))\gamma_1(\beta(\alpha(h)))$. Then ρ is clearly a monomorphism. For $A \leq H$ let $X(A)$ be the set of elements in the base group of H^* such that the projection onto the $0'$th-component is contained in $\gamma_0(\prod_{i\in I}\beta_i(A))$. Identifying H with $\rho(H)$ we see immediately that (b) and (c) hold. Now $\langle X(H)^{H^*}\rangle$ is the base group of H^* and so (d) holds. To prove (a) let $h \in H^\#$. Put $N = [h, \gamma_1(K)]$. Then N is a normal subgroup of $\gamma_1(K)$ and is not contained in the base group of $\gamma_1(K)$. Thus by 7.1 N contains $\gamma_1(B(K))$. Furthermore, $K/B(K) \cong S$ is simple and so $N = \gamma_1(K)$. Since $N \leq \langle h^{H^*}\rangle$ and H^* acts transitively on J, we get $\gamma_j(K) \leq \langle h^{H^*}\rangle$ for all $j \in J$. Hence $\langle h^{H^*}\rangle$ is the base group of H^* and so contains H. □

enaslfsg **Theorem 7.3** *There exists a locally finite, simple group G with an ascending chain*

$$1 \lhd M_1 \lhd M_2 \lhd M_3 \dots$$

of subgroups such that $G = \bigcup_{i=1}^{\infty} M_i$. In particular, G is not absolutely simple.

Proof: Let G_1 be any nontrivial perfect finite group, and inductively let $G_{i+1} = G_i^*$ and X_i any function from the subgroups of G_i to the subgroups of G_{i+1} which fulfills 7.2. Let $G = \bigcup_{i=1}^{\infty} G_i$. Then G is a locally finite group and we claim that G is simple. Indeed, let $x, y \in G$ with $x \neq 1$. Pick i with $x, y \in G_i$. Then by (a) in 7.2

$$y \in G_i \leq \langle x^{G_{i+1}}\rangle \leq \langle x^G\rangle.$$

Thus $\langle x^G\rangle = G$ and G is simple.

Put $M_{1,1} = 1$, $M_{1,2} = G_1$ and inductively, $M_{n+1,j} = X_n(M_{n,j})$, for $1 \leq j \leq 2n$, $M_{n+1,2n+1} = \langle X(G_n)^{G_{n+1}}\rangle$ and $M_{n+1,2n+2} = G_{n+1}$. Then by induction and 7.2, $M_{n+1,i} \lhd M_{n+1,i+1}$ for all $1 \leq i \leq 2n+1$ and $M_{n+1,i} \cap G_n = M_{n,i}$ for all $1 \leq i \leq 2n$. Put $M_i = \bigcup_{n \geq \frac{i}{2}} M_{n,i}$. Then $G_n \leq M_{2n}$, $G_n \cap M_i = M_{n,i}$ for all $i \leq 2n$, $M_i \lhd M_{i+1}$ and $G = \bigcup_{i=1}^{\infty} M_i$. □

Chapter 8

Conjugacy centralizer property

Conjugacy

Definition 8.1 *Let $N \leq G$.* ccp

(a) *N has the* conjugacy centralizer property *if for all $F \in \mathcal{F}$ there exists $n \in N$ with $[F, F^n] = 1$.*

(b) *N has the* centralizer property *if $F \leq C_G(F)N$ for all $F \in \mathcal{F}$.*

Lemma 8.2 *Let N be a normal subgroup of G with the conjugacy centralizer property. Then N has the centralizer property.* ccpcp

Proof: Let $f \in F \in \mathcal{F}$ and pick $n \in N$ with $[F, F^n] = 1$. Then
$f = n^{-1}fnn^{-1}f^{-1}nf = f^n n^{-1} n^f \leq F^n N \leq C_G(F)N$. □

Lemma 8.3 *Let $N \leq G$ have the centralizer property. Then* cp

(a) *If $M \leq G$ is N-invariant, then $M \trianglelefteq G$.*

(b) *N is normal in G.*

(c) *If $A \trianglelefteq N$, then $A \trianglelefteq G$.*

(d) *$G' \leq N$.*

Proof: (a) Let $m \in M$ and $g \in G$. Then $\langle g, m \rangle \leq C_G(\langle g, m \rangle)N$ and so we can write $g = cn$ with $c \in C_G(m)$ and $n \in N$. Then $m^g = m^n \in M$ and so $M \trianglelefteq G$.

(b) and (c) follows from (a).

(d) Let $F \in \mathcal{F}$. Then $FN/N \leq C_G(F)N/N \leq C_{G/N}(FN/N)$. Thus G/N is locally abelian and so abelian. □

Chapter 9

Transitive groups of finitary permutations

Groups

The purpose of this section is to describe all transitive groups of finitary permutations.

Definition 9.1 *Let G be acting on a set Ω and $g \in G$.* finitaryp

(a) $\mathrm{Supp}_\Omega(g) = \{\omega \in \Omega \mid \omega \neq \omega^g\}$.

(b) *The* degree $\deg_\Omega(g)$ *of g on Ω is defined as* $|\mathrm{Supp}_\Omega(g)|$.

(c) *G acts* finitarily *on Ω provided that all elements in g have finite degree on Ω.*

(d) *A* block system *(also called* system of imprimitivity*) for G on Ω is a G-invariant proper partition $\mathcal{D}$ of G. (here G-invariant means that $\Delta^g \in \mathcal{D}$ for all $g \in G$ and proper means $\mathcal{D} \neq \Omega$ and $\mathcal{D} \neq \{\Omega\}$.*

(e) *A* block *(also called a* set of imprimitivity*) for G, is a proper subset Δ of Ω such that for each $g \in G$, $\Delta^g = \Delta$ or $\Delta \cap \Delta^g = \emptyset$ holds.*

(f) *If G has a block system on Ω, G is called* imprimitive*. Otherwise G is* primitive*.*

(g) *G acts* almost primitively *on Ω if G acts primitively on Ω or acts transitively on Ω and primitively on some block system for G on Ω.*

(h) *G acts* totally imprimitively *on Ω if there exists an ascending chain of finite blocks for G on Ω with*

$$\Delta_1 \subset \Delta_2 \subset \Delta_2 \subset \dots$$

such that $\Omega = \bigcup_{i=1}^{\infty} \Delta_i$.

(i) *Let $\mathcal{D}$ be a partition of Ω. Then* $\mathrm{m}(\mathcal{D}) := \inf_{\Delta \in \mathcal{D}} |\Delta|$.

(j) *Let* $\Delta \subseteq \Omega$. $G_\Delta = C_G(\Delta) = \{g \in G \mid \omega^g = \omega$ *for all* $\omega \in \Delta\}$. $G^\Delta = N_G(\Delta)/G_\Delta \leq Sym(\Delta)$.

We remark that if G acts on Ω, G also acts on the set of subsets of Ω. In particular, if $\mathcal{D}$ is a set of subsets of Ω, then $C_G(\mathcal{D})$ denotes the group $\{g \in G \mid D^g = D$ for all $D \in \mathcal{D}\}$. The easy proofs of the next three lemmas are left to the reader.

ubeb **Lemma 9.2** *Let* G *be acting on the set* Ω *and let* $\mathcal{B}$ *be a chain of blocks. Then* $\bigcup \mathcal{B}$ *is a block or equals* Ω. □

premx **Lemma 9.3** *Let* G *be acting transitively on the set* Ω *and fix* $\omega \in \Omega$. *Then the map* $H \to \omega^H$ *is a one-to-one correspondence between subgroups* $G_\omega < H < G$ *and blocks containing* ω.

In particular, G *is primitive if and only if* G_ω *is a maximal subgroup of* G. □

tiec **Lemma 9.4** *Totally imprimitive groups of finitary permutations are countable.* □

sip **Lemma 9.5** *Let* G *be transitive and finitary on* Ω. *Let* $\mathcal{D}$ *be a block system for* G *on* Ω *and* $g \in G$.

(a) *If* g *acts non-trivially on* $\mathcal{D}$, *then* $\mathrm{m}(\mathcal{D}) \leq \frac{1}{2}\deg(g)$.

(b) *Put* $N = \langle g \in G \mid \deg(g) < 2\,\mathrm{m}\,(\mathcal{D})\rangle$. *Then* $N \leq C_G(\mathcal{D})$.

(c) *All blocks for* G *on* Ω *are finite.*

Proof: (b) and (c) follow immediately from (a). For (a) pick $\Delta \in \mathcal{D}$ with $\Delta^g \neq \Delta$. As Δ is a block $\Delta \cap \Delta^g = \emptyset$. Thus g fixes none of the elements in Δ and also none of the elements of Δ^g. Hence $\Delta \cup \Delta^g \subset \mathrm{Supp}(g)$ and so $\deg(g) \geq 2|\Delta|$. As G is transitive on Ω and on $\mathcal{D}$, $|\Delta| = \mathrm{m}(\mathcal{D})$ and so also (a) is proved. □

apoti **Theorem 9.6** *Let* G *be a transitive group of finitary permutations.*

Then G *is either almost primitive or totally imprimitive.*

Proof: We may assume that G is imprimitive and does not have a maximal block. Then each block is properly contained in a larger block and so there exists an infinite chain

$$\Delta_1 \subset \Delta_2 \subset \Delta_3 \subset \ldots$$

of blocks for G on Ω. By 9.5c, all Δ_i's are finite. By 9.2, $\bigcup_{i=1}^\infty \Delta_i$ is a block or equals Ω. The first case is impossible as $\bigcup_{i=1}^\infty \Delta_i$ is infinite. So $\bigcup_{i=1}^\infty \Delta_i = \Omega$ and G is totally imprimitive. □

The next lemma follows easily from 9.5 and we leave the details of the proof to the reader.

dti **Proposition 9.7** *Let G be a totally imprimitive and transitive subgroup of* $\mathrm{FSym}(\Omega)$ *and let Ξ be a chain of blocks for G on Ω with $\Omega = \bigcup \Xi$.*

(a) *Let A be a finite subset of Ω. Then there exists a $\Delta \in \Xi$ with $A \subseteq \Delta$.*

(b) *For a block Δ put $N(\Delta) = C_G(\Delta^G)$. Then $G = \bigcup_{\Delta \in \Xi} N(\Delta)$.*

(c) *Let Δ be a block for G. Then $N(\Delta)$ is a sub-direct (restricted) product of the isomorphic finite groups $N(\Delta)/C_{N(\Delta)}(\Lambda)$, $\Lambda \in \Delta^G$.* □

We remark that if G is a non-primitive, transitive group of finitary permutations either each block lies in a maximal block, or G is totally imprimitive and has no maximal blocks. Indeed this follow from 9.7 and the proof of 9.6.

Lemma 9.8 *Let X be a group acting on the set Ω. Let A and B be finite subsets of Ω and suppose that each orbit of X on Ω has size larger than $|A||B|$. Then there exists a $g \in X$ with $A^g \cap B = \emptyset$.* disjoint

Proof: **Case 1** *All orbits of X have infinite length.*

The prove is by induction on $|A|$. If $A = \emptyset$, there is nothing to prove. So assume the result for any set A' with $|A'| < |A|$. For a contradiction suppose that no element g with the required property exists. That is, each conjugate of A meets B (non-trivially).

Claim For any subset C of Ω with $|C| \leq |A|$, only finitely many conjugates of A contain C.

The claim is proved by induction on $|A| - |C|$. It starts trivially when $|A| - |C| = 0$. So suppose that $|C| < |A|$ and that the claim holds for all $|C'|$ with $|C| < |C'| \leq |A|$. By the first induction we may assume (conjugating C if necessary) that $C \cap B = \emptyset$. By the second induction for each of the finitely many points $b \in B$, only finitely many conjugates of A contain $C \cup \{b\}$. So only finitely many conjugates of A contain C and meets B. By assumption each conjugate of A meets B and so the claim is proved.

Now taking $C = \emptyset$ in the claim we see that A has only finitely many conjugates. But this is a clear contradiction to the hypothesis in Case 1 that X has no finite orbits.

Case 2 *All orbits of X on Ω are finite.*

Let $\Omega_1, \ldots, \Omega_r$ be the orbits that meet A. Then

$$\sum_{i=1}^{r} |A \cap \Omega_i| = |A|$$

$$|\Omega_i| > |A| \cdot |B|$$

Since X acts transitively on Ω_i each element of Ω_i lies in the same number (say u_i) of conjugates of A. Also each conjugate of A meet Ω_i in the same number of elements as A, that is $|A \cap \Omega_i|$. Counting the pairs (ω, D) with $\omega \in D$ and $D \in A^X$ in two different ways we get $|\Omega_i| \cdot u_i = |A^X||A \cap \Omega_i|$. Thus $u_i = \frac{|A^X||A\cap\Omega_i|}{|\Omega_i|}$. This means that there exist at most $u_i|\Omega_i \cap B| = \frac{|A^X||A\cap\Omega_i||B\cap\Omega_i|}{|\Omega_i|}$ conjugates of A which meet $B \cap \Omega_i$ non-trivially. Since $|B \cap \Omega_i| \leq |B|$ and $|\Omega_i| > |A| \cdot |B|$ we have $\frac{|A^X||A\cap\Omega_i||B\cap\Omega_i|}{|\Omega_i|} < \frac{|A^X||A\cap\Omega_i|}{|A|}$. Summing over all i we see that there exist less than

$$\sum_{i=1}^{r} \frac{|A^X||A \cap \Omega_i|}{|A|} = \frac{|A^X|}{|A|} \sum_{i=1}^{r} |A \cap \Omega_i| = \frac{|A^X|}{|A|}|A| = |A^X|$$

conjugates of A that meet B non-trivially. But this means that some conjugate of A meets B trivially. This completes the proof in Case 2.

General Case

Let Ω_0 be the union of the infinite X-orbits and Ω_1 the union of the finite X orbits. Let A_i and B_i be the intersection of Ω_i with A and B, respectively. By Case 2 applied to Ω_1 there exists $x \in X$ with $A_1^x \cap B_1 = \emptyset$. Let $Y = C_X(A_1^x)$. Since A_1^x is finite and a^X is finite for all $a \in A_1^x$, Y has finite index in X. Thus the orbits for Y on Ω_0 are still infinite and we can apply Case 1 to Y and Ω_0 to get $y \in Y$ with $A_0^{xy} \cap B_0 = \emptyset$. But $A_1^{xy} = A_1^x$ and so $A_1^{xy} \cap B_1 = \emptyset$ and $A^{xy} \cap B = \emptyset$. This completes the proof of the proposition also in the general case. □

We remark that the bound $|A||B|$ in the previous lemma is the best possible. To see this, let C and D be any finite sets and $X = \mathrm{Sym}(C) \times \mathrm{Sym}(D) \leq Sym(C \times D)$. Let $c \in C$, $d \in D$, $A = C \times \{d\}$ and $B = \{c\} \times D$. Then the conjugates of A under X are of the form $C \times \{d'\}$ with $d' \in D$, so they meet B in (c, d').

ntnsg **Lemma 9.9** *Let G be transitive and finitary on the infinite set Ω.*

(a) *G is neither abelian nor finite.*

(b) *Let N be an intransitive normal subgroup of G. Then all orbits for N on Ω are finite. Moreover G/N acts transitively and finitarily on the infinite set of orbits of N on Ω. In particular, G/N is neither finite nor abelian.*

Proof: (a) G cannot be finite. If G is abelian, let g be an element acting non-trivially on Ω. Then $\mathrm{Supp}_\Omega(g)$ is a finite G-invariant set, a contradiction.

(b) Since the orbits of N form a system of imprimitivity, each of the orbits is finite. As Ω is infinite, N has an infinite number of orbits. The remaining assertions now are readily verified. □

trnope **Theorem 9.10** *Let G be transitive and finitary on the infinite set Ω.*

(a) *G' acts transitively on Ω.*

(b) *Let N be a transitive, subnormal subgroup of G. Then $G' \leq N$ and and N is normal.*

(c) *G' is perfect and has no proper subgroups of finite index.*

Proof: (a) Since G/G' is abelian, 9.9(b) implies that G' is transitive.

(b) By induction on the defect of N in G we may assume that N is normal in G. Thus in view of 8.2 and 8.3 it remains to verify that N has the conjugacy centralizer property. For this let $F \in \mathcal{F}$ and put $\Delta = \mathrm{Supp}(F)$. As G is finitary, Δ is finite. So by 9.8 applied with $X = N$ and $A = B = \Delta$, there exists $n \in N$ with $\Delta \cap \Delta^n = \emptyset$. Since $\mathrm{Supp}(F^n) = \Delta^n$, this clearly implies $[F, F^n] = 1$. Thus N has the conjugacy centralizer property and (b) holds.

(c) By (a) G'' is transitive and so by (b) $G' \leq G''$. Thus G' is perfect. Let N be a subgroup of finite index in G'. Without loss, N is normal in G'. Then N is transitive and again by 9.9b $G' \leq N$. □

Lemma 9.11 *Let X be a group acting primitively on a set Ω and let $a \in \Omega$.* pj

(a) *Suppose that X_a has k orbits on Ω and X_a has an orbit of length m on $\Omega - a$. If k and m are finite, then $|\Omega| \leq \sum_{i=0}^{k-1} m^i$.*

(b) *If $g \in X$ acts non-trivially on Ω then the number of orbits of X_a on Ω is at most* $\deg(g)$.

Proof: (a) Let $a \neq b \in \Omega$ with $|b^{X_a}| = m$. Define a directed graph on Ω by $\alpha \to \beta$ if and only if $(\alpha, \beta) = (a^x, b^x)$ for some $x \in X$. Let $\Delta(a)$ be the set of elements in Ω that can be reached from a via a directed path. Let $\Delta_i(a)$ be the set of elements in $\Delta(a)$ that are of distance i from a. Then $|\Delta_0(a)| = |\{a\}| = 1$, $|\Delta_1(a)| = |b^{X_a}| = m$ and by induction $|\Delta_{i+1}(a)| \leq m|\Delta_i(a)| \leq m^{i+1}$. Clearly each $\Delta_i(a)$ is a union of X_a-orbits and so at most k of the $\Delta_i(a)$'s are not empty. It follows that

$$|\Delta(a)| = \sum_{i=0}^{k-1} |\Delta_i(a)| \leq \sum_{i=0}^{k-1} m^i.$$

In particular, $\Delta(a)$ is finite. If $c \in \Delta(a)$ then clearly $\Delta(c) \subseteq \Delta(a)$. Since X is transitive, $|\Delta(a)| = |\Delta(c)|$ and, since $\Delta(a)$ is finite, we conclude $\Delta(a) = \Delta(c)$. But this implies that $\Delta(a)$ is a block and, as X is primitive, we conclude $\Delta(a) = \Omega$. Thus (a) holds.

(b) Without loss g does not fix a. Since X is primitive, X_a is by 9.3 a maximal subgroup of X. Thus $X = \langle X_a, g \rangle$ and so g does not normalize any of the orbits of X_a on Ω. But then each orbit of X_a meets $\mathrm{Supp}(g)$ non-trivially and so there are at most $\deg(g)$ orbits.□

Theorem 9.12 *Let X be a primitive subgroup of* $\mathrm{Sym}(\Omega)$ *and suppose that X contains a non-trivial element of finite degree d and that* jordan

$$|\Omega| > \sum_{i=0}^{d-1} (d-1)^{2i}.$$

Then $\mathrm{Alt}(\Omega) \leq X$.

Proof: Let $1 \neq g \in X$ with $\deg(g) = d$. By assumption $|\Omega| > d$ and so there exists $a \in \Omega$ with $a = a^g$. By 9.11b, X_a has at most d orbits on Ω. If X_a has an orbit of length less or equal to $(d-1)^2$ on $\Omega - a$, then 9.11a implies $|\Omega| \leq \sum_{i=0}^{d-1}(d-1)^{2i}$, a contradiction.

Hence all orbits of X_a on $\Omega - a$ have length larger than $(d-1)^2$. Let t be a conjugate of g in X with $a \neq a^t$, that is $a \in \mathrm{Supp}(t)$. Then we can apply 9.8 to $X = X_a$, $\Omega = \Omega - a$ and $A = B = \mathrm{Supp}(t) - a$ to obtain $h \in X_a$ with $(\mathrm{Supp}(t) - a) \cap (\mathrm{Supp}(t) - a)^h = \emptyset$. As h fixes a, $\mathrm{Supp}(t) \cap \mathrm{Supp}(t)^h = \{a\}$. A straight-forward calculation now shows that $[t, t^h]$ is a 3-cycle. $\mathrm{Alt}(\Omega) \leq X$ is now a consequence of exercise 14. □

psio **Corollary 9.13** *Let G be a primitive subgroup of $\mathrm{FSym}(\Omega)$ with Ω infinite. Then $G = \mathrm{Alt}(\Omega)$ or $G = \mathrm{FSym}(\Omega)$.* □

sifipu **Corollary 9.14** *Let G be an infinite, simple group of finitary permutations. Then G is an alternating group.*

Proof: Let $G \leq \mathrm{FSym}(Omega)$. Since G is simple, it acts faithfully on each of its orbits. So we may assume that G is transitive. Suppose that G is is totally imprimitive. Then by 9.7b, G is the union of proper normal subgroups, a contradcition to the simplicity of G. Hence by 9.6 G is almost primitive on Ω. Thus G is primitive or acts primitively on some bock system $\mathcal{D}$. In the latter case, the simplicity forces G to be faithful on $\mathcal{D}$ and so in any case G is a primitive group of finitary permutations. As G is infinite we can apply 9.13 to conclude that G is an alternating or a symmetric group. As G is simple, G is alternating.□

Proposition 9.15 *Let G be an almost primitive subgroup of $\mathrm{FSym}(\Omega)$ with Ω infinite. Let B be a maximal block for G on Ω and $\mathcal{D} := B^G$. Put $W = G^B \wr \mathrm{FSym}(\mathcal{D})$ and view W as a subgroup of $\mathrm{FSym}(\Omega)$ with $G \leq W$. Then*

$$W' \leq G \leq W.$$

Proof: The goal is to show that G has the centralizer property in W. The proposition then follows from 8.3.

Let $\mathcal{E}$ be any finite subset of $\mathcal{D}$ and $\Delta = \bigcup \mathcal{E}$.

Claim: $G^\Delta = W^\Delta$

Let $E \in \mathcal{E}$. Then by definition of W, $G^E = W^E$. Hence there exists a finite subgroup F of $N_G(E)$ with $F^E = W^E$. Pick a finite subset $\mathcal{S}$ of $\mathcal{D}$ with $\mathrm{Supp}(F) \subseteq \bigcup \mathcal{S}$. As G induces at least $\mathrm{Alt}(\mathcal{D})$ on $\mathcal{D}$, there exists a g in $N_G(E)$ with $\mathcal{E} \cap \mathcal{S}^g = \{E\}$. Put $H(E) = F^g$. Then $H(E)$ induces W^E on E and centralizes all other $E' \in \mathcal{E}$. Put $H = \langle H(E) \mid E \in \mathcal{E}\rangle$. Then $H^\Delta \cong \times_{E \in \mathcal{E}} W^E \cong C_W(\mathcal{E})^\Delta$. Moreover, $G^\mathcal{E} = \mathrm{Sym}(\mathcal{E}) = W^\mathcal{E}$ and so the claim holds.

Next let F be any finite subgroup of W. Since W is finitary we can choose our $\mathcal{E}$ as above with $\mathrm{Supp}(F) \subseteq \Delta$. Then by the claim $F \leq C_W(\Delta) N_G(\Delta)$. Clearly $C_W(\Delta) \leq C_W(F)$ and so $F \leq G C_W(F)$. So G indeed has the centralizer property and the proposition is proved.□

Chapter 10

Linear periodic groups in characteristic 0

LinChar0

Definition 10.1 *Let X be a group, n a positive integer and K a field. Then X is* linear *of degree n over K if X is isomorphic to a subgroup of $GL_n(K)$. X is* linear *if it is linear of some degree over some field.* dlin

Lemma 10.2 *Let $X \leq GL_n(K)$. Then there exists a finitely generated subgroup H of X, such that the images of KX and KH in $\mathrm{End}_n(K)$ are equal. In particular, X and H have the same submodules in K^n, and X is completely reducible if and only if H is.* locirr

Proof: The image of KX in $\mathrm{End}_n(K)$ is a vector space of dimension at most n^2 over K and spanned by the images of X. So it is spanned by the images of at most n^2 elements. Take H to be the group generated by these elements. □

Lemma 10.3 *Suppose that $G \leq GL_n(K)$ and $\gcd(|G|, operatornamecharK) = 1$. Then G is completely reducible.* masch

Proof: By 10.2 we may assume that G is finite. Let W be proper KG-subspace of $V = K^n$. Then there exits a K-subspace U of V with $V = W \oplus U$. Define a K-linear map $\pi : V \to W$ by $\pi(w + u) = w$ for all $w \in W, u \in U$. Then define $\phi : V \to W$ by $\phi(v) = \sum_{g \in G} \pi(v^g)$. Since ϕ_W is just multiplication by $|G|$ and since $\gcd(|G|, operatornamecharK) = 1$, ϕ_W is an isomorphism. Thus $V = W \oplus \ker\phi$. Let $h \in G$. Then $\phi(v^h) = \phi(v)$ and so $\ker\phi$ is a KG-module. By induction on the dimension both W and $\ker\phi$ are direct sum of irreducible KG-modules. Hence V is completely irreduicble as KG-module. □

Proposition 10.4 *Periodic linear groups are locally finite.* perloc

Proof: TO BE CONTINUED

Definition 10.5 *Let V be a vector space over the field F and σ an automorphism of F of order 2. Let $f : V \times V \to K$ be a map.* duni

(a) *f is called a* unitary form *provided that*

  (a) *f is linear in the first coordinate.*

  (b) *f is σ-linear in the second coordinate.*

  (c) *$f(u, w) = f(w, u)^\sigma$ for all $u, w \in V$.*

  $GU(V, f)$ denotes the group of invertible linear transformations fixing f).

(b) *f is called a* hermitian form *if f is unitary with $F = \mathbb{C}$ and σ is the complex conjugation.*

(c) *f is a* positive definite hermitian form *provide that f is a hermitian form and $f(u, u) > 0$ for all $0 \neq u \in V$.*

The next lemma is easily proved by induction on $\dim U$ and we will leave the details to the reader.

ortsum **Lemma 10.6** *Let f be a positive definite hermitian form on V and U a finite dimensional subspace of V. Then $V = U \oplus U^\perp$, where $U^\perp = \{v \in V \mid f(u, v) = 0 \text{ for all } u \in U\}$.*

invher **Lemma 10.7** *Let G be acting on the finite dimensional vector space V over $\mathbb{C}$. Then there exists a positive definite G-invariant hermitian form V.*

Proof: Suppose first that G is finite. Let f be any positive definite hermitian form on G. For $g \in G$ define $f^g(u, v) = f(u^g, v^g)$. Then $\sum_{g \in G} f^g$ does the trick.

For the general case we may assume without loss that G is irreducible. By 10.2 there exists a finite, irreducible subgroup F of G. By the finite case there exists such an F-invariant positive definite form f on V. We claim that f is already G-invariant.

For this let be $F \leq H \in \mathcal{F}$ and h be an H-invariant positive definite hermitian form on V. Let $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ be the dual of V^*. Define

$$\phi_f : V \to V^*, \phi_f(w)(v) = f(v, w)$$

.

Then $\phi_f(w)^g(v) = \phi_f(w)(v^{g^{-1}}) = f(v^{g^{-1}}, w) = f(v, w^g) = \phi_f(w^g)(v)$. Thus $\phi_f \in \mathrm{Hom}_{\mathbb{C}G}(V, V^*)$. By Schur's lemma and since $\mathbb{C}$ is algebraically closed $\mathrm{Hom}_{\mathbb{C}G}(V, V^*) = \mathbb{C} \cdot \phi_f$. Now H is also an F-invariant hermitian form and so $\phi_f = \lambda \cdot \phi_h$ for some $\lambda \in \mathbb{C}$. Thus $f(u, v) = \lambda h(u, v)$. Since h is H-invariant this implies that f is H-invariant.

This is true for any finite H containing F and so f is indeed G-invariant. □

norm **Definition 10.8** *Let f be a positive definite hermitian form on V. For $v \in V$ put $||v|| = \sqrt{f(v, v)}$. For $A \in \mathrm{End}_{\mathbb{C}}(V)$ put $||A|| = \sup\{\frac{||Av||}{||v||} \mid 0 \neq v \in V\}$.*

trie **Lemma 10.9** *Let f be a positive definite hermitian form on V.*

(a) $f(v,w)+f(w,v)\leq 2||v||\cdot||w||$.

(b) $||v+w||\leq||v||+||w||$.

(c) $||A+B||\leq||A||+||B||$.

(d) *If $A\in GU(V,f)$ then $||A||=1$ and $||A-1||\leq 2$.*

(e) *Suppose that $A\in GU(V,f)$ and there is a $0\neq v\in V$ with $f(Av,v)=0$. Then $||A-1||\geq\sqrt{2}$.*

(f)
$$||AB||\leq||A||\cdot||B||$$

Proof: Let $v,w\in V$ and put $a=||v||$, $b=||w||$ and $t=f(v,w)+f(w,v)$.
(a) Let $r\in\mathbb{R}$. Then
$0\leq f(v+rw,v+rw)=a^2+rt+r^2b^2$. Choosing $r=-\frac{a}{b}$ we conclude that $a^2-\frac{a}{b}t+a^2\geq 0$. Thus $t\leq 2ab$ and a holds.
(b) $f(v+w,v+w)=a^2+t+b^2\leq a^2+2ab+b^2=(a+b)^2$.
(c) Follows from (b) and the definition of $||A||$.
(d) As f is A-invariant, $||Av||=||v||$. Moreover, $||(A-1)v||=||Av-v||\leq||Av||+||v||=2\cdot||v||$.
(e) If $f(Av,v)=0$, then $f(Av-v,Av-v)=2f(v,v)$.
(f) follows immediately from the definition of $||A||$. □

Lemma 10.10 *Let f be a positive definite hermitian form on the finite dimensional K-vectorspace V and $A,B\in GU(V,f)$.* norcom

(a) *V is the orthogonal sum of the eigenspaces of A.*

(b)
$$||AC||=||C||=||CA||$$
for all $C\in\operatorname{End}_{\mathbb{C}}(V)$.

(c) $||[A,B]-1||\leq 2\cdot||A-1||\cdot||B-1||$.

(d) *If $||B-1||<\sqrt{2}$, $||A-1||\leq\frac{1}{2}$ and $[B,A,k]=1$,* [1] *for some $k\leq 1$, then $[B,A]=1$.*

Proof: (a) As $\mathbb{C}$ is algebraically closed, there exists a non-zero eigen-vector v. Then by 10.6 $V=Kv\oplus v^{\perp}$, and (a) follows by induction on $\dim V$.
(b) is clear.
(c) $||[A,B]-1||=||A^{-1}B^{-1}AB-1||=||AB-BA||=||(A-1)(B-1)-(B-1)(A-1)||\leq\leq||(A-1)(B-1)||+||(B-1)(A-1)||\leq 2||A-1||\cdot||B-1||$.

[1] $[B,A,2]:=[[B,A],A]$ and $[B,A,n+1]:=[[B,A],A]$

(d) By (c) and induction, $||[B,A,k]-1|| \leq \sqrt{2}$ for all k. So by induction on k we may assume that $k=2$. So suppose that $[B,A] \neq 1$. Then $A \neq A^B$. Since A commutes with $[B,A] = A^{-B}A$, A also commutes with A^B. As $A \neq A^B$ there exists some eigenspace D for A on V with $D \neq D^B$. Since A^B commutes with A, A^B leaves D invariant and so D is the orthogonal sum of the eigenspaces for A^B on D. Hence $D = (D \cap D^A) \oplus (D \cap D^{B\perp})$. So there exists $0 \neq v \in (D \cap D^{B\perp})$. But then $v^B \in D^A$ and $v \perp v^B$. Thus 10.9e implies $||B-1|| \geq \sqrt{2}$. □

linjor **Theorem 10.11** *There exists a function $J : \mathbb{N} \to \mathbb{N}$ so that whenever G is a periodic subgroup of $GL_n(\mathbb{C})$, then G has an abelian normal subgroup of index at most $J(n)$.*

Proof: Let $V = \mathbb{C}^n$. By 10.7 there exists a G-invariant hermitian form on V. Let $\mathcal{A} = \{g \in G \mid ||g-1|| < \frac{1}{2}\}$ and $H = \langle \mathcal{A} \rangle$.

Step 1 *H is abelian.*

Let $a, b \in \mathcal{A}$ and put $a_k = [a,b,k]$. By 10.4 $\langle a, b \rangle$ is finite. Hence there exists a k with $||a_k - 1||$ minimal. Suppose that $a_k \neq 1$. Then $||a_k - 1|| \neq 0$ and so by 10.10c

$$||a_{k+1} - 1|| = ||[a_k, b] - 1|| \leq 2 \cdot ||a_k - 1|| \cdot ||b - 1|| < ||a_k - 1||,$$

a contradiction to the minimal choice of a_k. Thus $a_k = 1$ and so $[a,b] = 1$ by 10.10d.

Step 2 *$|G/H|$ is bounded by a function of n.*

Let $\mathcal{R}$ be a transversal to H in G and $r, s \in \mathcal{R}$. Since $Hr \neq Hs$, $rs^{-1} \notin H$ and so

$$||s - r|| = ||1 - rs^{-1}|| \geq \frac{1}{2}.$$

It follows that the spheres of radius $\frac{1}{5}$ around the members of $\mathcal{R}$ are disjoint. As the elements of $\mathcal{R}$ have norm 1, all these spheres are contained in a sphere of radius $\frac{6}{5}$. Hence $|G/H| = \mathcal{R}$ is bounded by the fraction of the volumes of spheres of radius $\frac{6}{5}$ and $\frac{1}{5}$ in the $2n^2$ dimensional metric space $\mathrm{End}_{\mathbb{C}}(V)$. □

Corollary 10.12 *Let G be a periodic subgroup of $GL_n(K)$, where K is a field of characteristic 0. Then G is an abelian normal subgroup of index less or equal to $J(n)$.*

Proof: We contend our self with a sketch of a proof. Suppose first that G is finite. Then it can be show that there exists an $F \leq K$ with $G \leq GL_n(F)$ and F is algebraic over $\mathbb{Q}$. But then $F \leq \mathbb{C}$, and so $G \leq GL_n(\mathbb{C})$. So the theorem holds by 10.11.

In the general case we see that all finite subgroups of G have an abelian normal subgroup of index at most $J(n)$. Let I be the set of finite subgroups of G. For $H \in I$ let T_H be the set of all abelian normal subgroups A of H with $|H/A| \leq J(n)$. Then T_H is not empty. Let $L \leq H \in I$ and $A \in T_H$. Then $A \cap L \trianglelefteq L$ and $|L/A \cap L| = |LA/A| \leq |H/A| \leq J(n)$. Thus $A \cap L \in T_L$ and we obtain a map, $\phi_{HL} : T_H \to T_L, A \to A \cap L$. Clearly ϕ_{HH} is the identity function on T_H. Also if $M \leq L$ and $A \in T_H$, then $M \cap A = M \cap (L \cap A)$ and so $\phi_{LM} \circ \phi_{HL} = \phi_{HM}$. So the T_H and ϕ_{HL} form an inverse limit system. Since each T_H

is finite, 11.2 shows that that the inverse limit system has an inverse limit, $(A_H)_{H\in I}$. Put $A = \bigcup_{H\in I} A_H$. We will show that A is an abelian normal subgroups of A and $|G/A| \leq J(n)$.

Let $H \in I$. We claim that $A_H = A \cap H$. Clearly, $A_H \leq A \cap H$. Let $a \in A \cap H$. Then there exists $L \in I$ with $a \in A_L$. Put $M = \langle H, L\rangle$. Then $M \in I$, $A_H = A_M \cap H$ and $A_L = A_M \cap L$. Since $a \in A_L$, $a \in A_M$ and so $a \in A_M \cap H = A_H$. So $A \cap H \leq A_H$.

Let $a, b \in A$ and $g \in G$. Put $H = \langle a, b, g\rangle$. Then $H \in I$ and so $\{a, b\} \in A \cap H = A_H$. Since A_H is a normal abelian subgroup of H we have $\langle a, b\rangle \leq A_H \leq A$, $a^g \in A_H \leq A$ and $[a, b] = 1$. Thus A is an abelian normal subgroup of G.

Let R be a transversal to A in G and S a finite subset of R. Put $H = \langle S\rangle$. Then $H \in I$. Since $As \neq At$ for all $s \neq t \in S$ and since $A_H = A \cap H$, $A_H s \neq A_H t$ for all $s \neq t \in S$. Thus $|S| \leq |H/A_H| \leq J(n)$. Since this holds for all finite subsets of R, $|R| \leq J(n)$ and so $|G/A| \leq J(n)$. □

Chapter 11

Inverse limits of finite sets

In this section we will show that inverse limits of finite sets are not empty.

Definition 11.1 *Let I be a directed set. Then an* inverse limit system *based on I is a tuple $(T_i \mid i \in I, \phi_{ij} \mid i \geq j \in I)$ such that* dinvlim

(a) *For $i \in I$, T_i is a non-empty set.*

(b) *For $i \geq j \in I$, $\phi_{ij} : T_i \to T_j$ is a function.*

(c) *For $i \in I$, ϕ_{ii} is the identity on T_i.*

(d) *For $i \geq j \geq k$, $\phi_{jk} \circ \phi_{ij} = \phi ik$.*

The inverse limit *of a inverse limits system is the set of all tuples $(t_i \mid i \in I)$ such that $t_i \in T_i$ and $\phi_{ij}(t_i) = t_j)$ for all $i \geq j \in I$.*

Proposition 11.2 *The inverse limit of an inverse limits system of finite sets is not empty.* inlifi

Proof: Let $\mathcal{A}$ be the set of all tuples $(J, t_j \mid j \in J)$ such $J \subseteq I$, $t_i \in T_i$ and

(*) *For all finite subsets F of J and all $F \leq s \in I$ there exists $t_s \in T_s$ so that $\phi_{sf}(t_s) = t_f$ for all $f \in F$.*

Order $\mathcal{A}$ by $(J, t_j \mid j \in J) \leq (K, s_k \mid k \in K)$ if $J \subset K$ and $t_j = s_j$ for all $j \in J$. It is easy to verify that the assumptions of Zorn's lemma are fulfilled and so there exists a maximal elements $(J, t_j \mid j \in J)$ in $\mathcal{A}$. If $J = I$ then (*) easily implies that $(t_j \mid j \in J)$ is in the inverse limit. So we may assume that $J \neq I$. Pick $i \in I \setminus J$ and put

$$X = \bigcap \left\{ \phi_{si}(\bigcap_{f \in F} \phi_{sf}^{-1}(t_f)) \mid F \subseteq J, |F| < \infty, F \leq s \in I \right\}$$

We claim that X is non empty. Since T_i is finite, there exists finite subsets $F_1, F_2, \ldots F_l$ of J and elements $s_1, s_2, \ldots s_l$ with $F_k \leq s_k$

$$X = \bigcap \left\{ \phi_{s_k i}(\bigcap_{f \in F_k} \phi_{s_k f}^{-1}(t_f)) \mid 1 \leq k \leq l \right\}$$

Let s be an upper bound for the $s_i's$ and $F = \bigcap_{k=1}^{l} F_k$. Then by (*) there exists $t_s \in T_s$ with $\phi_{sf}(t_s) = t_f$ for all $f \in F$. Put $x = \phi_{si}(t_s)$. Then for all k and all $f \in F_k$ $x = \phi_{s_k i}(\phi_{ss_k}(t_s))$ and $\phi_{s_k f}(\phi_{ss_k}(t_s)) = \phi_{sf}(t_s) = t_f$. Hence $\phi_{ss_k}(t_s) \in \bigcap_{f \in F_k} \phi_{s_k f}^{-1}(t_f)$ and $x \in X$. Thus X is indeed not empty.

Now let $t_i \in X$ and put $J^* = J \cup \{i\}$. It's clear from the definition of X that $(J^*, t_j \mid j \in J^*)$ fulfills (*), a contradiction to the maximality of $(J, t_j \mid j \in J)$. □

Chapter 12

Ultra-products and a little bit of model theory

sUltra

In this section we will prove Malcev's Theorem which asserts that a group with a local system of linear groups of bounded degree is itself linear. The main tool in th proof are ultra-products. The most natural setting for ultra products is model theory. So we will use the occasion to introduce the basic definitions in model theory.

Definition 12.1 dmodth

(a) *A* first-order language *is a set $\mathcal{FL}$ together with a partition $(\mathcal{FS}, \mathcal{RS}, \mathcal{CS}, \mathcal{LS})$ of $\mathcal{FL}$. The elements of $\mathcal{FL}$, $\mathcal{FS}$, $\mathcal{RS}$, $\mathcal{CS}$ and $\mathcal{LS}$ are referred to as* symbols, function symbols, relations symbols, constant symbols *and* logical symbols *Also $\mathcal{LS} = \{), (, \wedge, \neg, \equiv, \forall, v_1, v_2, v_3 \ldots, v_n, \ldots\}$. The v_i's are called variables and $\forall$ a quantifier. The functions symbols are assumed to be partitioned into 1-placed,2-placed, 3-placed, ... function symbols and the relation symbols are partitioned into 1-placed, 2-placed,... relation symbols. $\mathcal{V}$ is the set of variables.*

(b) *A model $\mathcal{A}$ for the language $\infty\mathcal{L}$ is a set A together with a function which assigns to each m-placed functions symbol s an function $A^m \leftarrow A$, and to each m-placed relation symbol an m-placed relation on A and to each constant symbol an element of A. A is called the* universe *of the model. The object assigned to a symbol is called the interpretation of the symbol in $\mathcal{A}$.*

(c) *A* term *is any sequence of symbols obtain in finitely many steps by the following procedure:*

  (a) *A variable is a term.*
  (b) *A constant symbol is a term.*
  (c) *If f is an m-placed function symbol and $t_1, t_2, \ldots, t_m$ are terms, then $f(t_1 t_2 \ldots t_m)$ is a term.*

(d) *An* atomic formula *is a sequence of symbols of one of the following two forms:*

   (a) $t_1 \equiv t_2$*, where* t_1 *and* t_2 *are terms.*

   (b) $R(t_1 t_2 \ldots t_m)$*, where* R *is an* m*-placed relation symbol and* $t_1, \ldots t_m$ *are terms.*

(e) *A* formula *is any sequence of symbols obtain in finitely many steps by the following procedure:*

   (a) *An atomic formula is a formula.*

   (b) *If* ϕ *and* ψ *are formulas, then* $(\phi \wedge \psi)$ *and* $(\neg\phi)$ *are formulas.*

   (c) *If* v *is a variable and* ϕ *is an formula then* $(\forall v)\phi$ *is a formula.*

(f) *Let* v *be a variable and* ϕ *a formula. If* v *is in the sequence* ϕ*, we say that* v *is a variable of* ϕ*. If* v *is a variable of* ϕ *and the last occurrence of* v *is of the form* $\forall v$ *then* v *is called a bound variable of* ϕ*. A variable of* ϕ *which is not bound is called a free variable of* ϕ*. Otherwise* v *is a bounded variable of* ϕ*.*

dquant **Definition 12.2** *Given formulas* ϕ *and* ψ *and a variable* v*.*

(a) $\phi \vee psi)$ *denotes the formula* $(\neg((\neg\phi) \wedge (\neg\psi)))$*.*

(b) $(\phi \leftarrow \psi)$ *denotes the formula* $((\neg\psi) \wedge \phi)$

(c) $(\psi \leftrightarrow \psi)$ *denotes the formula* $((\phi \leftarrow \psi) \wedge (\psi \leftarrow phi))$*.*

(d) $(\exists v)\phi$ *denotes the formula* $(\neq (\forall v)(\neg\psi)))$*.*

dsatmod **Definition 12.3** *Let* $\infty\mathcal{L}$ *be a first order language and* $\mathcal{A}$ *a model for* $\infty\mathcal{L}$ *with universe* A*. Let* E *be a function from* $\mathcal{V} \to A$*.*

(a) *Let* t *be a term. The the value* $t\,|_E$ *of* t *at* E *is defined as follows:*

   (a) *If* t *is a variable, then* $t\,|_E = E(t)$*.*

   (b) *If* t *is a constant, then* $t\,|_E$ *is the interpretation of* t *in* $\mathcal{A}$*.*

   (c) *If* $t = F(t_1 \ldots t_m)$*, for a function symbol* F *and terms* $t_1, \ldots, t_m$*, then*

$$t\,|_E = F^*(t_1\,|_E, \ldots, t_m\,|_E).$$

   where F^* *is the interpretation of* F *in* $\mathcal{A}$*.*

(b) *Let* ϕ *be a formula. Then the statement* E *satisfies* ϕ *in* $\mathcal{A}$ *(written* $\mathcal{A} \models \phi\,|_E$*) is defined as follows.*

   (a) *If* ϕ *is the atomic formula* $t_1 \equiv t_2$ *for terms* t_1*,* t_2 *when* $\mathcal{A} \models \phi\,|_E$ *if and only if* $t_1 \mid E = t_2 \mid E$*.*

(b) *If ϕ is the $R(t_1 \dots t_m)$ for a relation symbol R and terms $t_1, \dots t_m$. Then $\mathcal{A} \models \phi \mid_E$ if and only if $(t_1 \mid_E, \dots, t_m \mid_E)$ fulfills the interpretation of$R$ in $\mathcal{A}$.*

(c) *If $\phi = \phi_1 \wedge phi_2$, then $\mathcal{A} \models \phi \mid_E$ if and only if both $\mathcal{A} \models \phi_1 \mid_E$ and $\mathcal{A} \models \phi_2 \mid_E$ hold.*

(d) *If $\phi = \neg\psi$, then $\mathcal{A} \models \phi \mid_E$ if an only if not $\mathcal{A} \models \psi \mid_E$.*

(e) *If $\phi = (\forall v)\psi$, then $\mathcal{A} \models \phi \mid_E$ if and only if $\mathcal{A} \models \phi \mid_{E*}$, for all functions $E^* : \mathcal{V} \to A$,which agree with E except maybe on v.*

The easy proof of the following lemma is left to the reader.

Lemma 12.4 *Let $\infty\mathcal{L}$ be a first order language, $\mathcal{A}$ a model for $\infty\mathcal{L}$ and $E, F : \mathcal{V} \to A$.* dov

(a) *If t is a term and E and F agree on the variables of t, then $t \mid E = t \mid F$.*

(b) *If ϕ is a formula and E and F agree on the free variable of ϕ then $\mathcal{A} \models \phi \mid_E$ if and only if $\mathcal{A} \models \phi \mid_F$.* □

In view of the previous lemma we can talk about $t \mid_E$, whenever t is a term and E is a function from the variables of t to A and about $\mathcal{A} \models \phi \mid_E$ whenever ϕ is a formula and E a function defined on the free variables of ϕ. In particular, if ϕ has no free variable, the statement $\mathcal{A} \models \phi$ already makes sense.

Definition 12.5 *Let $\mathcal{FL}$ be a first order language and $\mathcal{A}$ a model for $\mathcal{A}$.* dtheory

(a) *ϕ is a* sentence *if ϕ has no free variables.*

AIb Let ϕ be a sentence then we say that $\mathcal{A}$ is a model of ϕ if $\mathcal{A} \models \phi$.

(c) *A* theory *for $\mathcal{A}$ is a set of sentences.*

(d) *A model for a theory Σ is a model for $\mathcal{FL}$ which is a model for each of the sentences in Σ.*

(e) *Two models $\mathcal{A}$ and $\mathcal{B}$ for $\mathcal{FL}$ are called elementary equivalent if for each sentence ϕ of $\mathcal{FL}$, $\mathcal{A} \models \phi$ if and only if $\mathcal{B} \models \phi$.*

Choose $\mathcal{FL} = \{\cdot, 1\} \cup \mathcal{LS}$, where " $\cdot$ " as 2-placed function symbol and "1" as a constant symbol. Then groups are exactly the models of the theory with the following sentences: (where we write $a \cdot b$ for $\cdot(ab)$

- $(\forall v_1)(\forall v_2)(\forall v_3) v_1 \cdot (v_2 \cdot v_3) \equiv (v_1 \cdot v_2) \cdot v_3$.
- $(\forall v_1) v_1 \cdot 1 \equiv v_1$
- $(\forall v_1)(\exists v_2) v_1 \cdot v_2 = 1$

Definition 12.6 *Let I be a set.* dfilter

(a) *Let $\mathcal{P}(I)$ is a power set of I, i.e the set of all subsets of I.*

(b) *A filter $\mathcal{D}$ on I is a subset of $\mathcal{P}(\mathcal{I})$ such that*

  (a) $I \in \mathcal{D}$.

  (b) *If $D, E \in \mathcal{D}$, then $D \cap E \in \mathcal{D}$.*

  (c) *If $D \in \mathcal{D}$ and $D \subseteq E \subseteq I$, then $E \in \mathcal{D}$.*

(c) *An ultra filter on I is a filter $\mathcal{D}$ on I so that for each $D \subseteq I$ exactly one of D and $I \subset D$ is in $\mathcal{D}$.*

Examples for filter on a set I are:
The *improper filter* $\mathcal{P}(I)$
The *trivial filter* $\{I\}$.
For $D \subseteq I$ the *principal filter* for D on I: $\{E \mid D \subset E \subseteq I\}$.
The *cofinite or Frechet filter* : $\{E \subset I \mid |I \setminus E| < \infty$.
A *proper filter* is any filter other than the improper filter

Its an easy exercise to show that on a finite set each filter is a principal filter. Also arbitrary intersections of filters are filters and unions of chains of filters are filters. In particular, if $\mathcal{E}$ is any subset of $\mathcal{P}(I)$ than there exists a smallest filter containing $\mathcal{E}$, namely the intersection of all the filters containing $\mathcal{E}$. This filter is called the filter generated by $\mathcal{E}$. We say that $\mathcal{E}$ has the finite intersection property if any intersection of finitely many members of $\mathcal{E}$ is not empty. The following proposition follows immediately from the definitions:

genfil **Lemma 12.7** *Let I be a set and $\mathcal{E} \subseteq \mathcal{P}(I)$. Then the filter generated by $\mathcal{E}$ is the filter*

$$\{D \mid \bigcap \mathcal{J} \subset D \subset I \text{ for some finite subset } \mathcal{J} \text{ of} \mathcal{E}.$$

In particular, the filter generated by $\mathcal{E}$ is proper if and only if $\mathcal{E}$ has the finite intersection property. □

ult=max **Lemma 12.8** *A filter is an ultra-filter if and only if its a maximal proper filter.*

Proof: Clearly a ultra-filter is a maximal proper filter. So let $\mathcal{D}$ be maximal proper filter and $E \subset I$ with $E \notin \mathcal{E}$. We need to show that $I \setminus E \in \mathcal{D}$. Since $\mathcal{E}$ is maximal, the filter generated by $\mathcal{D} + E$ is improper. By 12.7 this means that $\mathcal{D} + E$ does not have the finite intersection property. Since finite intersection of members of $\mathcal{D}$ are still in $\mathcal{D}$ we conclude that there exists $D \in \mathcal{D}$ with $D \cap E =$. Hence $D \subset I \setminus E$ and so $I \setminus E \in \mathcal{D}$. □

Corollary 12.9 *Every proper filter, and so also every subset of $\mathcal{D}$ with the finite intersection property, lies in an ultra filter.*

Proof: By Zorn's lemma every filter lies in a maximal filter. By 12.8 this maximal filter is an ultra filter. □

dultprod **Definition 12.10** *Let I be a set, $\mathcal{D}$ a filter on I and for $i \in I$ let A_i be a non-empty set. Let $A = \prod_{i \in I} A_i = \{\langle a_i | mid a_i \in A_i, i \in I \rangle\}$ be the cartesian product of the A_i's.*

The equivalence relation $\equiv_{\mathcal{D}}$ on A is defined by

$$\langle a_i \mid i \in I \rangle \equiv_{\mathcal{D}} \langle b_i \mid i \in I \rangle \text{ if and only if } \{i \in I \mid a_i = b_i\} \in \mathcal{D}$$

For $a \in A$ let $a_{\mathcal{D}}$ be the equivalence class of $\equiv_{\mathcal{D}}$ containing a. Let $\prod_{\mathcal{D}} A_i$ be the set of equivalence classes of $\equiv_{\mathcal{D}}$.

$\prod_{\mathcal{D}} A_i$ is called the reduced product *of A_i modulo $\mathcal{D}$ A reduced product modulo $\mathcal{D}$ is called an ultra-product of A_i modulo $\mathcal{D}$ if $\mathcal{D}$ is an ultra filter. If $A_i = B$ for all $i \in I$, then $\prod_{\mathcal{D}} B$ is called an ultra-power of B.*

Since over-sets of filter sets are filter sets $\langle a_i \mid i \in I \rangle \equiv \langle b_i \mid i \in I \rangle$ if and only if there exists $D \in \mathcal{D}$ with $a_i = b_i$ for all $i \in \mathcal{D}$. This observation will be usefully in any concrete calculation since in it might be rather difficult to determine the exact sets on which a_i and b_i agree

Definition 12.11 *Let I be a set, $\mathcal{FL}$ a first order language and $\mathcal{A}_i, i \in I$ a model for $\mathcal{FL}$. Let A_i be the universe of $\mathcal{A}_i$ and for a non-logical symbol s let s_i^* be the interpretation of s in $\mathcal{A}_i$. Then the* reduced product *$\prod_{\mathcal{D}} \mathcal{A}_I$ is the model for $\mathcal{FL}$ describes as follows.*

(a) *The universe A of $\prod_{\mathcal{D}} \mathcal{A}_i$ is $\prod_{\mathcal{D}} A_i$.*

(b) *If R is an m-placed relation symbol, when the interpretation R^* for R in $\prod_{\mathcal{D}} \mathcal{A}_I$ is defined by*

$$R^*(a^1, \dots a_m) \text{ if and only if } \{i \in I \mid R_i^*(a^1(i), \dots a^m(i)\} \in \mathcal{D}$$

where $a^k = \langle a^k(i) \mid i \in I \rangle_{\mathcal{D}}$

(b) *If F is an m-placed function, symbol then the interpretation F^* for F in $\prod_{\mathcal{D}} \mathcal{A}_i$ is defined by*

$$F^*(a^1, \dots, a^m) = \langle R_i^*(a_i^1, \dots, a_i^m) \mid i \in I \rangle_{\mathcal{D}}$$

where $a^k = \langle a_i^k) \mid i \in I \rangle_{\mathcal{D}}$.

(c) *If c is an constant symbol then the interpretation c^* for c in $\prod_{\mathcal{D}} \mathcal{A}_i$ is defined by*

$$c = \langle c*_i \rangle \mid i \in I \langle_{\mathcal{D}}$$

.

The reader should convince herself that this is well defined, i.e. does not depend on the particular choice of the representatives $\langle a^k(i) \mid i \in I\rangle$ for a^k.

fundult **Theorem 12.12** *Let $\mathcal{D}$ be an ultra filter on I and $\mathcal{A} = \prod_{\mathcal{D}} \mathcal{A}_i$ be an ultra product of models of the first order language $\mathcal{FL}$. Let $E_i : \mathcal{V} \leftarrow A_i$ be a functions and $E = \prod_{\mathcal{D}} E_i : \mathcal{V} \leftarrow A, v \leftarrow \langle E_i(v) \mid i \in I\rangle_{\mathcal{D}}$.*

(a) *Let t be a term of $\mathcal{FL}$, then*

$$t_E = \langle t_{E_i} \mid i \in I\rangle_{\mathcal{D}}$$

(b) *Let ϕ be a formula in $\mathcal{FL}$. Then*

$$\mathcal{A} \models \phi_E \text{ if and only } \{i \mid \mathcal{A}_i \models \phi \mid_{E_i}\} \in \mathcal{D}$$

(c) *Let Σ be a finite theory. Then $\mathcal{A}$is a model for Σ if and only if there exists $D \in \mathcal{D}$ so that for all $i \in \mathcal{D}$, $\mathcal{A}_i$ is a model for Σ.*

Proof: (a) This is merely an unwinding of the definition and the details are left to the reader.

(b) As usual this is proved using the inductive definition of a formula. We will carry out this analysis for formulas of the form $(\forall v)\phi$ and $\neq \phi$ an d leave the remaining easier cases to the reader. So suppose that (b) holds for the formula ϕ.

Let v be a variable. We wish to show that (b) also holds for $(\forall v)\phi$.

By definition $\mathcal{A} \models ((\forall v)\phi) \mid_E$
if and only if for all $\hat{E} : \mathcal{V} \leftarrow \mathcal{A}$ such that $\hat{E}$ agrees with E on $V - v$ one has

$$\mathcal{A} \models \phi_{\hat{E}}.$$

As (b) holds for ϕ this is equivalent to

$$(*) \qquad \{i \mid \mathcal{A}_i \models \phi \mid_{\hat{E}_i}\} \in \mathcal{D}$$

for all such $\hat{E}$ and where we may choose the $\hat{E}_i$ to agree with E_i on $V - v$.

Let $D = \{i \mid \mathcal{A}_i \models (\forall v\phi) \mid_{E_i}\} \in \mathcal{D}$. Then for any $\hat{E}$, D is a subset of the set in (*). So if $D \in \mathcal{D}$, (*) holds.

Conversely suppose that $D \notin \mathcal{D}$. Then by definition of $\models (\forall v)\phi \mid_{E_i}$, for each $i \in I \setminus D$ there exists $\hat{E}_i$ so that $\hat{E}_i$ agrees with E_i on $V - v$ and we do not have $\mathcal{A}_i \models \phi \mid_{\hat{E}_i}$. Put $\hat{E}_d == E_d$ for $d \in D$ and put $\hat{E} = \prod_{\mathcal{D}} \hat{E}_i$. For this choice of $\hat{E}$ the set in (*) is a subset of D and so as $D \notin \mathcal{D}$ (*) does not hold. This completes the proof of (b) for formulas of the form $(\forall v)\phi$.

Next we consider the formula $\neq \phi$.

By definition $\mathcal{A} \models\neq \phi \mid_E$ if and only if not $\mathcal{A} \models \phi_E$. As (b) holds for ϕ, this is equivalent to

$$\{i \mid \mathcal{A}_i \models \phi) \mid_{E_i}\} \notin \mathcal{D}$$

As $\mathcal{D}$ is an ultra-filter (this is the only place there the ultra filter assumption is used) this holds if and only if

$$\{i \mid \text{ not } \mathcal{A}_i \models \phi) \mid_{E_i}\} \in \mathcal{D}$$

By definition of $\models \neg\phi \mid_E$ this is the same as

$$\{i \mid \mathcal{A}_i \models (\neg\phi) \mid_{E_i}\} \in \mathcal{D}$$

(c) By (b), (c) holds a single sentence, As a $\mathcal{D}$ is closed under finite intersection (c) holds for arbitrary finite theory (or more general for any theory with a finite set of axioms). □
bigskip

As an immediate consequence of 12.12 we obtain that an ultra products of groups is again a group and an ultra-product of fields is a field. Also if n is a given positive integer and each A_i is a set of size at most n, then any ultra-product of A_i is a set of size n. Indeed, a set has size at most n if and only if it satisfies the sentence

$$(\exists v_1 \ldots v_n)(\forall v_{n+1})(v_1 \equiv v_{n+1} \vee \ldots \vee v_n \equiv v_{n+1})$$

To be able to talk about groups,fields and vector spaces at the same time we add an additional set Ξ to a first order language ,where each element of Ξ stands for one of the structures. The universe of now is of the form $(A_\xi | \xi \in \Xi)$ where each A_ξ is a set. The function symbols are now $(m_\xi | \mid \xi \in \Xi) - \xi_0)$-placed (with $\sum_{\xi \in Xi} m_\xi)$ finite) where such a symbol is interpreted by a function $\prod_{\xi\in\Xi} A_\xi^{m_\xi} \to A_{\xi_0}$. The relation symbols are treated similarly. The variables are of the form $v_n^\xi, n \in \mathbb{N}, \xi \in Xi$. To evaluate terms v_n^ξ is only allowed to be replaced to element of A_ξ, that is the only consider functions $E : \bigcup \mathcal{V} \bigcup A_\xi$ with $E(v_k^\xi) \in A_\xi$. All the assertions about first order language we made so far are still true in this more general set-up. Indeed, it is not difficult to see that this set can be expressed in terms of the old set up.

If V_i is vector space over the field K_i, we now obtain $\prod_\mathcal{D} A_i$ is a vector space over the field $\prod_\mathcal{D} K_i$. Now if each A_i has dimension at most n, then $\prod_\mathcal{D} A_i$ has dimension at most n over $\prod_\mathcal{D} K_i$. Indeed a vector space over a field has dimension at most n if the following sentence is fulfilled:

$$(\exists v_1 \ldots v_n)(\forall v_{n+1})(\exists k_1 \ldots k_n)v_{n+1} \equiv k_1 \cdot v_1 + k_2 \cdot v_2 + \ldots + k_n v_n)$$

where the v_i's are vector space variables and the k_i's a field variables.

If $g \in GL_K(V)$ define $\deg(g) = \deg_V(g) = \dim[V, g] = \dim V/C_V(g)$. Furthermore $\mathrm{pdeg}_V(g) = \inf\{\deg k \cdot g \mid o \neq k \in K\}$. The statements $\deg_V(x) \leq n$ and $\mathrm{pdeg}_V(g) \leq n$ can be expressed in the first order language of groups acting on vector-space . For example $\mathrm{pdeg}_(g) \leq n$ reads as

$(\exists v_1 \ldots v_n)(\exists k_{n+1})(\neg(k_{n+1} \equiv 0) \wedge ((\forall v_{n+1})(\exists k_1 \ldots k_n) k_{n+1} v_{n+1}^g \equiv k_1 \cdot v_1 + k_2 \cdot v_2 + \ldots + k_n v_n + v_{n+1})$

We also have a natural embedding $\prod_{\mathcal{D}} GL_{K_i}(V_i) \to GL_{\prod_{\mathcal{D}} K_i}(\prod_{\mathcal{D}} V_i$. This induces an embedding $\prod_{\mathcal{D}} PGL_{K_i}(V_i) \to PGL_{\prod_{\mathcal{D}} K_i}(\prod_{\mathcal{D}} V_i$. Indeed, $\langle g_i \rangle_{\mathcal{D}}$ acts as a scalar $\langle k_i \rangle \mathcal{D}$ on $\prod_{\mathcal{D}} V_i$ if and only if there exists $D \in \mathcal{D}$ with $k_i = g_i$ for all $i \in D$. But this is the same as saying that $\langle g_i \rangle_{\mathcal{D}}$ is in the kernel of the map from $\prod_{\mathcal{D}} GL_{K_i}(V_i)$ to $\prod_{\mathcal{D}} PGL_{K_i}(V_i)$. If $\operatorname{pdeg} V_i(g_i) \leq n$ for all $i \in D \in \mathcal{D}$, then we get $\operatorname{pdeg}_{\prod V_i}(\langle g_i \rangle_{\mathcal{D}}) \leq n$.

Let X be a group, $(L_i \mid i \in I)$ a local system for X and for each $i \in I$, let $\phi_i : L_i \to H_i$ be a group homomorphism. For $i \in I \leq I^{\geq I} = \{j \in I \mid j \geq i\}$. Since I is directed, $\{I^{\geq i} \mid i \in I\}$ as the finite intersection property. Hence there exists an ultra-filter $\mathcal{D}$ on I so that $I^{\geq i} \in \mathcal{D}$ for all $i \in I$.

Let $H = \prod_{\mathcal{D}} H_i$. Extend ϕ_i to a map (but not a homomorphism) $X \to H_i$ by $\phi_i(x) = 1$ if $x \in X \setminus L_i$. Define

$\phi : X \to H, x \to \langle \phi_i(x) \langle_{\mathcal{D}}$.

Let $x, y \in X$ and pick i with $\langle x, y \rangle \leq L_i$. Then $\phi_j(xy) = \phi_j(x)\phi_j(y)$ for all $j \in I^{\geq i}$. Since $I^{\geq i} \in \mathcal{D}$ this implies $\phi(xy) = \phi(x)\phi(y)$. Hence ϕ is a homomorphism. In general it seems to a non-trivial task to determine the kernel of ϕ and in general the kernel depends on the choice of the ultra filter. But there is an easy sufficient condition which guarantees $\phi(x) \neq 1$. Namely if there exists $i \in I$ so that $\phi_j(x) \neq 1$ for all $j \geq i$ then $\phi(x) \neq 1$. In particular if all ϕ_i are one to one, then

For locally finite X (or more generally for arbitrary groups if instead of finite groups, finitely generated groups are considered) we give a similar conditions which ensures that ϕ is one to one:

Suppose that $\{(L_i, M_i) | i \in I\}$ is a sectional cover and that $\ker \phi_i = M_i$. For $i \in I$ let $I^{\leftarrow i} = \{j \in I \mid L_i \leq L_j, L_i \cap M_j = 1\}$. The definition of a sectional cover implies that $\{L^{\leftarrow i} \mid i \in I$ as the finite intersection property. Hence we can choose an ultra-filter $\mathcal{D}$ with $L^{\leftarrow i} \in \mathcal{D}$ for all $i \in I$. Then clearly ϕ is one to one.

malcev **Theorem 12.13** *Let X be a group with a local system of linear groups of degree n. Then X is linear of degree n.*

Proof: Let $\mathcal{L}_\rangle \mid \rangle \in \mathcal{I}$ be the local system such that there exist field K_i and embeddings $\phi_i \to GL_n(K_i)$. By the preceding discussion X can be embedded into $\prod_{\mathcal{D}} \to GL_n(K_i)$ which in turn is embedded into $GL_n(\prod_{\mathcal{D}} K_i)$. □

With the same sort of argument we obtain:

joncev **Theorem 12.14** *Let X be a group and $\{(L_i, M_i) \mid i \in I\}$ be a sectional cover for G. For $i \in I$ let V_i be a vector space over the field K_i and $\phi_i : L_i/M_i \to PGL_{K_i}(V_i)$ an embedding. Furthermore let $Y \subset X$ so that for all $y \in Y$ there exists an integer d_y and a finitely generated subgroup F_y so that for all $i \in I$ with $F_y \leq L_i$ and $F_y \cap M_i = 1$ one has* $\operatorname{pdeg}_{V_i}(\phi_i(y)) \leq d_y$.

Then there exists a vector space V over the field K and an embedding $\phi : X \to PGL_K(V)$ so that $\operatorname{pdeg}_V(\phi(y)) \leq d_y$ *for all $y \in Y$.* □

Chapter 13

Periodic, simple, linear groups

slinear

In this chapter we present and partly prove the classification of the periodic, simple, linear groups.

Definition 13.1 *A* projective representation *of the group X is homomorphism of X into $PGL_K(V)$, where V is a vector space over the field K. $\dim_K V$ is called the* degree *of the projective representation.* dpr

For example $PSL_n(K)$ has projective representation of degree n. This indicates that a projective representation can be a more natural object to look at as a linear representation. Also we remark that any group with a faithful projective representation of finite degree is linear (but maybe of larger degree). Indeed if V is a projective representation for X, then $X \otimes X^*$ provides a linear representation, where X^* is the dual of X.

Definition 13.2 *Let H be a finite group. Then $\mathrm{mdeg}(H)$ is the minimal positive integer d so that there exists a field K and finite groups $B \trianglelefteq T \leq GL_d(K)$ such that $T \cap K \leq B$ and $M \cong H$.* md

Lemma 13.3 *Let q be a prime, M a finite q-group with $1 \neq M' \leq \Phi(M) \leq Z(M)$. Let $|M/Z(M)| = q^n$.* irrext

(a) *If $Z(M)$ is cyclic, then M has an abelian subgroup of rank at least $\frac{n}{2}$.*

(b) *Let K be a field and V faithful irreducible KM-module. Then V has dimension at least $q^{\frac{n}{2}}$.*

Proof: (a) Let $x \in M$. Then the commutator map induces defines an isomorphism from $M/C_M(x)$ into $[M,x] \leq Z(M)$. Since $M/C_M(x)$ is elementary abelian and $Z(M)$ is cyclic this implies $|M/C_M(x) \leq q$. An easy induction argument shows that $|M/C_M(A)| \leq |A/A \cap Z(M)|$ for all $A \leq M$. Since $C_M(A) \leq A$ for any maximal abelian subgroup of M, (a) holds.

(b) Since M is irreducible, $Z(M)$ is cyclic Also if $1 \neq z \in Z(M)$, then $C_V(z)$ is an M submodule and so $C_V(z) = 0$. Without loss K is algebraically closed.

Let A be a maximal abelian subgroups of M. Then $Z(M) \leq A$ and A is normal in M. Let W be a Wedderburn component for A on V and let $B = N_M(W)$. Then $[A, B] \leq M' \leq Z(M)$. As A is abelian and K is algebraically closed, A has an eigenvector on W and so acts as scalars on W. Thus $[A, B]$ centralizes W. Put $C_{Z(M)}(W) = 1$ and so $[A, B] = 1$. The maximality of A implies $B \leq A$. Hence $N_M(W) = A$ and $\dim V = \dim W \cdot M/A|$. It remains to show that $|M/A| \geq |A/Z(M)|$. For this pick $B \leq M$ with $M = AB$ and $A \cap B = Z(M)$. Then as seen above $|M/C_M(B)| \leq |B/Z(M)| = |M/A|$. Thus also $|A/C_A(B)| \leq |M/A|$. Put $C_A(B) \leq C_M(AB) = Z(M)$ and so $|A/Z(M)| \leq M$. □

minsim **Lemma 13.4** *Let H be a non-cyclic, finite, simple group, $d = \operatorname{mdeg}(H)$ and $B \trianglelefteq T \leq GL_d(K)$ finite groups with $H \cap K \leq B$ and $T/B \cong H$. Then T acts primitively on K^d and $B \leq K$.*

Proof: Suppose that T normalizes a proper subspace W in $V = K^d$. By minimality of d neither $T/C_T(W)$ not $T/C_T(V/W)$ have composition factor isomorphic to H. But neither does the abelian group $C_T(W) \cap C_T(V/W)$.

Hence T is irreducible on V. Suppose next that Δ is a system of imprimitivity for T on V. Since $C_T(\Delta)$ is not irreducible it cannot have composition factor isomorphic to H. Thus T^Δ does. But $T^\delta \leq Sym(\Delta) \leq Sym(n)$ and $Sym(n)$ is isomorphic to a subgroup of $GL_{d-1}(K)$ via the even permutation module, a contradiction.

So T is indeed primitive on V.

Suppose now that $B \not K$ and pick a counter example with $|T|$ minimal. Without loss K is algebraically closed. Put $Z = T \cap K$ and let q be a prime dividing the order of B/Z. Let Q be a Sylow q-subgroup of B. By the Frattini argument $T = BN_T(Q)$ and so $N_T(Q)/N_M(Q) \cong H$. Thus by minimality of $|B|$, Q is normal in T. Let M be a subgroup of Q minimal with respect to $M \not\leq Z$ and $M \trianglelefteq T$. As T is primitive Clifford's theorem implies that V is the direct sum of isomorphic irreducible M-modules. If M would be abelian then as K is algebraically closed we get $M \leq Z$, a contradiction.

Thus M is not abelian. By minimality of M. $\Phi(M) \leq Z$ and $Z(M) \leq Z$. Put $\bar{M} = M/M \cap Z$.By minimality of T and as $M \leq C_T(\bar{M})$ the latter group has no composition factor isomorphic to H. Hence $T/C_T(\bar{M})$ has a composition factor isomorphic to H. Let $\bar{M} = q^m$. Note that $\bar{M}$ is a vector space over $GF(q)$ and so by minimality of d, $d \leq m$. On the other hand by 13.3, $d \geq q^{fracm2}$. So $\overset{q^m}{\leq} m^2$. As $m > 1$, $q = 2$ and $m \leq 4$. But then $(\operatorname{Aut}(M)/Inn(M))\infty \leq Sp_4(2)^\infty \cong PSL_2(\bar{9})$. Hence $d \leq 2$, $m = 2$ and $Aut(M)$ is solvable, a contradiction.

boundq **Lemma 13.5** *Let A be a elementary abelian r subgroup of $X = GL_n(K)$ with $r \neq operatornamecharK$. Then A has rank at most n and $|N_X(A)/C_X(A)| \leq n!$.*

Proof: Without loss K is algebraically closed. Let $\mathcal{E}$ be the set of eigenspaces for A on K^n. Then $|\mathcal{E}| \leq n$. Let $E \in \mathcal{E}$. Then $A/C_A(E)$ is cyclic and so A has rank at most

$|\mathcal{E}|$. Since A acts as scalars on E we get that $[N_X(E), A]$ centralizes E and so $[C_X(\mathcal{E}), A]$ centralizes $K^{=}\bigoplus \mathcal{E}$. Hence $N_X(A)/C_X(A)$ is is isomorphic to a section of $\mathrm{Sym}(\mathcal{E})$ and so has order at most $|\mathcal{E}|!$. □

Proposition 13.6 *Let G be a infinite, periodic,linear, simple group and d the minimal degree of a projective representation for G.* lsls

(a) *Let $\mathcal{K}$ be any Kegel cover for G. Let $\mathcal{J}$ consists of whose $K \in \mathcal{K}$ with $\bar{K}$ perfect and* $\mathrm{mdeg}(\bar{K}) = d$*. Then $\mathcal{J}$ is Kegel cover for G, $\mathcal{K} \setminus \mathcal{J}$ is not a Kegel cover and H_J is simple for all $J \in \mathcal{J}$.*

(b) *G is the union of an ascending chain $G_1 \leq G_2 \leq G_3 \ldots$ of finite simple subgroups with* $\mathrm{mdeg}(G_i) = d$ *for all i.*

Proof: (a) Without loss $\mathcal{K}$ has non cyclic factors.Suppose that $\mathcal{K} \setminus \mathcal{J}$ is a Kegel-cover for G. Then by 13.4 all the factors of this Kegel cover have a projective representation of degree less than d. By Malcev's theorem 12.13 we conclude that also G has a projective representation of degree less then d. A contradiction.

So $\mathcal{K} \setminus \mathcal{J}$ is not a Kegel cover and thus $\mathcal{J}$ is a Kegel cover. Let $J \in \mathcal{J}$. Since H_J is has a projective representation of degree d and $d = \mathrm{mdeg}\,\bar{J}$. Hence H_J is simple by 13.4

(b) By exercise 16. G is countable. Thus (b) follows from (a). □

In order to proceed we proudly present:

The classification of finite simple groups

Any finite simple groups is isomorphic to one of the following groups where p is a prime, $q = p^k$ and n, k are positive integers.

- C_p
- $Alt(n)$, $n \geq 5$
- $A_n(q) = PSL_{n+1}(q)$, $(n, q) \neq (1, 2), (1, 3)$.
- ${}^2A_n(q^2) = PSU_{n+1}(q^2)$, $n \leq 3$, $(n, q) \neq (3, 2)$
- $B_n(q) = P\Omega_{2n+1}(q)$, $n \geq 3$, p odd.
- ${}^2B_2(q) = Sz(q)$, $p = 2$, $k > 1$ odd.
- $C_n(q) = PSp_{2n}(q)$, $n \geq 2$, $(n, q) \neq (4, 2)$.
- $D_n(q) = P\Omega^+_{2n}(q)$, $n \geq 4$
- ${}^2D_n(q^2) = P\Omega^-_{2n}(q)$, $n \geq 4$.
- ${}^3D_4(q^3)$

- $E_6(q), E_7(q), E_8(q)$
- ${}^2E_6(q^2)$
- $F_4(q)$
- ${}^2F_4(q)$, $p = 2$, $k > 1$ odd, ${}^2F_4(q)'$
- $G_2(q)$, $q \neq 2$.
- ${}^2G_2(q)$, $p = 3, k > 1$.
- $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_2, J_3, J_4, HS, McL, Suz, Ly, He, Ru,$
  $O'N, Co_3, C0_2, C0_1, Fi_{22}, Fi_{23}, Fi'_{24}, F_3, F_5, F_3, F_1$
- $DDELNPPRST$

All of these groups are simple. And they are pairwise non-isomorphic except for $Alt(5) \cong PSL_2(4) \cong PSL_2(5)$, $PSL_2(9) \cong Alt(6)$, $PSL_3(2) \cong PSL_2(9)$ and $PSp_4(3) \cong PSU_4(2^2)$.

lslt **Proposition 13.7** *Let G be a infinite, periodic,simple linear group.*

(a) *There exists a prime p, an positive integer n, a Lie-type X and an ascending chain $G_1 < G_2 < G_3 < \ldots$ of finite simple subgroups of G such that $G = \bigcap_{i=1}^{\infty} G_i$ and $G_i \cong X_n(p^{k_n})$. Moreover, any linear representation of G has to be over a field in characteristic p.*

(b) *If G is a linear representation over the field K, then $operatornamecharK = p$.*

Proof: Let $G_1 \leq G_2 \leq ldots$ be as 13.6. Using 3.3 we will select a subsequence which fulfills (a). (b) will be proved along the way. First of all we may assume that none. of the G_i's is one of the sporadic groups. Also suppose that K is a field so that G has a faithful projective representation of degree d over K. Let $p = operatornamecharK$.

Let r be any prime with $r \neq p$ and let L be any simple section of one of the G_i's. By 13.4 L has faithful projective representation of degree d. L is linear of degree d^2 and so L has no elementary abelian subgroups of rank larger than d^2. In particular there is bound on the size of the alternating sections. Hence only finitely many of the G_i's can be alternating groups. So we may assume that $G_i \cong X(i)_{n_i}(q_i)$, $q_i = p_i^{k_i}$.

We will use the following facts about groups of Lie type without proof:

1. $X_n(q)$ has a section isomorphic to $Alt(n)$
2. $X_n(q)$ has an elementary abelian subgroup A with $|A| = q$ and $|N(A)/C(A)| \geq \frac{q-1}{2}$

¿From 1. we conclude that there are only finitely many distinct n_i's. Also there are only finitely many $X(i)$'s and so by 3.3 we may assume that for all i and some X and n, $X(i) = X$ and $n_i = n$. By 2. and 13.5 there is a bound on the q_i's with $p_i \neq operatornamechar K$. Thus, $p = operatornamechar K$ is a prime and $p_i = p$ for all but finitely many i. □

bglv **Lemma 13.8** *Let K be a field of characteristic $p \geq 0$, V a finite dimensional vector space over K, $\mathcal{B}$ a basis for V, $G = GL_V(K)$ and $S = SL_H(V)$. Let W be a K-subspace of V and put $Q_W = C_G(W) \cap C_G(V/W)$*

(a) *$Q_W \leq SL_K(W)$ and $Q_W \cap K = 1$.*

(b) *$N_G(W)/Q_W \cong GL_K(W) \times GL_K(W)$,*
$N_S(W)/W \cong \{(a, b) | a \in GL_K(W), b \in GL_K(V/W), \det(a)\det(b) = 1\}$.

(c) *Q_W is an elementary abelian p-group (where an elementary abelian 0-group is defines to be a torsion free abelian group). As $\mathbb{Z}N_G(W)/Q_W$-module Q_W is isomorphic to $\mathrm{Hom}_K(V/W, W)$.*

(d) *If W is a proper subspace of V, then $C_G(Q_W) = Q_W K^{\#}$ and $C_{PSL_K(V)}(Q_W) = Q_W K^{\#}/K^{\#}$*

(e) *$SL_K(V) = \langle Q_{Kb} \mid b \in \mathcal{B} \rangle$.*

(f) *Q_B acts transitively on the points of $V \setminus B$.*

(g) *Let $V = A \oplus B$ with $\dim A = 1$. Then $SL_K(V) = \langle Q_A, Q_B \rangle$.*

(h) *Suppose that $\dim_K V \geq 3$ or $|K| \geq 4$. Then every proper normal subgroup of $SL_K(V)$ is contained in K and $PSL_K(V)$ is simple.*

Proof: (a) Obvious.

(b) There is an obvious homomorphism from $\phi : N_G(W) \to GL_K(W) \times GL_K((V/W)$ and Q_W is the kernel of this homomorphism. Also if $V = W \oplus U$,then $GL_K(W) \times GL_K(U) \leq GL_K(V)$, and ϕ restricted to this subgroup is an isomorphism. Hence ϕ is onto. This proves the first part of (b). The second is obvious.

(c) For $q \in Q_W$ define $q^* \in \mathrm{Hom}(V/W, W)$ by $q * (v + W) = [v, q]$. As q centralizes V/W, $[v, q]$ indeed lies in W and since q centralizes W this well defined. Conversely if $\alpha \in \mathrm{Hom}(V/W, W)$ define $\alpha^* \in Q_W$ be $\alpha^*(v) = v + \alpha(v + W)$.

(d) Let $g \in C_G(Q_W)$. Let H be a hyper-plane (that is a subspace of codimension 1) of V/W and U a 1-space in W. Pick $\alpha \in \mathrm{Hom}(V/W, W)$ with $\ker \alpha = H$ and $\Im \alpha = U$. Then $\ker \alpha^g == H^g$ and $\Im \alpha^g = U^g$. Since $\alpha^* = \alpha^g$ we also have $\alpha = \alpha^g$ and so $H^g = H$ and $U^g = U$. Since this is true for all such H and U it is easy to see that g acts as an scalar on W and on U. $\alpha g = g\alpha$ implies that these scalars are the same. Thus the first part of (d) holds. The second follows from the first and (a).

(e) Let $n = \dim KV$. If $n = 1$, there is nothing to prove. So suppose $n \geq 2$. Let $H = \langle Q_{Kb} \mid b \in \mathcal{B}$. Let $a \in \mathcal{B}$, $\mathcal{B}_0 = \mathcal{B} - a$, $H_0 = \langle Q_{Ka} \cap C_G(a) \mid a \in \mathcal{B}_0 \rangle$ and $V_0 = K\langle \mathcal{B}_0 \rangle$. Let $b \in \mathcal{B}$. By induction on n, $H_0 = SL_K(V_0)$.

Step 1 If $v + Ka = w + Ka$, then $v^q = w$ for some $q \in Q_{Ka}$.

Pick $\alpha \in \mathrm{Hom}_{V/Kb,Kb}$ with $\alpha(v + Ka) = w - v$. Then $v^{\alpha^*} = w$.

Step 2 H acts transitively on $V^\#$.

By step 1 every 1-space in $V \setminus K_a$ is conjugate under Q_{Ka} to am element of V_O. Since H_0 acts transitively one the 1-spaces of V_0, we conclude that $Q_{Ka}H_0$ acts transitively on the 1-spaces different from 1. Moreover, Q_{Kb} does not fix Ka and so H is transitive one the 1-spaces of V. It remains to show that for all $\lambda \in K^\#$, a and $\lambda \cdot a$ are conjugate under H. By step 1 (sometimes applies to b in place of a), a and $a + b$ are conjugate under Q_{Kb}, $a + b$ and $\lambda \cdot a + b$ are conjugate under Q_{Ka}, and $\lambda \cdot a + b$ and $\lambda \cdot a$ are conjugate under Q_{Kb}.

Step 3 Q_{Ka} acts transitively on the complements to Ka in V.

Let W be any complement to Ka in V. Let $\beta(v_0 + Ka)$ be the projection of v_0 onto Ka via $V = Ka \oplus W$. Then $\beta \in \mathrm{Hom}_{V/Kb,Kb}$ and $v_O + \beta(v_0) \in W$ for all $v_0 \in V_0$. Thus $V_0^{\beta^*} = W$ and Step 3 is established.

Let $s \in SL_V(K)$. By Step 2 and Step 3 there exists $h \in H$ so that sh centralizes a and normalizes V_0. Thus $sh \in SL_K(V_0) \leq H_0 \leq H$ and $s \in H$.

(f) This is just the dual statement to Step 3 in (e).

(g) By (f) $V = \langle A^{Q_B} \rangle$ and so A^{Q_B} contains a basis. Thus by (e) $\langle Q_A^{Q_B} \rangle = SL_K(V)$.

(g) Let N be a normal subgroup of S. Let A be a 1-space and $M = N_S(A)$. Clearly $SL_K(V)$ acts double transitive on the 1-spaces and so M is a maximal subgroup of G. If $N \leq M$, then as N is normal in S and S is transitive, S normalizes all the 1-spaces in V. Thus $N \leq K$. So suppose $N \not\leq M$. Since M normalizes Q_A we conclude that $MQ_A \trianglelefteq S$. Thus $S = NQ_A$ and $[Q_A, M] \leq N$. Suppose that $[Q_A, M] \neq Q_A$. Since $C_G(A)$ acts irreducibly on $Q_A^\#$ we conclude that $[Q_A, M] = 1$. Thus by (d) $M \leq Q_AK$. This implies $\dim V = 2$ and $|K| \leq 3$, a contradiction to the assumptions. Hence $Q_A = [Q_A, M] \leq N$ and $S = N$. □

mdegsl **Lemma 13.9** *Let K be a field in positive characteristic p. Let $n > 1$ an integer. Let $\phi : PSL_n(K) \leftarrow PGL_F(W)$ be a embedding with $operatorname{char}K = operatorname{char}F$ and $\dim_F W \leq n$. Then $\dim FW = n$. Moreover if A is point or a hyper-plane in V, then $\phi(Q_A) \leq Q_B$ for a unique point or hyper-plane B in W.*

Proof: By induction on n. Let $p = operatorname{char}F$. Since Q_A is a p-group, Q_A centralizes proper subspace B in W. If $n = 2$, B has to be a point and we are done. So suppose $n > 2$. Note that $N_{SL_n(K)}(A)/Q$ has a subgroup isomorphic to $SL_{n-1}(K)$. Induction and the simplicity of $PSL_{n-1}(K)$ (respectively the structure of $PSL_2(2)$ and $PSL_2(3)$ implies either B or W/B has dimension at least $n - 1$. Thus B is a point or a hyper-plane and $Q_A \leq Q_B$. □

pslslt **Theorem 13.10** *Let G and $G_1 \leq G_2 \leq \ldots$ be as in 13.7(a) with $X = A$. Let F be the subfield of $\bar{\mathbb{F}}_p$ generated by the $\mathbb{F}_{p^{k_i}}$. Then $G \cong A_n(F)$.*

Proof: Let $d = n = 1$. Let $F_i = \mathbb{F}_{p^{k_i}}$ and $V_i = F_i^d$. Let V_∞ and F_∞ be a suitable ultra product of the V_i and F_i, respectively. Then V_∞ is a projective G-module of degree d over F_∞. Put $G_\infty = G$. Let $Z_i = \{k \in F_i \mid k^d = 1\}$. Then for $i < \infty$, $G_i \cong PSL_{K_i}(V_i) \cong SL_{K_i}(V_i)/Z_i$. $|Z_i| \leq d$ and so passing to a subsequence we may assume $|Z_i| = z$ for all $1 \leq i < \infty$ and some z. Then also $|Z_\infty| = z$.

Let $V_1 = A_1 \oplus B_1$ for a point A_1 and a hyper-plane B_1. Then for $1 \leq i \leq \infty$, $Q_{A_1} \leq Q_{A_i}$ for some point or hyper-plane A_i in V_i. Replacing V_i by its dual if necessary we may assume that A_i is a point. Also let B_i be the point or hyper-plane of V_i with $Q_{B_1} \leq Q_{B_i}$. Then $G_i = \langle Q_{A_1}, Q_{B_1} \rangle$ normalizes $A_i + B_i$ and so $V_i = A_i \oplus B_i$. Note also that if $i \leq j \leq \infty$, then $Q_{A_i} \leq Q_{A_j}$. Fix $1 \leq i < \infty$. Let $M_i = N_{G_i}(A_i)$ and M_i^* the inverse image of $M_i \in SL_{F_\infty}(V_\infty)$. Let K_i be the subfield of F_∞ generated by the scalars induced by M_i^* on A_∞. The p-elements in M_i^* centralize A_∞. Also $M_i/O^{p'}(M_i) \cong F_i^{\#}/Z_i$, $M_i^*/Z_\infty \cong M_i$ and $Z_i \cong Z_\infty$. Thus K_i is isomorphic to a subfield of F_i. Let $O \neq a \in A_\infty$. Let W_i be the K_iG_i submodule of V_∞ spanned by a. Also let $t_1, t_2, \ldots t_d \in G_1$ so that $V_1 = \oplus_{i=1^d} A_1^{t_i}$ and let $a_j = a^{t_j}$

Claim For all $i < \infty$, $W_i = K_i a \oplus (W_i \cap B_\infty)$

Let $g \in G_i$. If $A_i^g B_i$, then $Q_{B_i} \leq M_i^g$ and so $A_i^g \leq C_{W_i}(Q_{A_i}) \leq B_\infty$. If $A_i^g \not\leq B_i$, then by 13.8f, $A_i^g = A_i^q$ for some $q \in Q_{B_i}$. Hence $g = mq$ for some $m \in M_i^*$. By definition of K_i, m normalizes $K_i a$ and so $K_i a^g = K_i a^q \leq K_i a[K_i, q] \leq K_i a(B_\infty \cap W_i)$.

Thus the claim is proved. It follows that $W_i \cap A_\infty = K_i a$ and so $[W_i, Q_{A_i}] \leq K_i a$. Hence $G_i = \langle Q_{A_i}^{t_j} \mid 1 \leq j \leq d$ normalizes $\oplus_{j=1}^d K_i a_j$. Hence $a_1, a_2, \ldots a_d$ is a K_i-basis for W_i. W_i is a faithful projective G_i module and so $|PSL_d(K_i)| \geq |PSL_d(K_i)$. Since K_i is a isomorphic to a subfield of F_i this implies that K_i and F_i are isomorphic. Moreover G_i maps isomorphically onto $PSL_{K_i}(W_i)$.

As $G = \bigcup G_i$ we have $W_\infty = \bigcup i = 1^\infty W_i$, $K_\infty = \bigcup K_i \cong F$ and $G = \bigcap PSL_{K_i}(W_i) = PSL_{K_\infty}(W_\infty) \cong PSL_d(F)$. □

The next theorem (or in some cases parts of it) was proved independently by Belyeav [?], Borovik [?], Hartley and Shute [?] and Thomas [?]. We refer the reader to whose papers for a proof.

Theorem 13.11 *Let G be an infinite, periodic,linear, simple groups. Then G is a groups of Lie type over an infinite,locally finite field. More precisely if $G_1 \leq G_2 \leq \ldots$ are as 13.7(a) and F is the subfield of $\bar{\mathbb{F}}_p$ generated by the $\mathbb{F}_{p^{k_i}}$. Then $G \cong X_n(F)$.* □ BBHST

Chapter 14

The maximal subgroups of $SL_2(K)$

masusl

In this section p is a prime and K is locally finite field in characterictic p, that is K is a subfield of the algebraic closure of $\mathbb{F}_p$. Let V be a 2-dimensional vector space over K , $L = \mathrm{SL}_K(V)$, $Z = Z(L) = \{\pm 1\}$ and $\bar{L} = L/Z(L) = \mathrm{PSL}_K(V)$. The goal of this section is to determine the maximal subgroup of L.

Lemma 14.1 *Let* $m = \sup\{i \mid \mathbb{F}_{p^{2^i}} \leq K\}$. ffson

(a) *F has a field extension of degree 2 if and only if* $m < \infty$.

(b) *Let F be a field extension fo degree* 2 *of K. Then*

  (a) $F : K$ *is Galois.*

  (b) *F is unique up to K-isomorphism. Namely F is the splitting field if* $x^{p^{2^{m+1}}} - 1$ *over K*

  (c) *Let* σ *the non-trivial K-automorphism of F. Then* $K = \{ff^\sigma \mid f \in F\}$.

Proof: Let E the algebraic closure of K. We start with describing the subfields of E. Let q be a power of p. Then $(ab)^q = a^q b^q$ and $(a+b)^q = a^q + b^q$ for all $a, b \in E$. Hence the q roots of the polynomial $x^q - x$ for a subfield $\mathbb{F}_q$ of E. Conversely, if T is a subfield of order q in E, then as $(T^\#, \cdot)$ is a groups of order $q-1$, $t^{q-1} = 1$ for all $o \neq t \in T$ and so $t^q - t = 0$ for $t \in T$. Thus $\mathbb{F}_q$ is a unique subfield of order q in E. Let T be any subfield and for a prime r let $m_r(T) = \sup\{r^i \mid \mathbb{F}_{p^{r^i}} \leq T\}$. Let N_K be the formal product $\prod_r m_T(r)$,there the product is taken of all the primes. For two such formal we can build the product, the greates comman divisor and leat commom multiple in an obviuos way. Given a formal product $N = \prod m_r$ let $\mathbb{F}_{p^N} = \bigcup_{n \in \mathbb{N}} \{\mathbb{F}_{p^n} \mid n \mathrm{divides} N\}$. Then $\mathbb{F}_{p^N}$ is a subfield of E with $N_{\mathbb{F}_{p^N}} = N$ and $T = \mathbb{F}_{p^{N_T}}$. Hence we obtain a one to one correspondence between subfield of E and the formal products. Given two formal products N, M. Then $\mathbb{F}_{p^N} \cap \mathbb{F}_{p^M} = \mathbb{F}_{p^{\gcd(N,M)}}$, The subfield of E generated by $\mathbb{F}_{p^N}$ and $\mathbb{F}_{p^M}$ is $\mathbb{F}_{p^{\mathrm{lcm}(N,M)}}$. Also $\mathbb{F}_{p^N} \leq \mathbb{F}_{p^M}$ if and only if N divides M, in which case the degree of this extension is M/N.

By the preceeding dicussion it remains to prove (bc) Since F is the union of its finite subfields and since all finite subfields are invariant under σ we may assume that F is finite. Let $q = |K|$. Then $x \to x^q$ is non trivial K-autmorphism of F and so $\sigma(f) = f^q$ for all $f \in F$. Since σ has order two, ff^σ is fixed by σ and so lies in K. The map $f \to ff^\sigma = f^{q+1}$ induces a homomorphism form the multiplicative group of F to the multiplicative group of K. Since F has at most $q+1$ $(q+1)$'th roots of unity, the cernel of this map has order at most $q+1$. As $|F^\#| = q^2 - 1 = (q-1)(q+1)$ and $|K^\#| = q-1$, this map must be onto.□

absl **Lemma 14.2** *Let A be an abelian subgroup of L. Then $C_L(A)$ is is the unique maximal abelian subgroup of L containing A, $N_L(A)$ is solvable of derived length two and one of the following holds:*

1. (a) *$\bar{A}$ is an elementary abelian p-group.*

   (b) *KA has exactly one proper submodule W in V.*

   (c) *$C_L(M) = Z(L)U$ where $U = C_L(W) \cong (K,+)$ is a maximal unipotent subgroup of L. With respect to some of basis V, $U = \{\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \mid k \in K\}$.*

   (d) *$N_L(A) = N_L(W)$, $N_L(W)/U \cong (K^\#, \cdot)$ and with respect to some basis of V, $N_L(A) = \{\begin{pmatrix} \lambda & 0 \\ k & \lambda^{-1} \end{pmatrix} \mid k \in K, \lambda \in K^\#\}$.*

2. (a) *A is a locally cyclic p' group.*

   (b) *KA has excactly two proper submodules W_1, W_2 on V*

   (c) *$C_L(A) = N_L(W_1) \cap N_L(W_2) \cong (K^\#, \cdot)$. With respect to some basis V, $C_L(A) = \{\begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix} \mid k \in K^\#\}$.*

   (d) *$N_L(A) = N_L(\{W_1, W_2\})$, $|N_L(A)/C_L(A)| = 2$ and $N_L(A)/C_L(A)$ inverts $C_L(A)$.*

3. (a) *A is a locally cyclic p'-group.*

   (b) *A acts irreducible on V.*

   (c) *Let $F = \mathrm{End}_{KA}(V)$. Then F is a field extension K of degree 2. Let $1 \neq \sigma \in \mathrm{Gal}(F,K)$, then $C_L(A) = F \cap L = \{f \in F \mid ff^\sigma = 1\}$.*

   (d) *$N_L(A) = N_L(F)$, $|N_L(A)/C_L(A)| = 2$ and $N_L(A)/C_L(A)$ inverts $C_L(A)$.*

Proof:

Case 1 A is not a p' group.

Let $1 \neq b \in B$ be an element of order p. Then $(b-1)^p = b^p - 1 = 0$. Hence there exists $v_2 \in V$ with $v_1 = v_2^{(b-1)} \neq neq 0$ and $v_1(b-1) = 0$. Then $W = C_V(b) = Kv_1$ is the only proper $K\angle b\rangle$-invarinant subspace of V. As A is abelian, A normalizes W and so 1b holds. Hence also $N_L(A)$ leaves W invariant and 1d holds. A simple calculation shows that

$C_L(b) = UZ(L)$ where U is as in 1c. Since $UZ(L)$ is abelian, $UZ(L) = C_L(A)$ and so 1c holds. Also $\bar{A} \leq \bar{U}$ and so 1a holds.

Case 2 A is a p' group and A is not irreducible.

Let W_1 be a proper KA-submodule in V. Let $a \in A \setminus Z(L)$. Then a acts as a scalar λ on W_1. Since $\det a = 1$, a acts as λ^{-1} on V/W_1. Suppose that $\lambda = \lambda^{-1}$. Then $\lambda = \pm 1$. Put then $a^p = \pm 1$ and as A is a p-group, $a = \pm 1$ a contradcition to $a \notin Z(L)$. So $\lambda \neq 1$. Let W_2 be the eigenspace of a corresponding to the eigen vector λ^{-1}. Then W_1 and W_2 are the only proper $K\langle a \rangle$ submodules in V and so 2b holds. Is is now easy to check that also 2c and 2d hold. Finally finite subgroups the multiplicative groups of field are cyclic and so also 2a holds.

Case 3 A is irreducible(and so also a p' group)

By Schur lemma F is a division ring. Clearly $K \leq F$ and as A is abelian $A \leq F$ and so $F \neq A$. Since V is a vector space over F we get $2 = \dim_K V = \dim_F V \cdot \dim_K F$. Hence F as degree 2 over K and V is 1-dimensional over F. In particular we may and do identify the K-space V with the K-spave F. Let $f \in F \setminus K$. Then $1, f$ is a K basis for V. Let $f^2 = -k - lf$ for some $k, l \in K$. So $m_f(x) = x^2 + lx + k$ is the minimal polynomial for f over K and the matrix for f with respect to the basis $1, f$ is $\begin{pmatrix} 0 & 1 \\ -k & -l \end{pmatrix}$. Thus $\det f = k$. As $f \in SL(V)$ if and only if $k = 1$.

In particular for $a \in A \setminus Z(L)$ we see that a and a^{-1} are the roots of $m_a(x)$, F contains all roots of $m_a(x)$ and $m_a(x)$ does not have a double root. Thus $F : K$ is a Galois extension.

Let σ be the non trivial element of $Gal(F : K)$. Then f, f^σ are the roots of $m_f(x)$ and so $k = ff^\sigma$. Hence $f \in L$ if and only if $ff^\sigma = 1$. Note that $\sigma \in GL_K(F) = GL_K(V)$. With rspect to the basis a, a^{-1} we see that $\det \sigma = -1$. So in order to show that $N_L(F) \not\leq C_L(F)$ we need find $f \in F$ with $\det f = -1$, for then $f\sigma \in N_L(F) \setminus C_L(F)$. This amounts to finding $f \in F$ with $ff^\sigma - 1$. But such an f exists by 14.1. Since $C_L(A) \leq F$, 3a holds and thus 3. holds in this case. □

Lemma 14.3 *Let $M \leq L$. Then one of the following holds.* solmn1

(a) *M has an abelian normal subgroup A with $A \not\leq Z(L)$ and the structure of $M \leq N_L(A)$ is described by 14.2. In particular, M is solvable of derived length at most 2.*

(b) *p is odd and there exists $Q \trianglelefteq M$ so that Q is a quaternion group of order eigth. Moreover $N_{\bar{L}}(Q) \cong Alt(4)$ or* Sym(4).

(c) *$\bar{M}^\infty$ is locally "finite,simple" and so simple.*

Proof: Suppose first that M has a solvable normal subgroup Q with $Q \not\leq Z(L)$. Choose Q of minimal derived length. If Q is abelian, then (a) holds. So suppose that Q is not abelian. Then by minimal choice $Z(Q) = Q' = Z(L)$ and p is odd. By 18.13 $Q/C_Q(Q) \leq 2^2 = 4$. If $|Q/Z(Q)| \leq 3$, $Q/Z(Q)$ is cyclic and so Q is abelian. So $|Q/Z(Q)| = 4$ and $|Q| = 8$. By 14.2, $C_2 \times C_2$ is nor a subgroup of L and so Q is a quaternion group. It is easy to see that

$C_L(Q) = Z(L)$ and so $N_{\bar{L}}(Q)$ is ismorphic to a subgroup of $\mathrm{Aut}(Q)$. Since $\mathrm{Aut}(Q) \cong \mathrm{Sym}(4)$ it rmains to verify that 3 divides $N_L(Q)$. We leave this as an exercise to the reader.

Note that we have probed that all solvable subgroups of L have derived length at most 4 and so all locally solvable subgroups of L are solvable. So we may assume that $\mathrm{LSol}(M) \leq Z(L)$. Let R be non trivial normal finite perfect subgroup of M and T a normal subgroup of R minimal with respect to $T \not\leq Z(L)$. Let S be some Sylow subgroup of T with $S \not\leq Z(L)$. Then by Frattini, $R = TN_R(S)$. Since $N_L(S)$ is solvable and R is perfect, $R = T$. We conclude that $\bar{R}$ is simple. □

BurTho **Lemma 14.4** *Let H be a finite group.*

(a) *Let r a prime and S a Sylow r-subgroup of G. If S is cyclic and $N_H(S) \leq C_H(S)$, then H has a normal p-complement, that is $H = NS$ for some normal p' subgroups N of H.*

(b) *Let $T < S \leq G$ with S a Sylow 2-subgroup of H. Let t be an involution in H and suppose that $|S/T| = 2$. Then either t is conjugate in H to an element of T or H has a normal subgroup of index two which does not contain t.*

(a) Since S is cyclic S has a faithful 1 dimensional module over $\mathbb{C}$. Let W be the $\mathbb{C}$ module induced from U. For $h \in H$ let $\mathrm{sign}(h) \in \{1, -1\}$ be the sign of h as a permutation on H/S and let $\det h$ the determinant of h as a linear map map on W.

If h is a r' element then no non-trivial power of h normalizes a conjugate of W under H and so as $\mathbb{C}\langle h\rangle$-module, W is isomorphic to the permutation module on H/S. Thus $\det h = \mathrm{sign}\, h$.

So suppose now that $s \in S$ with $|h| = p$. Let λ be scalar induced by s on U. Let $g \in H$ so that s fixes Sg. Then both $s \in S \cap S^g$ and so both S and S^g are Sylow p-subgroups of $C_H(s)$. Hence $S^g = S^c$ for some $c \in C_H(s)$. Thus $gc^{-1} \in N_H(S) = C_H(S) \leq C_H(s)$ and so $g \in C_H(s)$. In particular s acts as λ on U^g. Let $m = |C_H(s)/S|$. We conclude that $\det s = \lambda^m \,\mathrm{sign}\, h$.

Define $\phi(h) = \det(h)\,\mathrm{sign}(h)$. Then ϕ is a homorphism form H into the multiplicative group of $\mathcal{C}$. Also by the above $\phi(h) = 1$ for all p' elements and $\phi(s) = \lambda^m \neq 1$ as λ is a p-root of unity and m is coprime to p. Hence $H = \ker\phi S$ and $\ker\phi \cap S = 1$.

(b) Consider ther action of H on $\Omega = H/T$ by right multiplication. Suppose first that H has a fixed point Th on Ω. Then $Tht = Th$ and so $t^{h_1} \in T$. So suppose that t acts fixed point freely on Ω. Then t as $\frac{|\Omega|}{2} = \frac{|H|}{|S|}$ orbits on Ω. Thus t induces an odd permutaion on Ω. Thus the normal subgroup of H consisting of all the elements inducing an even permutation on Ω has index two and does no contain Ω. □

ssl **Lemma 14.5** *Let $M \leq L$ so that $\bar{M}$ is non-abelian, finite simple simple. The one of the following holds.*

1. *There exists a finite subfield K_M of K and 2-dimensional K_M subspace V_M with $KV_M = V$ and $M = SL_{K_M}(V_M)$.*

2. *p is odd and $\bar{M} \cong \mathrm{Alt}(5)$.*

Proof: Let $A_1, A_2, \dots A_r$ be representatives for the conjucagcy classes of maximal abelian p' subgroups of M. Also let $\bar{A}_0$ be a Sylow p subgroup of $\bar{M}$.

1. Every non trivial element of $\bar{M}$ is contained in exactly one conjugate of exactly on of the $\bar{A}_i$'s, $0 \leq i \leq r$. ssl-1

Let $x \in M \setminus Z(L)$ and put $A = \langle x \rangle$. Then A is abelain. If A is not a p' group then by 14.2 $\bar{A}$ is a p-group and so A is contained in a conjugate of A_O. If A is a p' group, then A is contained a maximal abelian p-prime subgroup of M, which by definition is a conjugate of one of the A_i's, $1 \leq i \leq r$. Finally if $A \leq A_i^x$ for some $x \in M$, then $A_i^x \leq C_M(A)$ which by 14.2 is abelian and either (modulo $Z(L)$ a p or a p' group. Thus $A_i^x = C_M(A)$ is unique.

Let $a_i = |\bar{A}_i|$, $m = |\bar{M}|$ and $r_i = |N_L(A_i)/A_i|$. Let $e = \gcd(2, p-1)$

2. $1 - \frac{1}{m} = \sum_{i=0}^{r} \frac{1}{r_i}(1 - \frac{1}{a_i})$ ssl-2

Fist note $|N_{\bar{L}}(A_i)| = r_i a_i$ and so A_i has $|L/N_L(A_i)| = \frac{m}{r_i a_i}$ conjugates. Each of these conjugates contains $a_i - 1$ non-trivial elements and so by 1. and as $\bar{M}$ as $m-1$ non-trivial elements:

$$m - 1 = \sum_{i=0}^{r} \frac{m}{r_i a_i}(a_i - 1).$$

Dividing boths sides by m gives 2..

3. $a_i \neq 1$ for at least two $0 \leq i \leq r$ ssl-2.5

Suppose $a_i \neq 1$ for exactly one i. Then as $a_i r_i \leq m$, 2.

$$1 - \frac{1}{a_i r_i} \leq 1 - \frac{1}{m} = \frac{1}{r_i}(1 - \frac{1}{a_i}) = \frac{1}{r_i} - \frac{1}{a_i} r_i \leq 1 - \frac{1}{a_i r_i}$$

Hence all inequalites are inequalities and we conclude, $a_i r_i = m$, $r_i = 1$ and $a_i = m$. Thus $\bar{M} = \bar{A}_i$ is abelian a contradiction.

4. ssl-3

(a) Let $1 \leq i \leq r$, then $r_i \leq 2$.

(b) If $a_0 \neq 1$, then $r_0 = a_j$ for some $1 \leq j \leq r$.

(a) follows directly from 14.2. For (b) suppose that $a_0 \neq 1$. Suppose that $r_0 = 1$. If $a_0 \leq 3$, then $\bar{A}_0$ is a cyclic Sylow p-subgroup of $\bar{M}$. Since $r_0 = 1$, $N_M(A_0) = A_0$ is abelian, a contradiction to 14.4. Thus $a_0 \geq 4$ and so $\frac{1}{r_0}(1 - \frac{1}{a_0} \geq \frac{3}{4}$. By 3. $r \geq 1$ and by (a) $\frac{1}{r_1}(1 - \frac{1}{a_1} \geq \frac{1}{2}(1\frac{1}{2}) = \frac{1}{4}$.

Thus 2. provides the contradiction

$$1 - \frac{1}{m} \geq \frac{3}{4} + \frac{1}{4} = 1$$

Hence $r_0 \neq 1$. Let r be a prime divisor of r_0 and T be a Sylow r subgroup of $N_M(A_0)$. Let $A = C_M(T)$. Then be 2., $A = A_i^g$ for some i and g in M. By 14.2(1) there exists a unique 1-space Y invariant under A_0. In particular also T normalises Y and by 14.2(2), T fixes exactly two 1-spaces and both of these are also invarinat under A. Thus A normalizes Y and so also A_0. Thus $A = N_{N_M(A_0)}(T)$. Since $N_M(A_O)/A_0$ is cyclic, A_0T is normal in $N_M(A_0)$ and so by a Frattini argument, $N_M(A_0) = A_0A$. Thus $r_0 = |\bar{A}| = a_i$.

ssl-4 **5.** Let r be a primes divisor of $\bar{M}$.

(a) r divides exactly one of the a_i's.

(b) Let S_r be a Sylow r subgroup of M. Then exactly one of the following holds:

1. r is odd or $p = 2$, and $S_r \leq A_i^g$ for some $g \in M$.
2. $r = 2$, p is odd and for some $g \in M$, $S_r \cap A_i^g$ is a Sylow r-subgroup of A_i^g and $|S_r/S_r \cap A_i^g| = 2$.

(c) $r_i = 2$ for all $i \geq 1$.

Let $\bar{a}$ be an element of order r in $\bar{M}$. Then $a \in A_i^g$ for some i and some $g \in M$. Without loss $a \in A_i$. Then r divides a_i. Let T_r be the Sylow r-subgroup of A_i. Let $t \in N_M(T_r)$. Then $T_r = T_r^g \leq A_i \cap A_i^g$ and so by 1., $A_i = A_i^g$. Thus $N_M(T_r) = N_M(A_i)$.

Supppose first that r is odd or $p = 2$. Without loss $T_r \leq S_r$. By 4., r does not divide r_i. So T_r is a Sylow p-subgroup of $N_M(T_r)$ and hence $N_{S_r}(T_r) = T_r$. Hence $T_r = S_r$. So (b) holds. Assume that r also divides a_j. Then also A_j contains a Sylow r-subgroup of M and so 2. implies $i = j$. Thus (a) holds. Suppose $i \geq 1$ and $r_i = 1$, then $N_M(T_r) = A_i$ is abelian. Since T_r is a cyclic Sylow r subbgroup of M, this contradicts 14.4. So (c) holds for this i.

Suppose next that $r = 2$ and p is odd. Without loss $\bar{a} \leq Z(\bar{S})$. Then $T_r = C_{S_r}(a)$. S_r acts on the coset $\bar{a} = aZ(L)$ and $aZ(L)$ contains only two elements. Thus $|S_r/T_r| = 2$ and $S_r \leq N_M(T_r)$. If $S_r = T_r$ we obtain the same contradiction as at the end of the last paragraph. Thus $S_r \neq T_r$ and $r_i = 2$. So (b) and (c) hold. Assume that $r = 2$ also divides a_j. Then $\bar{A}_j$ contains an involution $\bar{b}$. As T_r has indes two on S_r, 14.4 implies that $b^g \in T_r \leq A_i$ for some g. Thus $b^g \in A_j^g \cap A_i$ and so by 2., $j = i$. So also (a) holds and all parts of 5. are proved.

ssl-5 **6.** $m = e \prod_{i=0}^{r} a_i$.

Clearly $m = \prod |\bar{S}_r|$, where the product is taken over all prime divisors r of m and S_r is a Sylow rp-subgroup of M. Thus 6. follows from 5.(b).

7. One of the following holds ssl-6

(a) $r = 3$

(b) $r = 2$ and $a_0 \neq 1$.

Since $r_i \leq 2$ for all $i \geq 1$ 2. implies

$$1 - \frac{1}{m} \geq r\frac{1}{2}\frac{1}{2}$$

hence $1 < \frac{r}{4}$ and $r < 4$.

Suppose that neither (a) nor (b) hold. Then $r \leq 2$ and $a_0 = 1$, or $r \leq 1$. So by 3. there exists exactly two a_i's (say i and j) with $a_i \neq 1$. Then $r_i \geq 2$ and $r_j \geq 2$ and so by 2.

$$1 - \frac{1}{m} = \frac{1}{r_i}(1 - \frac{1}{a_i} + \frac{1}{r_j}(1 - \frac{1}{a_j} \leq \frac{1}{2}(1 - \frac{1}{a_i} + 1 - \frac{1}{a_j})$$

Thus

$$\frac{1}{a_i} + \frac{1}{a_j} \leq \frac{1}{m}$$

By 6., $m = ea_ia_j$ so multiplication with m implies

$$e(a_j + a_i) \leq 1$$

A contradiction, as e, a_i and a_j ar all larger or equal to a.

8. Suppose that $r = 3$. Then $r_0 = 1$, $p > 5$, $\{a_1, a_2, a_3\} = \{2, 3, 5\}$ and $m = 60$. In particular, $\bar{M} \cong \mathrm{Alt}(5)$. ssl-7

Since $r = 3$, 2. implies:

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} = 1 + \frac{2}{m} + \frac{2}{r_0}(1 - \frac{1}{a_0}) > 1$$

Suppose that $\{a_1, a_2, a_3\} \neq \{2, 3, 5\}$. Then since the a_i's are relatively prime

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{7} = \frac{21 + 14 + 6}{42} = \frac{41}{42} < 1$$

a contradiction.

Thus $\{a_1, a_2, a_3\} = \{2, 3, 5\}$. Since p does not divide any a_i for $i \geq 1$, $p > 5$. In particualar, p is odd and $m = 2 \cdot 2 \cdot 3 \cdot 5 \cdot a_0$ and so $m = 60a_0$. Also

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} = \frac{15 + 10 + 6}{30} = \frac{31}{30}$$

$$\frac{1}{60} = \frac{1}{60a_0} + frac1r_0(1 - \frac{1}{a_0})$$

But this clearly implies, $a_0 = 1$ and so $m = 60$.

ssl-8 **9.** Suppose that $r = 2$ and $a_0 \neq 1$. Put $q = a_0$ and choose notation so that $r_0 = a_1$. Then one of the follwing holds:

(a) $ea_1 = q - 1, ea_2 = q + 1$, $Z(M) = Z(L)$ and $|M| = q(q + 1)(q - 1)$.

(b) $q = p = 3$, $a_1 = 2$, $a_2 = 5$ and $m = 60$. In particular, $\bar{M} \cong Alt(5)$.

Without loss $A_1 \leq N_M(A_0$. By 1. $C_{\bar{M}} \leq \bar{A}_0$ for all $1 \neq a \in \bar{A}_0$ and so all orbits of $\bar{A}_1$ on $\bar{A}_0^{\#}$ have length a_1. Thus

$$a_1 \text{ divides} q - 1 \tag{1}$$

By 2.

$$1 - \frac{1}{eqa_1a_2} = \frac{1}{a_1}(1 - \frac{1}{q}) + \frac{1}{2}(1 - \frac{1}{a_1}) + \frac{1}{2}((1 - \frac{1}{a_2})$$

Thus

$$\frac{1}{2a_2} - \frac{1}{eqa_1a_2} = \frac{1}{2a_1} - \frac{1}{a_1q}$$

Put $d = \frac{2}{e} = \gcd(2, p)$ and multiply the previous equation with $2a_1a_2q$. We obtain:

$$qa_1 - d = qa_2 - 2a_2 = (q - 2)a_2 \tag{2}$$

Thus $q - 2$ divides $qa_1 - d$ and so also $qa_1 - d - a_1(q - 2) = 2a_1 - d$. Since $a_1 > 1$ and $d \leq 2$, $2a_2 - d > 0$ and thus $q - 2 \leq 2a_1 - d$ and so

$$a_1 \geq \frac{q - 1}{2} + \frac{d - 1}{2} \tag{3}$$

Suppose first that $p = 2$. Then $d = 2$ and (3) and (1) imply that $a_1 = q - 1$ and by (2), $a_2 = \frac{q(q-1)-2}{q-2} = \frac{(q+1)(q-2)}{q-2} = q + 1$. Thus 9. (a) holds in this case. Soppose next that p is odd. Then $d = 1$ by (3) and (1), $a_1 = \frac{q-1}{2}$ or $a_1 = q - 1$. If $a_1 = \frac{q-1}{2}$ then by (2) $a_2 = \frac{q\frac{q-1}{2}-1}{q-2} = \frac{1}{2}\frac{q(q-2)}{q-2} = \frac{q+1}{2}$ and again 9. holds. So suppose $a_1 = \frac{q-1}{2}$. Thus by (2) $a_2 = q + 1 + \frac{1}{q-2}$. This implies $q = 3$, $a_1 = 2$ and $a_2 = 5$. Hence 9. is proved.

ssl-9 **10.** Suppose that 9.(a) holds. Then there exists a subfield K of F and a 2-dimensional F subspace W of V with $V = KW$ such that $M = SL_2(W)$, where $SL_F(W)$ is (in the canonical way) viewed as a subgroup of $SL_K(V)$.

Let U be the 1-dimensional subspace normalizes by A_0 and choose notation so that $N_M(A_0) = N_M(U) = A_0A_1$. For $a \in A_1$ let $\lambda(a) \in K$ be defined by $u^a = \lambda(a)u$ for all $u \in U$. Let $F = \lambda(a) \mid a \in A_1 \cup \{0\}$. Since $|A_1| = q-1$ and λ is a homorphism, $F^{\#}$ is cyclic of order q. Thus $f^q = f$ for all $f \in F$ and as q is a power of p, $F \cong \mathbb{F}_q$ is a subfield of U. Let $w \in N_M(A_1) \setminus A_1$. Then $N_M(A_O) \cap N_M(A_0^w) = N_M(U) \cap N_M(U^w) = A_1$. In particular, $|A_0^{wA_0}| = |A_0| = q$. On the otherhand, $|A_O^M| = |M/N_M(A_0)| = |q(q-1)(q+1)/q(q-1)| = q+1$ and so $A_0^{wA_0} \cup \{A_0\} = A_O^M$. Hence $\{1\} \cup wA_0$ is a transversal to A_OA_1.Thus

$$M = A_0A_1 \cup A_0A_1wA_1 \tag{4}$$

Let Y be a 1-dimensional F subspace of U. Let $W = \langle Y^M \rangle$. Then W as an FM-submodule of V. Since Y is A_0A_1 invariant,(4) implies $W = \langle Y + Y^{wA_0} \rangle = Y + Y^w + [Y^w, A_0]$

Since $[V, A_0] \leq U$ and $Y \leq U$ we conclude that $W = (W \cap U) + Y^w$. Thus $W \cap U^w = (W \cap U \cap U^w) + Y^w = Y^w$ and so $W \cap U = Y$ and $W = Y + Y^w$. Hence W is 2-dimensioal over F and $M \leq SL_2(W)$. As $|SL_2(W) = q(q-1)(q+1) = |M|$ we conclude $M = SL_2(W)$ and is proved. □ ssl-9

Lemma 14.6 *Let $p = 3$ or $p > 5$ and $M \leq L$ with $\bar{M} \cong \mathrm{Alt}(5)$. Then $M = N_L(M)$.* altsl

Proof: Note that $Z(L) \leq M$. Let $R = N_M(L)$ and let T be the kernel of the actions of R on the five Sylow 2-subgroups of M. If $T \not Z(L)$ we conclude that $N_L(T)/T$is solvable, a contradcition to $M \leq N_L(T)$ and $M \cap T = Z(L)$. Thus $T = Z(L)$ and so $\bar{R}$ is isomorphic to a subgroup of $\mathrm{Sym}(5)$. Thus $R = M$ or $\bar{R} \cong \mathrm{Sym}(5)$. Let A be a Sylow 5-subgroup of M. Then A is abelian and as $p \neq 5$ we conclude from 14.2 that $|N_R(A)/C_R(A)| \leq 2$. This excludes the case $\bar{R} \cong \mathrm{Sym}(5)$. Thus $R = M$. □

Lemma 14.7 *Let $M \leq SL_2(V)$ so that $\bar{M}$ is infinite simple. Then there exists an infinite subfield F of K and a 2-dimensional F-subspace W of V with $M = SL_2(W)$ and $V = KW$.* issl

Proof: By 14.3 $\bar{M}$ is locally " finite ,simple". As M is countbale we conclude that M is an ascending union of finite subgroups

$$M_1 < M_2 < M_3 < M_4 < \ldots M_k <$$

such that $\bar{M}_k$ is simple. Without loss $\bar{M}_k$ is non-abelian and has order larger than 60. Thus by 14.5 $M_k = SL_2(F_k)$ for some finite subfield F_k of K. Let $k \leq j$. Then $M_k \leq M_j$ and so by 14.5 applied to $K = M_j$, F_k is a subfield of F_j. Let $F = \bigcup_{k=1}^{\infty} F_k$. Then F is an infinite subfield of K. Let a be an element of order p in M_1 and $0 \neq\in u \in C_V(a)$. Put $W_k = \langle F_k u^{M_k} \rangle$. Then as seen in the proof of 14.5 **??**, W_k is 2-dimensional over F_k and $M_k = SL_{F_k}(W_k)$. Put $W = \langle Fu^M = \bigcup_{k=1}^{\infty} W_k$. Then W is 2 dimensional over F and $M = \bigcup_{k=1}^{\infty} M_k = \bigcup_{k=1}^{\infty} SL_{F_k}(W_k) = \bigcup_{k=1}^{\infty} SL_F(W)$. □

Lemma 14.8 *Let F be a subfield of K and W a 2-dimensional subspace F subspace of V with $V = KW$. Let $M = SL_K(W)$* nslf

(a) $N_L(M) = GL_K(W)F^\# \cap L = \{\lambda b \mid b \in GL_F(W), \lambda \in F^\#, \lambda^2 \det b = 1\}$.

(b) *Put* $F^{\frac{1}{2}} = \{k \in K \mid k^2 \in F\}$. *Then*

$$|N_L(M)/M| = |F^{\frac{1}{2}\#}/F^\#| \leq 2$$

Proof: Let $M = SL_F(W)$ and $a \in GL_K(V)$. Then $M^a = SL_F(W^a)$. Hence $M = M^a$ if and only of M normalizes W^a. Let u be a non-trivial unipotent element in M, $0 \neq w \in C_W(u)$ and $0 \neq y \in C_{W^a}(u)$. Then $y = \lambda w$ for some $\lambda \in K$. Then W^a and λW are both 2-dimensional FM-submodules of V and containe $y = \lambda w$. Thus $W^a = \lambda W$. Hence $a\lambda^{-1} \in N_{GL_K(V)}(W) = GL_F(W)$ and $a = b\lambda$ for some $b \in GL_K(W)$. Moreover $a \in L$ if and only if $1 = \det a = \lambda^2 \det b$. Thus (a) holds.

To prove (b) define $\phi : N_L(M) \to F^{\frac{1}{2}\#}/F^\#$, $\lambda b \to \lambda$. We claim that ϕ is a well defined homomorphism, is onto and has kernel L. Indeed, suppose that $\lambda b \mu c$. Then $\lambda\mu^{-1} = cb^{-1} \in GL_F(W) \cap K^\# = F^\#$ and so $\lambda F^\# = \mu F^\#$. Also as $\lambda^2 \det b = 1$, $\lambda \in F^{\frac{1}{2}}$. So ϕ is well defined. Clearly ϕ is a homomorphism. Let $\lambda \in F^{\frac{1}{2}\#}$. Then there exists $b \in GL_F(W)$ with $\det b = \lambda^{-2}$ and so $\lambda b \in N_L(M)$ and ϕ is onto. If λb is in the kernel of ϕ if and only if $\lambda \in F$. And that is the case if and only if $\lambda b \in GL_F(W) \cap L = SL_F(W)$. So the kernel of ϕ is M. Note that $F^{\frac{1}{2}\#}$ isa multiplicative subgroup of $K^\#$ and so is locally cyclic. It folloes that $F^{\frac{1}{2}\#}/F$ is a locally cyclic group, in which the square of every element is trivial. Thus $F^{\frac{1}{2}\#}/F$ has order at most two, and the proof of (b) is completed. □

masusl **Theorem 14.9 (Dickson's List)** *Let M be a maximal subgroup of L. Then one of the follwing holds.*

(a) $M = N_L(U)$, *U a 1-dimensional K subspace of W.*

(b) $M = N_L(\{U_1, U_2\})$, *U_1, U_2 distinct 1-dimensional K-subspaces of V.*

(c) $M = N_L(F)$, *where F is field extension of degree teo of K.*

(d) $M = N_L(SL_F(W))$, *where F is a subfield of K, and W is a 2-dimensional F-subspace of V with $V = FW$.*

(e) *p is odd and $M = N_L(Q)$, where $Q \leq L$ with $Q \cong Q_8$.*

(f) *$p = 3$ or $p > 5$, and $\bar{M} \cong \mathrm{Alt}(5)$.*

Proof: This merely summarizes the results of this chapte. We also refer the reader to the lemmas in this section for a more detailed descriptions of structure of M. □

Chapter 15

The generalized Fitting subgroup of periodic, linear group

This section is devoted to the general structure of periodic linear groups. Our aim goal is to show that modulo the unipotent radical, the generalized Fitting subgroup of a periodic linear group has finite index. As an application we show that Jordan Hoelder Theorem holds for countable, periodic, linear groups.

Definition 15.1 *Let X be a group.* dgf

(a) *X is called* quasi simple *if X is perfect, $X \neq 1$ and $X/Z(X)$ is simple.*

(b) *X is* (absolutely semisimple *if X is a direct sum of (absolutely) simple groups.*

(c) *A* component *for X is a quasi simple, subnormal subgroup of X.*

(d) *The* layer $\mathrm{E}(X)$ *is the subgroup of X generated by the components of X.*

(e) *The* Hirsch-Plotkin radical $\mathrm{LN}(X)$ *is the subgroup of X generated by all the locally nilpotent normal subgroup of X.*

(f) *The generalized Fitting subgroup* $\mathrm{F}^*(X)$ *of X is the group* $\mathrm{LN}(X)\,\mathrm{E}(X)$

We remark that the Hirsch-Plotkin radical is always locally nilpotent so $\mathrm{LN}(X)$ is the largest locally nilpotent normal subgroups of X. A proof for this non-trivial fact can be found in [**?**]. On the other hand for locally finite groups this is readily verified and we leave the details to the reader.

Lemma 15.2 *Let X be group.* comp

(a) *Let Y be a subnormal subgroup of X and K a component. Then $K \leq Y$ or $[Y, K] = 1$.*

(b) *Distinct components of X commute. In particular $E(X)/Z(E(G) \cong \bigoplus\{K/Z(K) \mid K$ a component of X.*

Proof: Note first that (a) implies (b). Indeed let K_1 and K_2 be components of X with $[K_1, K_2] \neq 1$. Then by (a) $K_1 \leq K_2$ and $K_2 \leq K_1$.

We prove (a) by induction on the defects of K and Y. If $K = G$, then either $Y \leq Z(K)$ or $YZ(K)/Z(K)$ is a subnormal subgroup of the simple group $K/Z(K)$. The latter implies $K = YZ(K)$ Thus $K = K' \leq Y$. So we may assume that $K \trianglelefteq \trianglelefteq L \triangleleft X$

If $Y = G$ we are done. So $Y \trianglelefteq \trianglelefteq N \triangleleft G$. If $K \leq N$, we are done by induction. Hence $[K, N] \leq L \cap N$ and $K \not\leq L \cap N$. Hence by induction $[L \cap N, K] = 1$. Thus $[N, K, K] = 1$ and the three subgroup lemma implies, $[N, K] = 1$ and so $[Y, K] = 1$.

ff*g **Proposition 15.3** *Let G be a finite group. Then $C_G(F^*(G) \leq F^*(G)$.*

Proof: Let $D = C_G(F^*(G))$. Clearly a component of D is a component of G and $F(D) \leq F(G)$. Thus $F^*(D) \leq F^*(G) \leq C_G(D)$ and $F^*(D) = Z(D)$. If $D = Z(D)$ we are done. So let $E/Z(D)$ be a minimal subnormal subgroup of D. Then $E/Z(D)$ is simple and so either $E/Z(D)$ is abelian or perfect. In the first case E is nilpotent and in the second case $E = E'Z(D)$ and E' is a component of G. Hence $E \leq F^*(D) \leq Z(D)$, a contradiction. □

siass **Lemma 15.4** *Let $\mathcal{S}$ be a set of perfect, simple groups and $X = \oplus\mathcal{S}$.*

(a) *If $N \trianglelefteq \trianglelefteq X$, then $N = \oplus\mathcal{N}$ for some $\mathcal{N} \subset \mathcal{S}$.*

(b) *If N is serial in X and X is absolutely semisimple, then $N = \oplus\mathcal{N}$ for some $\mathcal{N} \subset \mathcal{S}$.*

Proof: Let $\mathcal{C}$ be a (subnormal) series on X with $N \in \mathcal{S}$. Let $S \in \mathcal{S}$ and $C \in \mathcal{S}$. Then $C \cap S$ is a subnormal (serial) subgroup of S and as S is (absolutely) simple. $C \cap S = 1$ or $C \cap S = S$. In the latter case $S \leq S$. Let T_S the intersection of the elements of $\mathcal{S}$ containing S and B_S the union of the elements which meet S trivially. Then (B_S, T_S) is a jump, $T_S \leq B_S$, $B_S \cap S = 1$ and since SB_S/B_S is a normal subgroup of T_S/B_S, $T_S = SB_S$. It $T_S = T_{S^*}$ for some other $S^* \in \mathcal{S}$ we conclude $[T_S, T_S] \leq [S, S^*]B_S \leq B_S$,a contradiction. Let $1 \neq n \in N$ and write $s = s_1 \cdot \ldots \cdot s_n$ with $1 \neq s_i \in S_i$ and distinct $S_1, \ldots S_n \in \mathcal{S}$. To complete the proof of the lemma it suffices to show that $S_i \leq N$ for all i. Without loss $T_{S_i} \leq B_{S_n}$ for all $i < n$. Then $n \notin B_{S_n}$ and so $N \not\leq B_{S_n}$. But then $T_{S_n} \leq N$ and the lemma is proved. □

sas **Proposition 15.5** *Let K be a quasi- absolutely simple, serial subgroup of G and Y a serial subgroup of G. Then $Y \leq K$ or $[K, Y] = 1$. In particular K is a component.*

Proof: Let $\mathcal{D}$ be a composition series for G with $Y \in \mathcal{D}$.

Suppose first that $G = K$. Completing the chain $\{D^\infty Z(K)/Z(K) \mid D \in \mathcal{D}\}$ with respect to intersections we obtain a series for $K/Z(K)$. As $K/Z(K)$ is absolutely simple we conclude that $Y = K$ or $Z(K) \leq Y$.

In the general case we conclude $K \leq D$ or $D \cap K \leq Z(K)$ for all $D_\in\mathcal{D}$. So there exists exists a jump (B, T) of $\mathcal{C}$ with $K \leq T$ and $T \cap B \leq Z(K)$. If $T \leq Y$, $K \leq Y$. So suppose $Y \leq B$. Let F be a finite subgroup of B. Then $K \cap F$ is subnormal in F and so

$[F, K\cap F, r] \leq K$, where is the defect for $K\cap F$ in F Hence $[F\cap B, K\cap F, r \leq B\cap K \leq Z(K)$ and the three subgroup lemma implies $[F \cap B, (F \cap K)^\infty = 1]$. Thus $K = K^\infty$ centralizes B. It follows that K commutes with its distinct conjugates. Hence K is normal in $\langle K^G\rangle$ □

Lemma 15.6 *Let X be linear of degree n then X has at most $\lfloor \frac{n}{2} \rfloor$ components.* nclg

Proof: Without loss $X = E(X)$ and say $X \leq GL_K(V)$ with $\dim_K V = n$ and K algebraically closed. If $n = 1$, $GL_K(V)$ is abelian and $X = 1$. So suppose $n \geq 2$If W is a proper KX submodule of V of dimension m. Clearly no component can centralize W and V/W. Moreover by induction at most $\frac{m}{2}$ components act non-trivially on V and at most $\frac{n-m}{2}$ on V/W. So we may assume that X acts irreducible. Let L be a component of X. Let $l \in L$ so that l does not act as a scalar on V and let W be an eigenspace for l on V. Any components R other than L centralize l and so normalize W. Moreover, since R is normal in X and so $C_V(R) = 0$ and $[V, R] = V$. But this implies $C_W(R) = 0$ and $[V/W, R] = V/W$. Thus R acts non-trivially on W and V/W. But at least one of W and V/W has dimension at most $\frac{n}{2}$. Thus by induction X has at most $\lfloor \frac{n}{4} \rfloor + 1$ components. □

Lemma 15.7 *Let d be a positive integer so that each $F \in \mathcal{F}$ has at most d non-abelian composition factors. Then any composition series for G has at most d non abelian factors.* fcbc

Proof: Let $\mathcal{C}$ be a composition series for G and (B_i, T_i), $1 \leq i \leq k$ by jumps of $\mathcal{C}$ with non-abelian factors. Then B_i/T_i is an non-abelian locally finite simple groups and thus is not locally solvable. Hence there exists finite subgroup $F_i \leq B_i$ so that F_iB_i/B_i is not solvable. Let $F = \langle F_i \mid 1 \leq i \leq k$. Then $F \cap T_i/F \cap B_i$ has a subgroup isomorphic to F_iB_i/B_i and so is not solvable. Hence $F \cap T_i/F \cap B_i$ has at least one non-abelian composition factor and so F has at least k non-abelian composition factors. Hence $k \leq d$. □

The following examples show that absolutely semisimple by finite groups can fail the Jordan Hölder theorem. For $n \in \mathbb{N}$ let K_n be an absolutely simple group and $1 \neq k_n \in K_n$. Let $B = \oplus K_n$ and define $\alpha_1 \in \mathrm{Aut}(X)$ by $(r_n)^\alpha = (r_n^{k_n^{-1}})$. Let $T = X \ltimes \langle \alpha_1 \rangle$ and inductively let $\alpha_{i+1} = \alpha_i k_i \in T$. Then α_i centralizes K_j for all $j < i$ and so $T_i = oplus_{t\geq i}\langle \alpha_i \rangle$ is a normal subgroups of T. Moreover $T_i/T_{i+1} \cong K_i$ and $\bigcup T_i = 1$. Hence T has a composition series without abelian composition factor. If the K_i are locally finite and the k_i have bounded order then T is locally finite.

Definition 15.8 *Let H be acting acting on a group X.* dnilact

(a) *H acts* unipotently *on X provided that $[X, H, k] = 1$ for some positive integer k.*

(b) *H acts* locally unipotently *on X provided that for each finitely generated subgroups F of H and each finitely generated subgroup L of X, F acts unipotently on $\langle X^F \rangle$.*

Lemma 15.9 *Let $G = MH$ with $M \trianglelefteq G$, $H \leq G$ and $C_G(M) \leq M$.* nilbynil

(a) *If M is nilpotent of class l and H acts unipotently of degree k on M , then G/M is nilpotent of class at most $kl - 1$ and G is nilpotent of class at most $kl + k - 1$.*

(b) *If M is locally nilpotent and H acts locally unipotently on M, then G is locally nilpotent.*

Proof: Clearly (a) implies (b).

(a) Let $1 = M_l \leq M_{l-1} \ldots \leq M_1 \leq M_0 = M$ be the upper central series for M. Define $L_{ik+r} = [M_i, H, r]$ for all $O \leq r < k$ and all $O \leq o \leq l$. Then $G = HM$ centralizes the factors of the series.

$$1 = L_{lk} \leq L_{lk-1} \leq \ldots L_1 \leq L_0 = M.$$

Let $G_s = [G, G, s-1]$ be the s'th term of upper-central series of G. We claim that $[L_u, G_s] \leq L_{u+s+1}$. If $s = 0$, then $G_0 = G$ and $[L_u, G] \leq L_{u+1}$. Assume by induction that $[L_u, G_{s-1}] \leq L_{u+s}$, for all u. Then

$$[L_u, G_{s-1}, G]] \leq L_{u+s+1} \text{ and} [L_u, G, G_{s-1}] \leq [L_{u+1}, G_{s-1}] \leq L_{u+s+1}.$$

Thus by the 3-subgroup lemma $[L_u, [G, G_{s_1}]] \leq L_{u+s+1}$. But $[G, G_{s-1}] = G_s$ and the claim is proved.

By the claim $[M, G_{kl-1}] = 1$. Since $C_G(M) \leq M$ we get $G_{kl-1} \leq Z(M)$. Hence G/M is nilpotent of class at most $kl-1$. Moreover, $G_{kl+k-1} = [G_{kl-1}, G, k] \leq [Z(M), H, k] = 1$ and so G is nilpotent of class at most $kl + k - 1$. □

abchar **Lemma 15.10** *Suppose that X has an abelian normal subgroup of finite index n. Then X has an abelian, characteristic subgroup of finite index bounded by a function of n.*

Proof: . Let A be a maximal abelian normal subgroup of index at most n in X. Let $\alpha \in \mathrm{Aut}(X)$. Then also A^α is a maximal abelian normal subgroup of index at most n. Then $C_A(A^\alpha)A^\alpha$ is an abelian normal subgroup of X and the maximality of A^α implies $A \cap A^\alpha = C_A(A^\alpha)$. But $C_A(A^\alpha) = C_A(AA^\alpha)$. Since $|G/A| \leq n$, the number of possibly AA^α is bounded by a function of n. Hence there are only boundedly many intersection $A \cap A^\alpha$, $\alpha \in \mathrm{Aut}(X)$. It follows that the index of the characteristic subgroup $\bigcap_{\alpha \in \mathrm{Aut}(X)} A^\alpha$ is bounded by a function of n. □

npabf **Lemma 15.11** *Let G be a locally nilpotent, periodic, linear group of degree n with trivial unipotent radical. Then G has an characteristic, abelian normal subgroup of index bounded by a function of n. In particular, G is solvable of derived length bounded by a function of n.*

Proof: By 15.10 it suffices to find a normal abelian subgroup of bounded index. By an inverse limit argument we may assume that G is finite. Let A be a maximal abelian normal subgroup of G. Since G is nilpotent, $C_G(A) = A$. Thus $|G/C_G(A)| \leq n!$ by 13.5 □

F*gp **Theorem 15.12** *Let G be a periodic, linear group of degree n over a field in characteristic p with trivial unipotent radical.*

(a) $C_G(F^*(G) \leq F^*(G)$.

(b) *G has an abelian normal subgroup and components $L_1, L_2 \ldots L_r$ so that each L_i is a group of Lie type over a locally finite field in characteristic p and the index $AL_1L_2 \cdots L_r$ in G is bounded by an function of n.*

(c) *The index of $F^*(G)$ in G is bounded by a function of n.*

(d) *The number of non-abelian composition factors (in any composition series) of G is bounded by a function of n.*

Proof: Suppose for the moment that we established that all four statement are equivalent. As (a) is true for finite groups, we conclude that (d) is true for finite groups. But then by 15.7 (d) is true in general and the theorem is established.

Suppose now that (a) holds. In several steps we show that (b) is true. In the following bounded always means bounded by a finite function of n

Step 1 The number of components of G is bounded

Thus is true by 15.6.

Step 2 Let H the the largest subgroup of G normalizing all the components of G. Then the index of H in G is bounded.

G/H acts faithfully in the set of components so G. Thus Step 2 follows from Step 1.

Step 3 Let I the the largest subgroup of G inducing inner automorphism on each of the components of G, (that is $I = C_G(E(G)E(G))$. Then the index of I in H is bounded.

By Step 1 it suffices to bound $N_H(L)/C_H(L)L$ for each component L in G. Here we use the classification of simple periodic linear groups. We conclude that up to boundedly many finite exceptions L is a group of Lie type $X_m(p^k)$, where is the characteristic of the underlying vector space and m is bounded. Next we use one more property of groups of Lie Types

- There exists an abelian p' groups C (the Cartan subgroup) in $X = X_m(p^k)$ such that $\mathrm{Aut}(X) = XN_{\mathrm{Aut}(X)}(C)$ and $|C_{Aut(X)}(X)X/X|$ is bounded by a function of m.

Put $R = N_G(L)$. Then we get $R = N_R(C)L$ and $C_R(C)L/LC_R(L)$ is bounded. By 13.5 also $R/C_R(C)L| = N_R(C)L/C_R(C)L \leq |N_R(C)/C_R(C)$ is bounded.

Step 4 Then there exists a characteristic, abelian subgroup of G which is bounded index in $C_G(E(G))$.

Let $J = C_G(E(G)$ and $L = \mathrm{LN}(G)$. Then $L \leq J$ and $C_J(L) \leq C_J(LE(G)) \leq C_G(F^*(G)) \leq L$. By 15.11, there exists a characteristic, abelian subgroup A of L of bounded index in L. Then $C_J(L/A)$ has bounded index in J. Also by 13.5 $C_J(A)$ has bounded index in J. Put $P = C_J(L/A) \cap C_J(A)$. Then P has bounded index in J. Since $[L, P] \leq A$, $[L, P, P] = 1$. Thus P acts unipotently on L and by 15.9 LP is locally nilpotent. Thus $P \leq L$ and so A has bounded index in P and so also in J. So Step 4 holds.

(b) now follows form Steps 2,3 and 4.

As LN(G) contains any abelian normal subgroup of G, (b) implies (c). As G has boundedly many components (c) implies (d).

It remains to show that (d) implies (a). This proof is very similar to the one in 15.3 But note first that as (a) and so also (b) holds for finite groups , any finite solvable subgroup of G is solvable of bounded derived length. Thus also any locally solvable subgroup of G is solvable (of bounded derived length).

Put $D = C_G(F^*(G))$.Then $F^*(D) = Z(D)$. Suppose $D \neq Z(D)$. By assumption $D/Z(D)$ has only finitely many non-abelian composition factors and so by 5.12, $D/Z(D)$ either has a non-trivial locally solvable normal subgroup, or a component. In the first case the previous paragraph implies that c that $D/Z(D)$ has non-trivial abelian normal subgroup. Thus in either case we get the same contradiction as in 15.3. □

jhpl **Corollary 15.13** *The Jordan Hölder theorem holds for all countable, periodic, linear groups. In particular it holds for all periodic linear groups with trivial unipotent radical.*

Proof: 15.12d and 5.13 □

Chapter 16

Non-linear, locally finite, simple groups with p-groups as centralisers

Definition 16.1 *Let X be a group, Y a subgroup of X and W a Y-set. Then* dcoin

$$W \Uparrow_Y^X = \{f : X \to W \mid f(xy) = f(x)^y \text{ for all } x \in X, y \in Y\}.$$

$W \Uparrow_Y^X$ is called the set coinduced *from W to X.*

Note Y acts on X by right multiplication and $W \Uparrow_Y^X$ consists of all Y-equivariant maps from X to W.

Lemma 16.2 *Let X be a group, Y a subgroup of X and W a Y-set. Put $V = W \Uparrow_Y^X$.* bcoin *Then*

(a) *X acts on V by $f^h(x) = f(hx)$ for all $x, h \in X$.*

(b) *Let I be a left transversal to Y in X. Then the map $\rho_I : V \to W^I, f \to f_I$ is a bijection. In particular, V and $W^{X/Y}$ are isomorphic as sets.*

(c) *Define*

$$\pi : V \to W, \pi(f) = f(1).$$

Then π is an onto Y-equivariant map.

(d) *Suppose that t is a fixed-point for Y on W. For $w \in W$ define*

$$\kappa_t(w) : X \to W, x \to \begin{cases} w^x & \text{if } x \in Y \\ t & \text{if } x \notin Y \end{cases}$$

Then

(a) *$\kappa_t(w) \in V$, and $\kappa_t : W \to V$ is a one to one Y-equivariant map.*

(b) *$\pi(\kappa_t(w)) = w$ for all $w \in W$.*

(c) *$\pi(\kappa_t(w)^x)) = t$ for all $w \in W$ and $x \in X \setminus Y$.*

(e) *Suppose in addition that W is a Y-group, that is Y is a group and for each $y \in Y$ the map $W \to W, w \to w^x$ is a homomorphism of group. Then the maps ρ_I, π and κ_1 all are homomorphism of groups.*

Proof: (a) We need to verify that $f^h \in V$ and $f^{hl} = (f^h)^l$ for all $f \in V, h, l \in X$. Let $x \in X$ and $y \in Y$. Since $f^h(xy) = f(h(xy)) = f((hx)y) = f(hx)^y = (f^h(x))^y$, $f^h \in V$. Also $f^{hl}(x) = f((hl)x) = f(h(lx)) = f^h(lx) = (f^h)^l(x)$ and so (a) holds.

(b) Let $x \in X$. Then $x = iy$ for some unique $i \in I$, $y \in Y$.

Let $f \in V$. Then $f(x) = f(iy) = f(i)^y$ and so f is uniquely determined by f_I. Thus $f \to f_I$ is one to one.

Let $g \in W^I$. Define $f : X \to W$ by $f(x) = f(i)^y$, It is easy to verify that $f \in V$ and $f_I = g$. Hence $f \to f_I$ is onto.

(c) Let $f \in V$, $y \in Y$. Then $\pi(f^y) = f^y(1) = f(y \cdot 1) = f(1 \cdot y) = f(1)^y = \pi(f)^y$. So π is Y-equivariant. By the axiom of choice, 1 is contained in some left transversal. Hence (b) implies that π is onto.

(d) Let $w \in W, y \in Y$ and $x \in X$. Then $x \in Y$ if and only if $xy \in Y$. Also by assumption $t = t^y$ and so

$$\kappa_t(w)(xy) = \begin{cases} w^{xy} & \text{if } xy \in Y \\ t & \text{if } xy \notin Y \end{cases} = \begin{cases} w^{xy} & \text{if } x \in Y \\ t^y & \text{if } x \notin Y \end{cases} = (\kappa_t(w)(x))^y$$

Thus $\kappa_t(w) \in V$.

Also $yx \in Y$ if an only if $x \in Y$ and so

$$\kappa_t(w^y)(x) = \begin{cases} (w^y)^x & \text{if } x \in Y \\ t & \text{if } x \notin Y \end{cases} = \begin{cases} w^{yx} & \text{if } yx \in Y \\ t & \text{if } yx \notin Y \end{cases} = \kappa_t(w)(yx) = (\kappa_t(w)^y(x)$$

Thus $\kappa_t(w^y) = (\kappa_t(w))^y$ and (da) holds.

Since $1 \in Y$, $\pi(\kappa_t(w)) = \kappa_t(w)(1) = w^1 = w$ and so (db) holds.

Let $x \in X \setminus Y$. Then $\pi(\kappa_t(w)^x) = \kappa_t(w)^x(1) = \kappa_t(w)(x \cdot 1) = \kappa_t(w)(x) = t$. So also (dc) holds.

Suppose now that W is a Y-group. It is readily verified that W^X is a group via $(fg)(x) = f(x)g(x)$. Moreover, for $f, g \in W \Uparrow_Y^X$, $x \in X$ and $y \in Y$ we have

$$(fg)(xy) = f(xy)g(xy) = f(x)^y g(x)^y = (f(x)g(x))^y = (fg)(xy)^y$$

So $fg \in W \Uparrow_Y$. Similarly $f^{-1} \in \Uparrow_Y$ and clearly $1 \in W \Uparrow_Y$. So $W \Uparrow_Y^X$ is a subgroup of W^X.

For any $J \subseteq X$, then restricted map $W^X \to W^J, f \to f\mid_J$ is a homomorphism. Thus ρ_I and π are homomorphism. $W \to W, w \to w^y$ and $W \to W, w \to 1$ are homomorphisms and so also κ_1 is a homomorphism. □

beta **Lemma 16.3** *Let X, Y, W, K be groups with $Y \leq X$ and $Y = W \ltimes K$ Then there exits a semidirect product $H = V \ltimes X$ and an embedding $\beta : Y \to H$ such that*

(a) $V \cong W^{Y/X}$ *as groups.*

(b) $V\beta(y) = Vy$ *for all* $y \in Y$.

(c) *Let $y \in Y$ with $y \notin k^W$, where $y = wk$, $w \in W, k \in K$. Then*

(a) $VC_H(\beta(y) = VC_X(y)$

(b) $\beta(y) \notin y^V$.

Proof: Let $y \in Y$ and $y = wk$ with $w \in W, k \in K$. Let ρ be the projection of Y onto K, that is $\rho(wk) = k$. Define $\sigma : Y \to \mathrm{Sym}(W)$ by $w^{\sigma(y)} = w^{\rho}(y)$. So $w^{\sigma(y)} = w^k$ and W becomes a Y-set W_σ. Put $V = W_\sigma \Uparrow_Y^X$. Then by 16.2 X acts on V and we can build the semidirect product, $H = V \ltimes X = \{(v, x) | v \in V, x \in X\}$. We view V and X as subgroups of H. So $H = VX$ and $V \cap X = 1$. Let $\pi : V \to W$ and $\kappa = \kappa_1 : W \to V$ be as in 16.2. Let $v \in V$.

1. $\pi(v^y) = \pi(v)^{\rho(y)} = \pi(v)^k$ and $\pi(v^{y^{-1}}) = \pi(v)^{k^{-1}}$. beta-2

By 16.2(c) and keeping in mind that the action of Y on W is given by σ, $\pi(v^y) = \pi(v)^\sigma(y) = \pi(v)^k$ and so the first part hold. Since $\rho(y^{-1}) = \rho(y)^{-1} = k^{-1}$, the second statement follows from the first.

2. Define beta-3

$$\beta : Y \to H, y \to (\kappa(w), y).$$

Then β is a monomorphism and $V\beta(y) = V \cdot (1, y)$.

Clearly β is one to one. For $i = 1, 2$ let $y_i \in Y$ and $y_i = w_i k_i$ with $w_i \in W$, $k_i \in K_i$. Then

$$y_1 y_2 = w_1 k_1 w_2 k_2 = w_1 w_2^{k_1^{-1}} k_1 k_2,$$

and so

$$\beta(y_1 y_2) = (\kappa(w_1 w_2^{k_1^{-1}}), y_1 y_2).$$

On the otherhand,

$$\beta(y_1)\beta(y_2) = (\kappa(w_1), y_1)(\kappa(w_2), y_2) = (\kappa(w_1)\kappa(w_2)^{y_1^{-1}}, y_1 y_2)$$

As κ is a Y-equivariant homomorphism,

$$\beta(y_1)\beta(y_2) = \kappa(w_1 w_2^{k_1^{-1}}), y_1 y_2)$$

and so indeed β is a homomorphism.

3. Let $(v,x) \in C_H(\beta(y))$. Then $\kappa(w)v^{y^{-1}} = v\kappa(w)^{y^{-1}}$. and $xy = yx$. beta-4

We compute

$$\beta(y)(v,x) = (\kappa(w),y)(v,x) = (\kappa(w)v^{y^{-1}}, yx)$$

and

$$(v,x)\beta(y) = (v,x)(\kappa(w),y) = (v\kappa(w)^{x^{-1}}, xy)$$

Thus 3. holds.

beta-5 **4.** Suppose that $VC_H(\beta(y)) \neq VC_Y(y)$. Then $y \in k^W$.

Since $\beta(C_Y(y)) \leq C_H(\beta(y))$, $VC_H(\beta(y) \geq V\beta(C_Y(y)) \overset{2.}{=} VC_Y(y)$ and so there exists $(v,x) \in C_H(\beta(y))$ with $x \notin Y$.

By 16.2(c), $\pi(\kappa(w)) = w$. By 1., $\pi(v^{y^{-1}}) = \pi(v)^{k^{-1}}$. Also since $x \notin Y$ and $\kappa = \kappa_1$, 16.2(dc) implies $\pi(\kappa(w)^{x^{-1}}) = 1$. So applying π to both sides of the equation in 3. we obtain

$$w\pi(v)^{k^{-1}} = \pi(v).$$

Put $r = \pi(v)$. Then $wkrk^{-1} = r$, $wk = rkr^{-1}$ and $y = wk = k^{r^{-1}} \in k^W$.

beta-6 **5.** If $\beta(y) \in y^V$, then $y \in k^W$.

Suppose that $\beta(y) = (1,y)^{(v,1)}$ for some $v \in V$. Then

$$(\kappa(w),y) = (v^{-1},1)(1,y)(v,1) = (v^{-1},y)(v,1) = (v^{-1}v^{y^{-1}}, y)$$

and so $\kappa(w) = v^{-1}v^{y^{-1}}$. Applying π to both sides we conclude $w = \pi(v^{-1})\pi(v)^{k^{-1}}$. Put $r = \pi(v)$ then $w = r^{-1}krk^{-1}$ and $y = wk = r^{-1}kr = k^r$. Thus 5. holds.

We are now in the position to prove the lemma. (a) follows from **??** (b),(e). By 2. (b) holds. By 4. (ca) and by ,5. (cb) holds. □

alpha **Lemma 16.4** *Let G be a finite group and Π a set of primes. Then there exist a finite abelian $\mathbb{Z}G$-module W and a monomorphism $\alpha : G \to W \ltimes G$ such that*

(a) *W is a Π-group.*

(b) *$W\alpha(g) = Wg$ for all $g \in G$*

(c) *$\alpha(g) \notin g^W$ for all non-trivial Π-elements g in G.*

(d) *If G is perfect, then $W = [W,G]$ and $W \ltimes G$ is perfect.*

Proof: Let m be the Π-part of $|G|$. Let $B = (\mathbb{Z}/m\mathbb{Z})^G$ and $H = \mathbb{Z}/m\mathbb{Z} \wr G$, where wreathed product is build with respect to regular action of G on G. Then B is the base group of H and $H = BG$. For $f \in B$ put $||f|| = \sum_{g \in G} f(g)$. Let $W = [B, G] = \{f \in B \mid ||f|| = 0\}$. Then W is a Π-group and (a) holds. Also if $G = G'$, the three subgroup lemma implies $[B, G, G] = [B, G]$ and so $W = [W, G]$ and WG is perfect. Thus (d) holds.

Let $b \in B$ be defined by $b(1) = 1$ and $b(g) = 0$ for all $g \in G^{\#}$. Define $\alpha : G \to WG, g \to g^b = [b, g^{-1}]g$. Then α is an monomorphism and (b) holds. It remains to prove (c). So let g be a non-trivial Π-element in G and suppose that $g^b = g^a$ for some $a \in W$. Put $n = |g|$ and $c = ba^{-1}$. Then $c \in C_B(g)$. Let I be a left transveral to $\langle g \rangle$. Then each g in G can be uniquely written as ig^k for some $i \in I$ and some $0 \leq k < n$. Since $c^g = c$, $c(i) = c(ig^k)$. Let $s = \sum_{i \in I} c(i)$. We conclude that $||c|| = ns$. Thus

$$1 = ||b|| = ||ac|| = ||a|| + ||c|| = 0 + ns = ns$$

in $\mathbb{Z}/m\mathbb{Z}$, a contradiction as n divides m. □

Lemma 16.5 *Let G and F be finite groups and Π a set of primes. Then there exist a finite group G^* with $G \leq G^*$ and normal subgroup V of Gsuch that:* beal

(a) *V is an abelian Π-subgroup and G^*/V is simple.*

(b) $G \cap V = 1$.

(c) *Let x be a Π-element in G. Then $C_{G^*}(x)$ has a normal solvable Π-subgroup $M(x)$ with $C_{G^*}(x) = M(x)C_G(x)$.*

(d) *G^* has a subgroup isomorphic to F.*

(e) *If G is perfect, G^* is perfect.*

Let α and $Y = A \ltimes G$ be as in 16.4. Let X be any finite simple group containing Y as a subgroup and such that X has a subgroup isomorphic to F. Let β and V be as in 16.3 applied with $K = G$. Let $G^* = V \ltimes X$ and $M^* = V$. Let g be a Π-element in G and put $\gamma = \beta\alpha$. Then by 16.4 $\alpha(g) \notin g^W$. So by 16.3, $C_{G^*}(\gamma(x)) \leq VC_Y(\alpha(x)$ Also $C_Y(\alpha(x)) \leq WC_G(x)$ and $VW\gamma(C_G(x)) = VWC_G(x)$. Thus

$$C_{G^*}(\gamma(x)) \leq VW\gamma(C_G(x))$$

Put $M(x) = C_{VW}(\gamma(x))$. Then $C_{G^*}(\gamma(x)) = M(x)C_G(x)$. Identifying G with its image in G^* under γ we see that all parts of the lemma hold. □

Lemma 16.6 *Let G be a locally finite group.* sobysi

(a) *Suppose G is absolutely perfect and (locally solvable) by simple. Then* $\mathrm{LSol}(G)$ *contains all proper normal subgroups of G.*

(b) *Suppose G is locally "absolutely perfect and (locally solvable) by simple" Then G is absolutely perfect and (locally solvable) by simple.*

(c) *If G has Kegel-cover $\mathcal{K}$ such that for all $(H, M) \in \mathcal{K}$, H is perfect and M is solvable, then G is simple.*

Proof: Let $L = \mathrm{LSol}(G)$.

(a) Let N be a normal subgroup of G with $N \not\leq L$. As G/L is simple, $G = NL$. Since G is absolutely perfect, G/N is the union of finite perfect subgroups. Now $G/N \cong L/L \cap N$ is locally solvable and so all finite subgroups of G/N are solvable. Thus $G/N = 1$ and $G = N$.

(b) Let N/L be a non-trivial normal subgroup of G/L. Then N is not locally solvable and so there exists a finite, non-solvable subgroup T of N. Let F be any finite subgroup of G. By assumption there exists an absolutely perfect and (locally solvable) by simple subgroup M of G with $\langle F, T\rangle \leq M$. Then $T \leq M \cap N \trianglelefteq M$. Thus $M \cap N \not\leq \mathrm{LSol}(M)$ and so by (a), $M \cap N = M$. Hence $F \leq M \leq N$ and $N = G$.

(c) By (b) it suffices to show that $L = 1$. Let F be a finite subgroup of L. Then there exists $(H, M) \in \mathcal{K}$ with $F \leq H$ and $F \cap M = 1$. Since H is perfect, $H \neq \mathrm{Sol}(H)$ and so $\mathrm{Sol}(H) = M$. Thus $F \leq L \cap H \leq M$ and $F = 1$. Therefore also $L = 1$. □

cc **Proposition 16.7** *Let G_0 be a finite, perfect group, Π a non-empty set of primes and for each positive integer n let F_n be a finite group. Then there exists a locally, finite simple group G with $G_0 \leq G$ and such that:*

(a) *If x is a Π-element of G, then $C_G(x)$ has a locally solvable, normal Π-subgroup $M(x)$ of finite index.*

(b) *If x is a Π element in G_0, then $C_G(x) = M(x)C_{G_0}(x)$.*

(c) *F_n is isomorphic to a subgroup of G.*

Proof: By induction on $n \in Z^+$ we will produce finite groups G_n and a normal subgroup M_n of G_N such that

cc-1 **1.**

1. G_n is perfect.
2. M_n is abelian and G_n/M_n is simple.
3. $G_{n-1} \leq G_n$ and $G_{n-1} \cap M_n = 1$.
4. If x is a Π element in G_{n-1}, then there exists a solvable normal Π subgroup $M_n(x)$ of $C_{G_n}(x)$ with $C_{G_n}(x) = M_n(x)C_{G_{n-1}}(x)$.
5. G_n has a subgroup ismorphic to F_n

Let $n \geq 0$ and suppose we already found $G_1, \ldots G_n$. Then G_n is perfect and so we can apply 16.5 to $G = G_n$ and $F = F_{n+1}$. Put $G_{n+1} = G_n^*$. Then by 16.5 1. to 5. hold.

Put $G = \bigcup_{i=1}^n G_n$. Then $\{(G_n, M_n) \mid n \geq 1\}$ is a Kegel cover for G and by 16.6, G is simple. Let $x \in G$ be a Π-element. Then $x \in G_n$ for some n. Put $M^n(x) = 1$ and inductively, $M^{m+1}(x) = M^m(x)M_{m+1}(x)$. It follows from 1.4. and induction that

2. Let $m \geq n$. Then $M^m(x)$ is a solvable, normal Π-subgroup of $C_{G_m}(x)$ and $C_{G_m}(x) =$ cc-2
$M^m(x)C_{G_n}(x)$.

Put $M(x) = \bigcup_{m=n}^{\infty} M^m(x)$. Then by 2., $M(x)$ is a locally solvable, normal Π-subgroup of $C_G(x)$ and $C_G(x) = M(x)C_{G_n}(x)$. Thus the proposition is proved. □

Corollary 16.8 *Let Π be a non-empty set of primes. Then there exists a non-linear, locally finite, simple group G such that* sobyfi

(a) *The centralizer of every non-trivial Π-element has a locally solvable Π-subgroup of finite index.*

(b) *There exists an element whose centralizer is a locally solvable Π-group.*

Proof: Let $p \in \Pi$. Let $G_0 = \mathrm{Alt}(2p+1)$ and x the product of two commuting p-cycles. Then $x \in G_0$, G_0 is perfect and $C_{G_0}(x) \cong C_p \times C_p$ and is a solvable Π-group. Apply 16.7 to this G_0 and with $F_n = \mathrm{Sym}(n)$. The resulting G is not linear and fulfills (a). Moreover, (b) holds for the element $x \in G_0 \leq G$. □

Corollary 16.9 msobfi

(a) *There exists a non-linear, locally finite, simple group such that the centralizer of every element is (locally solvable) by finite..*

(b) *Let p be a prime. Then there exists a non-linear, locally finite, simple group with an element whose centralizers is a p-group.*

Proof: Apply 16.8(a) with Π the set of all primes and 16.8(b) with $\Pi = \{p\}$. □

Chapter 17

Examples of simple and of characteristically simple groups

Let I be a set. Then the power set $\mathcal{P}(I)$ is a algebra over the field $\mathbf{F}_2$, where addition is given by symmetric difference and multiplication by intersection. Then empty set is the O and I is the 1 in this algebra. Let $\mathcal{A}$ be a subalgebra of $\mathcal{P}(I)$ with 1, i.e $\mathcal{A}$ is a non-empty set of subsets of I such that

1. If $A \in \mathcal{A}$ then $I \setminus A \in \mathcal{A}$.
2. If $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.

Note that then also $A \cup B$ and $A \setminus B$ are in $\mathcal{A}$ for all $\mathcal{A}$. Let H be a group and let

$$R = R(H, \mathcal{A}) = \{f \in H^I \mid |f(I)| < \infty, f^{-1}(i) \in \mathcal{A} \text{ for all } i \in I\}.$$

Let $\mathcal{D}$ be the set of finite partions of I into members of $\mathcal{A}$. Then $f \in H^I$ lies in $R(H, A)$ if and only if $\Delta_f = \{f^{-1}(i) \mid i \in I\} \in \mathcal{D}$. For $A \in \mathcal{A}$ let R_A consist of whose function $f \in H^I$ which are constant on A and 1 on $I \setminus A$. For $A \in \mathcal{A}$ let $\mathcal{D}_A$ be the set of finite partitions of A into members of $\mathcal{A}$. Also for $A \in \mathcal{A}$ and $\Delta \in \mathcal{D}_A$ let R_Δ consist of whose function which are constant on each member of Δ and are 1 on $I \setminus A$. For $f \in H^I$ let $\mathrm{Supp}_I(f) = \mathrm{Supp}(f) = \{f^{-1}(H^\#)$. Define a partial order on $\mathcal{D}$ by defining $\Delta \leq \Delta^*$ if and only if each member of Δ^* is contained in a member of Δ. For $\Delta, \Delta^* \in \mathcal{D}$ define $\Delta \wedge \Delta^* = \{D \cap D^* \mid D \in \Delta, D^* \in \Delta^*\}$. Then $\Delta \leq \Delta \wedge \Delta^*$ and so $\mathcal{D}$ is a directed set.

Lemma 17.1 basicR

(a) *Let $\Delta, \Delta^* \in \mathcal{D}$ with $D \leq D^*$. Then $R_\Delta \leq R_{\Delta^*} \leq R$.*

(b) *Let $A \in \mathcal{A}$. Then $R_A \cong H$.*

(c) *Let $\Delta \in \mathcal{D}$. Then $R_\Delta = \oplus_{D \in \Delta} R_D \cong H^\Delta$.*

(d) *Let* $f \in R$ *and* $\Delta \in \mathcal{D}$. *Then* $f \in R_\Delta$ *if and only if* $\Delta_f \leq \Delta$.

(e) $\{R_\Delta \mid \Delta \in \mathcal{D}$ *is a local system for* R. *In particular, if* H *is locally finite then* R *is locally finite.*

Proof: This is readily verified. □

cnoid **Lemma 17.2** *Suppose that* H *is a non-trivial perfect, simple group. Let* $\mathcal{B}$ *be an ideal in* $\mathcal{A}$ *and put* $R_\mathcal{B} = \{f \in R \mid \mathrm{Supp}(f) \in \mathcal{B}$ *The the map* $\mathcal{B}\langle N_\mathcal{B}$ *is a one to one correspodence between the ideals in* R *and the normal subgrousp of* R.

Proof: Note first that $\mathcal{B}$ is an ideal in $\mathcal{A}$ is equivalent to saying that $\mathcal{B}$ is a subset of $\mathcal{A}$ with the following two properties:

1. If $A \in \mathcal{A}$, $B \in \mathcal{B}$ with $A \subset B$, then $A \in \mathcal{B}$.
2. If $B, B^* \in \mathcal{A}$ with $B \cap B^* \emptyset$, then $B \cup B^* \in \mathcal{B}$.

Let $f, g \in R_\mathcal{B}$ and $r \in R$. Since $\mathrm{Supp}(f) = \mathrm{Supp}(f^r) = \mathrm{Supp}(f^{-1}$, $f^{-1}, f^r \in R_\mathcal{B}$. Let $D \in \Delta_f \wedge \Delta_g$. Then f, g and fg are constant on D and $D \in \mathcal{A}$. Suppose that fg is not 1 on D, then at lest one of f and g is not one on D. Hence $D \subseteq \mathrm{Supp}(f)$ or $\mathrm{Supp}(g)$ and so by 1. $D \in \mathcal{B}$. As $\mathrm{Supp}(fg)$ is the finite disjoint union of such D's, 2. implies $\mathrm{Supp}(fg) \in \mathcal{B}$ and so $R_\mathcal{B}$ is a normal subgroup of R.

Next let M be any normal subgroup of R and put $\mathcal{E} = \{Supp(m) \mid m \in M\}$. We need to show that $\mathcal{E}$ is an ideal in $\mathcal{A}$ and $M = R_\mathcal{E}$. Let $m \in M$ and $E = \mathrm{Supp}(m)$. Let $\Delta \in \mathcal{D}_E$ and put $\Delta^* = \Delta \wedge \Delta_f$. Then $m \in R_{\Delta^*}$ and and m projects non truvially onto R_{D^*} for all $D^* \in \Delta^*$. Hence by 15.4, $R_{D^*} \leq \langle m^{R_{\Delta^*}} \leq M$. It follows that $R_\mathcal{D} \leq R_{\mathcal{D}^*} \leq M$. Hence M contains all elements of R with support E and if $A \in \mathcal{A}$ with $A \leq E$, then $A \in \mathcal{E}$. Finally if $m_1, m_2 \in M$ withe $\mathrm{Supp}(m_1) \cap \mathrm{Supp}(m_2) = 1$, then $\mathrm{Supp}(m_1 m_2) = \mathrm{Supp}(m_1) \cap \mathrm{Supp}(m_2)$ and also 2. holds. Thus $\mathcal{E}$ is indeed an ideal in $\mathcal{A}$ and $M = R_\mathcal{E}$. □

minide **Lemma 17.3**

(a) *Suppose that* $\mathcal{B}$ *is a minimal ideal of* $\mathcal{A}$. *Then* $\mathcal{B} = \{\emptyset, B\}$ *for some minimal elements of* $\mathcal{A}$.

(b) *Suppose that* H *is a non-trivial perfect, simple group and that* M *is a minimal normal subgroups of* R. *Then* $M = R_B$ *for some minimal member of* $\mathcal{A}$.

(a) Let $E = \bigcup \mathcal{B}$, $e \in E$ and $\mathcal{C} = \{D \in \mathcal{B} \mid e \notin D\}$. Then $\mathcal{C} \neq \mathcal{B}$. As $\mathcal{B}$ is an ideal it easy to see that also $\mathcal{C}$ is an ideal. So by minimality of $\mathcal{B}$, $\mathcal{C} = \{\emptyset\}$. Hence $e \in B$ for all $\emptyset \neq B \in \mathcal{B}$ and all $e \in E$. Thus $B = E$ and (a) holds.

(b) follows from (a) and 17.2.

Rsesi **Lemma 17.4** *Suppose that* H *is a non-trivial, perfect simple group. Then* R *is semisimple if and only if* I *is the disjoint union of finitely many minimal members if* $\mathcal{A}$.

Proof: Suppose first that $R = \bigoplus_{j\in J} R_j$ for some nontrivial simple groups $R_j, j \in J$. Then by 17.3 $R_j = R_{B_j}$ for some minimal member B_j of $\mathcal{A}$. Let $1 \neq f \in R_I$. Then $f \in R = \bigoplus_{j\in J} R_j$ and so $f = r_1 r_2 \ldots r_n$ where $r_i \in R_{j_i}$ for some $j_i \in J$. It follows that $I = \bigcup_{i=1}^n B_{j_i}$ and so the "only if" statement of the lemma is proved.

Conversely, let I be the disjoint union of $A_1, \ldots, A_n$ where each A_i is a minimal member of $\mathcal{A}$. Then each $A \in \mathcal{A}$ is a union of some of the A_i's and so $R = \oplus_{i=1}^n R_{A_i} \cong H^n$. □

Definition 17.5 dcaaut

(a) *Let I_1 and I_2 sets and for $i = 1, 2$ let $\mathcal{A}_i$ a subalgebra of $\mathcal{P}(I_i)$ and $\phi : I_1 \leftarrow I_2$ a map. ϕ is called* homomorphism *from $(\mathcal{A}_1, I_1)$ to $(\mathcal{A}_2, I_2)$ provided that $\phi(\mathcal{A})_1 \subseteq \mathcal{A}_2$. ϕ is an* isomorphism *if ϕ is a bijection and maps $\mathcal{A}_1$ onto $\mathcal{A}_2$. ϕ is an* automorphism *if $I_1 = I_2, \mathcal{A}_1 = \mathcal{A}_2$ and ϕ is an ismorphism.* $\mathrm{Aut}(\mathcal{A}, I)$ *is the group of automorphism of $(\mathcal{A}, I)$.*

(b) *Let S be a subgroup of* $\mathrm{Aut}(\mathcal{A}, I)$*. Then the* canonical action *of S on $R(H, \mathcal{A})$ is defined by $f^s(i) = f(i^{s^{-1}})$ for all $f \in R(H, \mathcal{A})$, $s \in S$. $\mathcal{A}$ is called S-*simple *if S leaves no proper ideal of $\mathcal{A}$ invariant.*

□

Lemma 17.6 *Suppose that $S \leq Aut(\mathcal{A}, I)$ and $\mathcal{A}$ is S-simple. Also suppose that H is non-trivial,perfect,simple. Then S normalizes no proper normal subgroup of R. In particular R is characteristically simple and R is a minimal normal subgroup of $R \rtimes S$.* Rchsi

Proof: Clear by 17.2 □

Lemma 17.7 *Let $S \leq Aut(\mathcal{A}, I)$ and for all $\Delta \in \mathcal{D}$ there exists $\Delta^* \in \mathcal{D}$ with $\Delta \leq \Delta^*$ such that $N_S(\Delta^*)$ acts transitively on Δ^*. Then $\mathcal{A}$ is S-simple.* essial

Proof: Let $\mathcal{B}$ be a non-zero ideal S-invariant ideal in $\mathcal{A}$. We need to show that $\mathcal{B} = \mathcal{A}$. For this let $A \in \mathcal{A}$ and $\emptyset \neq B \in \mathcal{B}$. Let Δ be the set containg the non-empty members of $A \setminus B, B \setminus A, A \cap B, I \setminus (A \cup B)$. Then $\Delta \in \mathcal{D}$ and so by assumption there exists $\Delta^* \in \mathcal{D}$ so that $\Delta \leq \Delta^*$ and $N_S(\Delta^*)$ acts transitively on Δ^*. Since either $B \setminus A$ or $A \cap B$ is not empty, Δ contains a memember of $\mathcal{B}$. As $\mathcal{B}$ is closed under $\mathcal{A}$-subsets also $\mathcal{D}^*$ contains a member of $\mathcal{B}$. As $\mathcal{B}$ is S-invarinat and $N_S(\mathcal{D}^*)$ is transitive on Δ^* all $\Delta^* \subseteq \mathcal{B}$. Since $\mathcal{D} \leq \mathcal{D}^*$, we conclude that $A \setminus B$ and $A \cap B$ and so also A is a disjoint union of members of $\mathcal{B}$. Thus $A \in \mathcal{B}$. As A was arbitrary, $\mathcal{B} = \mathcal{A}$. □

As an application we now use the above machinery to prove:

Proposition 17.8 *There exists a locally finite group G and a minimal normal subgroup R of G so that R is not semi simple.* nsemn

Proof: Let $\mathcal{K}$ be an infinite set of finite sets. Let $I = \prod \mathcal{K}$. Let H be any locally finite, non-abelian, simple group. For subset $\mathcal{E}$ of $\mathcal{K}$ put $I_\mathcal{E} = \prod \mathcal{E}$. We will always identify I with $I_\mathcal{E} \times I_{\mathcal{K}\setminus\mathcal{E}}$.

$E \subset \prod \mathcal{E}$ put $A_E = E \times \prod \mathcal{K} \setminus \mathcal{E} \subset I$. Let $\mathcal{A}$ consists of all the subsets of the form A_E, where $\mathcal{E}$ runs through the finite subsets of $\mathcal{K}$ and E through the subsets of $\prod \mathcal{E}$. It is easy to see that $\mathcal{A}$ is a subalgebra of $\mathcal{P}(I)$ with $I \in \mathcal{A}$. As $\mathcal{K}$ is infinite, $\mathcal{A}$ has no minimal elements and by 17.4 $R = R(H, \mathcal{A})$ is not semisimple.

Next we will define a locally finite subgroup S of $\mathrm{Aut}(\mathcal{A}, I)$ which makes $\mathcal{A}$, S-simple. Then by 17.6 $G = R \rtimes S$ fulfills the conclusion of the lemma.

Again let $\mathcal{E}$ be a finite subset of $\mathcal{K}$. Define a monomorphsim $\phi_\mathcal{E} : \mathrm{Sym}(I_\mathcal{E}) \leftarrow \mathrm{Sym}(I)$, $\phi_\mathcal{E}(\pi) : (a, b) \leftarrow \phi(a), b)$ for all $a \in I_\mathcal{E}$, $b \in I_{\mathcal{K}\setminus\mathcal{E}}$ and $\pi \in \mathrm{Sym}(I_\mathcal{E})$. Put $S_\mathcal{E} = \phi_\mathcal{E}(\mathrm{Sym}(I_\mathcal{E}$. Then it is easy to see that $S_\mathcal{E}$ is a subgroup of $\mathrm{Aut}(\mathcal{A}, I)$. Moreover, if $\mathcal{E}^* \subset \mathcal{E}$ then $S_{\mathcal{E}^*} \leq S_\mathcal{E}$ and so if we let S be the union of the $S_\mathcal{E}$, where $\mathcal{E}$ runs through the finite subsets of $\mathcal{K}$, then S is a locally finte subgroup of $\mathrm{Aut}(\mathcal{A}, I)$.

It remains to show that $\mathcal{A}$ is S-simple. For this it suffices to verify the assumptions of 17.7. So let $\Delta \in \mathcal{D}$. Then $\Delta = \{A_{E_i} \mid 1$
$leqi \leq n\}$, for some finite subsets $\mathcal{E}_i$ of $\mathcal{K}$ and some $E_i \subset I_{\mathcal{E}_i}$. Put $\mathcal{E} = \bigcup_{i=1}^n \mathcal{E}_i$ and $\Delta^* = \{\{a\} \times I_{I\setminus\mathcal{E}} \mid a \in I_\mathcal{E}$. Then $\Delta \leq \Delta^*$, $S_\mathcal{E}$ normalizes Δ^* and induces $\mathrm{Sym}(\Delta^*0)$ on Δ^*. In particular, $S_\mathcal{E}$ and so also $N_S(\Delta^*)$ acts transitively on Δ^*. So we assumptions of 17.7 are indeed verified and the proposition is proved. □

For later use let as denote the group S constructed in the proof of the previous lemma by $\mathrm{Diag}(\prod \mathcal{K})$, the diagonal group on $\prod \mathcal{K}$. Note here that all orbits of $S_\mathcal{E} \cong \mathrm{Sym}(I_\mathcal{E}$ on $I = \prod \mathcal{K}$ are isomorphic to I_E, so $\mathrm{Sym}(I_\mathcal{E}$ is "diagonally" embedded into $\mathrm{Sym}(\prod \mathcal{K})$. Also the redaer should notice that $\mathrm{Diag}(\prod \mathcal{K})$ does depend on $\mathcal{K}$ and not only on $\prod \mathcal{K}$ as a set. Also let $A_\mathcal{E} = \phi_\mathcal{E}(\mathrm{Alt}(I_\mathcal{E})$ and $\mathrm{Diag}^*(\mathcal{K})$ be the union of the $A_\mathcal{E}$, where as usual $\mathcal{E}$ runs through the finite subsets of $\mathcal{K}$. The easy proof of the following propostion is left to the dedicated reader.

simdia **Lemma 17.9** $\mathrm{Diag}^*(\mathcal{K})$ *is a locally finte simple group.* $|\mathrm{Diag}(\mathcal{K})/\mathrm{Diag}^*(\mathcal{K})| \leq 2$ *with equality if and only if only if only finitely many of the* $K \in \mathcal{K}$ *have even size.* □

Chapter 18

Finitary groups

In this section we will investigate two kinds of finitary groups: locally solvable ones and locally finite, simples one of alternating type.

Definition 18.1 *Let X be a group, K a field and V a KX-module.* dfinitary

(q) *V is called a finitary KX-module if $[V,g]$ is finite dimensional for all $x \in X$.*

(b) *X is called a finitary group, if there exists a field K and a faithful, finitary KX-module.*

(c) *Let $x \in X$. Then* $\deg_V(x) = \dim_K[V,x]$.

(d) $\mathrm{FGL}_K(V) = \{x \in GL_K(V) \mid \deg_V(x) < \infty\}$. □

Proposition 18.2 *Let X be a group, K a field and V a faithful KX-module. Then X has largest normal subgroup $\mathcal{LU}(X)$ acting locally unipotent on V.* LUnip

Proof: Let Y the semidirect product VX and let N be a normal subgroup of X. If YN is locally nilpotent, it is easy to see that N acts locally unipotently on V. Conversely if N acts locally unipotently on V, then by 15.9 VN is locally nilpotent. Since $V \leq \mathrm{LN}(Y)$, $\mathrm{LN}(Y) = V(\mathrm{LN}(Y) \cap X)$. Hence $\mathrm{LN}(Y) \cap X$ is the largest normal subgroup of X acting unipotently on Y. □

For the remainder of this section we assume that K is a field, V is a vector space over K, X is a subgroup of $\mathrm{FGL}_K(V)$ and $\mathcal{F}$ is the set of finitely generated subgroups of X.

Lemma 18.3 *Let $t \in X$ and W a K-subspace of V,* finblo

(a) *If $C_W(t) = 0$, then* $\dim_K W \leq \deg_V(t)$.

(b) *If $W \cap W^t = 0$, then* $\dim_K W \leq \deg_V(t)$.

(c) *Let k be a positive integer and Δ be a block system for X on W so that $\dim_K W = k$ for all $W \in \Delta$. Let s the number of orbits for t on Δ. Then*

$$\frac{1}{2}|\operatorname{Supp}_\Omega(t) \leq k \cdot \frac{1}{2}|\operatorname{Supp}_\Omega(t)| \leq k \cdot (|\Delta| - s) \leq \deg_V(t).$$

Proof: (a) $W \cong W/C_W(t) \cong W/W \cap C_V(t) \cong W + C_V(t)/C_V(t) \leq V/C_V(t)$. As $\dim V/C_V(t) = \deg_V(t)$, (a) holds.

(b) Since $C_W(t) \leq W \cap W^t = 0$, (b) follows from (a).

(c) Let $W_1, \dots W_s$ be representatives for the orbit fof t on Δ. Let $W = \sum_{i=1}^s W_i$. Since each $W \in \Delta$ is conjugate one of the W_i's and since $V = \sum \Delta$, $V = \langle W^{\langle t \rangle} \leq W + [V, t]$. Thus

$$k \cdot |\Delta| = \dim_K V = \leq \dim_K W + \dim_K [V, t] = s \cdot k + \deg_V(t).$$

Thus it remains to show that $|Delta| - s \geq \frac{1}{2} \operatorname{Supp}_\Delta(t)$. For thus let $\Delta_1, \dots \Delta_s$ be the orbits of t on Δ with $\Delta_1, \dots, \Delta_r$ the orbits with $|\Delta_i| > 1$, i.e. $\Delta_i \leq \operatorname{Supp}(t)$. Then $\frac{1}{2}|\Delta_i| \leq |\Delta_i| - 1$ for all $1 \leq i \leq r$ and so

$$|\Delta| - s = \sum_{i=1}^{s} |\Delta_i| - 1 = \sum_{i=1}^{r} |\Delta_i| - 1 \geq \sum_{i=1}^{r} \frac{1}{2}|\Delta_i| = \frac{1}{2}|Supp_\Delta(t)|.\Box$$

Ffd **Lemma 18.4** *Let $F \in \mathcal{F}$. Then there exists a finite dimensional KF-submodule U in V and $C \leq C_V(F)$ with $V = U \oplus C$. In particular,F is a linear group.*

Proof: Let $F = \langle I \rangle$ for some finite subset I of F. Then $C_V(F) = \bigcap_{i \in I} C_V(i)$ and $[V, F] = \sum_{i \in I} [V, i]$. Hence as X is finitary, $V/C_V(F)$ and $[V, F]$ are finite dimensional. Thus there exists a finite dimensiona K-subspace U of V with $[V, F] \leq U$ and $V = U + C_V(F)$. Since $[V, F] \leq U$, U is F-invariant. Let C be a complement to $C_U(F)$ in $C_V(F)$. Then $V = U \oplus C$ and the lemma is proved. □

lufu **Lemma 18.5** *Suppose that X acts locally unipotently on V and V is finite dimnsional. Then X acts unipotently on V. In particular, if $V \neq O$, then $V \neq [V, X]$*

Proof: Without loss $V \neq 0$. Let $F \in \mathcal{F}$. Then F acts unipotently on V and so $[V, F] \neq V$. Pick $F \in \mathcal{F}$ so that $[V, F]$ has maximal K-dimension. Then for all $F \leq F^* \in \mathcal{F}$, $[V, F] = [V, F^*]$ and so $[V, X] = [V, F] \neq V$. That X is unipotent on V now follows by induction on $\dim_K V$. □

unpcen **Lemma 18.6** *Let $\mathcal{D}$ be an KX-composition series on V. Then $\mathcal{LU}(X)$ is the largest subgroup of X acting trivially on all of the factors of $\mathcal{D}$.*

Proof: Let W be factor of $\mathcal{D}$ and N a normal subgroup of X acting locally unipotent on W. Suppose that $[W, N] \neq O$. Then as X is irreducible on V, $W = [W, N]$. Let $0 \neq w \in W$. Then $w \in W = \langle \langle w^X \rangle, N]$ and so there exists $F \in \mathcal{F}$ with $[w \in [\langle w^F \rangle, F \cap N]$. Put $U \langle w^F \rangle$.

Then $w \in [U, F\cap N]$ ans since F normalizes $[U, F\cap N]$, $U = [U, F\cap N]$. But $F\cap N$ acts locally unipotent on U and we obtain a contradiction to 18.5. Hence N centralizes W.

Conversely let N be a subgroup of X which centralizes all the factor of $\mathcal{D}$. Let U be a finite dimensional subspace of V and F a finitely generated subgroup of N. Put $W = U[V, F]$. Then W is finite dimensional and F invariant. Moreover F centralizes all factors of the series $\mathcal{D}\cap W$ and so F avts unipotently on W. Thus N acts locally unipotently on V □

Lemma 18.7 *Let $H \leq X$ so that $d = \dim_K[V, H]$ is finite and put $L = \langle H^X\rangle$.* sbbd

(a) *Let $\mathcal{C}$ be a series for KX on V. Then at most d factors of L are non centralized by L.*

(b) *There exists a KX series $\mathcal{D}$ on V so that*

 (a) $|\mathcal{D}| \leq 2d+2$.

 (b) *If W is a factor of $\mathcal{D}$, then either L act trivially on W or W is irreducible as KX-module.*

Proof: Let $E = [V, H]$ and let $\mathcal{C}$ be any series for KX on V. Then $\mathcal{C}\cap E$ is a finite series

$$1 = E_0 < E_1 < \dots E_r = E$$

with $r \leq d$. For $0 \leq i \leq r$, put $T_i = \bigcap\{C \in \mathcal{C} \mid C \cap E = E_i$ and $B_{i+1} = \bigcup\{C \in \mathcal{C} \mid C\cap E = E_{i+1}$. Then

$$1 = T_0 \leq B_1 < T_1 \leq B_2 \dots B_r < T_r \dots B_{r+1} = V$$

is an KX-series on V. We claim that for all $1 \leq i \leq r$, (B_i, T_i) is a jump of $\mathcal{C}$. For this note first that $E \cap B_i = E_{i-1}$ and $E\cap T_i = E_i$. Let $C \in \mathcal{C}$ with $B_i \leq C \leq T_i$. Then $E_{i-1} \leq E\cap C \leq E_i$. As (E_{i-1}, E_i) is a jump of $E\cap\mathcal{C}$ we comclude $E\cap C = E_{i-1}$ or $E_\cap C = E_i$. In the first case $E \leq B_i$ and in the second $T_i \leq C$. Thus $C = B_i$ or T_i.

Thus (B_i, T_i) is indeed a jump and so T_i/B_i is a factor of $\mathcal{C}$. Next we show that for all $1 \leq i \leq r$, L centralizes B_i/T_{i-1}. For this note that $[B_i, H] \leq B_i \cap E \leq E_{i-1} \leq T_{i-1}$. Hence H and so also L acts trivally on B_i/T_{i-1}. It follows that all factors of $\mathcal{C}$ other than the (B_i, T_i) are centralized by L. Thus (a) holds. Choosing $\mathcal{C}$ to be a composition series, we get that T_i/B_i is irreducible as KX-module and so also (b) is proved. □

Corollary 18.8 *Let $H \leq X$ so that $d = \dim_K[V, H]$ is finite and put $L = \langle H^X\rangle$ and $\mathcal{LU}(L)$ is unipotent of degree at most $2d+1$ and so also nilpotent of class at most $2d$.* untx

Proof: By 18.7 there exists a series of length at most 2d+1 so that each factor of is either irreducible as KX-module or centralized by L. Hence by 18.6 $\mathcal{LU}(L)$ centralizes all the factors and so is unipotent of degree at most 2d+1. Thus by 15.9 $\mathcal{LU}(L)$ is nilpotent of class at most $2d$. □

Lemma 18.9 *Let $x, y \in X$. Then $\deg_V[x,y] \leq 2 \cdot x$.* degcom

Proof: $[x,y] = x^{-1}x^y$ and so $[V,[x,y]] \leq [V,x] + [V,x^y]$ □

degprp **Lemma 18.10** *Let H be a primitive of $\mathrm{Sym}(\Omega)$ with $\mathrm{Sol}(H) \neq 1$ Then $|\Omega| \leq 2|\operatorname{Supp}_\Omega(h)$ for all $1 \neq h \in H$.*

Proof: Let $1 \neq A$ be an abelian normal subgroup of H. An orbit of A is a block for H, A is transitive. Let $\omega \in \Omega$. By a Frattini argument, $H = H_\omega A$. Also $C_{H_\omega}(A)$ fixes $\Omega = \omega^A$, elementwise and so $C_{H_\omega}(A) = 1$. In particular, $A_\omega = 1$ and so $\mathrm{Supp}(a) = \Omega$ for all $1 \neq a \in A$. So we may assume that $h \notin A$. Now $C_H(A) = C_{H_\omega}(A)A = A$ and so ther exists $b \in A$ with $a = [b,h] \neq 1$. Thus $|Omega| = |Supp_\Omega(a)| \leq 2|\operatorname{Supp}_\Omega(h)$. □

crsu **Lemma 18.11** *Let H be a solvable group. Then H has a characteristic subgroup N with $C_G(N) \leq N$ such that N is nilpotent of class at most two.*

Proof: By Zorn's Lemma H has a maximal characteristic, abelian normal subgroup A. Next let N/A be maximal characteristic, abelian normal subgroup of $C_N(A)/A$. Then N is characteristic in H. Suppose that $B = C_H(N) \not\leq N$. Since Since $A \leq B \cap N \leq Z(B)$, the maximality of A implies $A = B \cap N$. Let E/A be last non-trivial term of the derived series of B/A. Then $[NE, NE] = N'[N,E]E' \leq A$ and the maximality of N/A implies $E \leq N$ and so $E \leq A$, a contradiction. □

degsol **Lemma 18.12** *Suppose that X is solvable and acts irreducible on V and that V is finite dimensional.*

(a) *If X is primitive then $\dim_K V \leq 4\deg_V(t)$ for all $1 \neq t \in X$.*

(b) *If X is generated by elements of degree less or equal to d, then $\dim_K V \leq 4d$.*

Proof: (a) By 18.11 there exists a normal subgroup N of X so that $C_X(N) \leq N$ and N is nilpotent of class at most N. Let $1 \neq a \in Z(N)$. Since X is primitive, $Z(N)$ is homogenous on V and so $V = [V,a]$. By **??** $a \in \langle t^{x_i} \leq i \leq 4\rangle$, where $x_i \in X$. Hence $\deg_V(a) \leq 4\deg_V(t)$ and so (a) is proved.

(b) By (a) we may assume that X acts imprimitively on V. Let Δ be a block system for X on V with $|\Delta|$ minimal. Then X acts primitively on Δ. As X is generated by elements of degree less r equal to d, there exists $t \in X$ with $\deg_V(t) \leq d$ acting non-trivially on Δ. By 18.10 $|\Delta| \leq 2|\operatorname{Supp}_\Delta(t)$. Let $k = \dim_K W, W \in \Delta$ By 18.3c, $k|Supp_\Omega(t) \leq 2\deg_V(t)$ and so

$$\dim_K V = k|\Delta| \leq k2 \cdot |\operatorname{Supp}_\Delta(t)| \leq 4\deg_V(t). \square$$

XY **Lemma 18.13** *Suppose that $n\dim_K(V)$ is finite and that $Y \leq GL_K(V)$ with $[X,Y] \leq K^\#$. Then $|X/C_X(Y)| \leq n^2$.*

Proof: Let T be a transversal to $C_X(Y)$ in X. View X and Y has subsets of $\mathrm{End}_K(V)$ and note that $\mathrm{End}_K(V)$ is a n^2 dimensinal vector space over K. Hence it suffices to show that T is linear independent over K. Otherwise let M be a minimal linear independent subset of T. Since none of the elements of T are 0, $|M|$ contains at least two elemenst a and b. Since $C_X(Y)a \neq C_Y(t)b$, $ba^{-1} \not\leq C_Y(t)$, there exists $x \in X$ with $(ba^{-1})^x \neq ba^{-1}$ and so $[b,x] \neq [a,x]$. For $x \in X$ and $m \in M$ let $\lambda_m = [m,x] \in K$. Then $\lambda_m = m^{-1}m^x$ and

(1)m $$m^x = \lambda_m$$

As M is minimal linear dependent,

(2) $$0 = \sum_{m \in M} \mu_m m$$

for some non-zero $\mu_m \in K$. Conjuating (2) by x and using (1) we obtain

(3) $$0 = \sum_{m \in M} \mu_m \lambda_m m$$

Multilying (2) by λ_a we obtain

(4) $$0 = \sum_{m \in M} \mu_m \lambda_a m$$

Subtracting (4) from (3) we conclude

(5). $$0 = \sum_{m \in M} \mu_m (\lambda_m - \lambda_a) m$$

Since $\mu_a \mu_a (\lambda_a - \lambda_a) = 0$, the minimality of M implies $\mu_m(\lambda_m - \lambda_a) = 0$ for all m. But as $\mu_b \neq 0$ and $\lambda_a \neq \lambda_b$, $\mu_b(\lambda_b - \lambda_a) \neq 0$, a contradiction. □

Lemma 18.14 *Suppose that X is solvable and $n = \dim_K V$.* solbf

(a) *If K is algebraicly closed and X is primitive, then $|X/Z(X)| \leq n^2!$.*

(b) *X has as normal subgroup A so that $|X/A| \leq n^2!$ and A' acts unipotenly on V.*

Proof: (a) Let N a a normal subgroup of X so that $C_X(N) \leq N$ and N has class at most 2. Since X is primitive and K is algebraicly closed, $Z(N) \leq K^{\#}$ and $Z(N) \leq Z(X)$. Since N has class at most 2, $[N,N] \leq Z(N)$. Thus by 18.13, $N/Z(N)| \leq n^2$. Let $Y =$

$|C_X(N/Z(N)|$. Then $|X/Y| \leq (|N/Z(N)| - 1)! \leq (n^2 - 1)!$. Also $[N, Y] \leq F$ and so by 18.13, $|Y/C_Y(N)| \leq n^2$. But $C_Y(N) \leq Z(N) \leq Z(X)$ and so $|X/Z(X)| \leq n^2(n^2-1)! = n^2!$.

(b) Without loss K is algebraicly closed.Suppose that W is a proper KX-submodule and put $m = \dim_K W$ By induction on n there exist normal subgroups A_1 and A_2 so that $C_H(W) \leq A_1$, $C_H(V/W) \leq A_2$, A_1' act unipotently on W, A_2' acts unipotently on V/W, $|H/A_1 \leq m^2!$ and $|H/A_2| \leq (n - m)^2!$. Put $A = A_1 \cap A_2$. Then A' acts unipotently on W and V/W and so also on V. Futhermore, $|H/A| \leq |H/A_1||H/A_2| \leq m^2!(n - m)^2! \leq n^2!$.

Suppose next that X acts irreducible on V and that Δ is a block system for H on V. Let $W \in \Delta$ and let $k = \dim_K W$. Since H has no non-trivial unipotent normal we conclude by induction that again by induction there exists exists normal subgroups B of $C_H(\Delta)$ so that $C_H(W) \cap C_H(\Delta) \leq B$, B' centralizes W and $|C_H(\Delta)/B \leq k^2!$. Let $m = |\Delta| = \frac{n}{k}$ and $A = \bigcap_{h \in H} B^h$. Then $|C_H(\Delta)/A| \leq (k^2!)^m$, $|H/C_H(\Delta)| \leq m!$ and $|H/A| \leq (k^2!)^m m! \leq (mk)^2! = n^2$.

Finally if X is primitive (b) follows from (a). □

bodeso **Corollary 18.15** *If X is solvable and V is finite dimensional, then the derived length of X is bounded by a function of* $\dim_K V$.

Proof: Since $\mathrm{Unip}(X)$ is nilpotent of class at most $\dim_K V$, this follows directly from 18.14 □

lsolnc **Theorem 18.16** *Let d be a positive integer and suppose that X is generated be elements of degree at most d. Then* $\mathrm{LSol}(X)/\mathcal{LU}(X)$ *is solvable of derived length bounded by a function of d.*

Proof: Let $L = \mathrm{LSol}(X), U = calLU(X) \leq L$ and $I = \{[l, x] \mid l \in l \in L, x \in X, \deg_V(x) \leq d$. Then $\deg_V i \leq 2d$ for all $i \in I$ and $H = \langle I \rangle [L, X]$. Since $L' \leq H$ it suffices to bound the derived length of HU/U in terms of d. By 18.15 there exists an postive integer r so that if F a linear groups of degree at most $8d$, then $F^{(}r)$ is unipotent. We will shot that $H^{(r)}$ is locally unipotent.

For this let J be a finite subset of I and put $H_J = \langle J \rangle$. Then $[V, X_J]$ is finite dimensional. Let W be any composition factor for H_J on W. Then $[W, H_J] = 0$ or $W = [W, H_J]$. Hence in any case W is finite dimensional. Since H_J is generated by elements of degree at most $2d$ on W we conclude from 18.12 that W has dimension at most $8d$. So $X_J^{(r)}$ is unipotent on W and so on all V. If $J^* \subset J$, then $H^{(}_{J^*}r) \leq H^{(}_{J}r)$. Since the H_J's form a local system for H, the $H^{(}_{J}r)$ form a local system for $H^{(r)}$. Thus $H^{(r)}$ is indeed locally unipotent. □

bosole **Corollary 18.17**

(a) *Let $H \leq X$ with $[V, H]$ finite dimensional. Then* $\mathrm{LSol}(\langle H^X \rangle)$ *is solvable of derived length bounded by a function of* $\dim_K[V, H]$.

(b) $\mathrm{LSol}(X)/calLU(X)$ *is the union of a countable ascending chain of solvable X-invariant subgroups.*

Proof: (a) follows from 18.16 and 18.8. (b) Let $X_i = \langle x \in X \mid \deg_V(x) \leq i \langle$. Then by 18.16, $\mathrm{LSol}(X_i)$ is unipotent by solvable. Also $\mathrm{LSol}(X_i) = \bigcup_{i=1}^{\infty} \mathrm{LSol}(X_i 0$ and so (b) holds. □

Lemma 18.18 *Let T, S be a finite groups, p a prime and T_p and S_p Sylow p-subgroups of T and S respectively. Let S be acting on the finite set I. Then $T_p \wr S_p$ is a Sylow p-subgroup of $T \wr S$.* sylwr

Proof: Clear by an order argument. □

Lemma 18.19 *Let n be a positive integer, p a prime, and $S_{n,p}$ a Sylow p-subgroups of* $\mathrm{Sym}(n)$. sylsym

(a) *If S is a primitive p-subgroup of* $\mathrm{Sym}(n)$, *then $|S| = n = p$.*

(b) *If $n = p^k$, then $S(p^k, p)$, is transitive and $S_{p^k,p} \cong S_{p^{k-1}} \wr C_p$.*

(c) *Let $n = \sum_{i=1}^{k} n_i p^i$ with $0 \leq i < p$. Then $S_{n,p} \cong \oplus S_{p^i,p}^{n_i}$.*

(d) *S_n has derived length $\lfloor \log_p n \rfloor$.*

(e) *Let $y \in \mathrm{Sym}(n)$ with $|y| = p$. Then there exists a p-subgroup S of $\mathrm{Sym}(n)$ with $y \in S$ and $\mathrm{der}\langle y^S \rangle \geq \lfloor \log_p \frac{\mathrm{Supp}(y)}{p} \rfloor$.*

Proof: (a) Let $A \leq Z(S)$ with $|A| = p$. Then $A \trianglelefteq S$ and as S is primitive, A is transitive. Thus $n = p$ and (a) holds. (b) Since $\mathrm{Sym}(p^k)$ contains a p^k-cycle, $S(p^k, p)$ is transitive. Let $\mathcal{D}$ be a system of blocks for $S(p^k, p)$ with $|\mathcal{D}|$ minimal. Then by (a), $|\mathcal{D}| = p$. Let $\Delta \in \mathcal{D}$. Then $S(p^k, p$ is a Sylow p-subgroup of $N_{\mathrm{Sym}(p^k)}(\mathcal{D}) \cong Sym(\Delta) \wr \mathrm{Sym}(\mathcal{D})$. So (b) follows from 18.18.

(c) Let $\Omega_1(i) \ldots \Omega_{m_i}(i)$ be the orbits of $S_{n,p}$ of length p^i. Then $S(n,p) \leq \oplus_{i,j} \mathrm{Sym}(\Omega_j(i)$ $leq\, \mathrm{Sym}(n)$. Let $T_j(i)$ be the projection of $S(n,p)$ onto $\mathrm{Sym}(\Omega_j(i)$ onto $\mathrm{Sym}(\Omega_j(i)$ and let $S_j(i)$ be a Sylow p-subgroup of $\mathrm{Sym}(\Omega_j(i)$ with $T_j(i) \leq S_j(i)$. Then $S(n,p) \leq \oplus_{i,j} T_j(i)$ and as the latter group is a p-subgroup of $\mathrm{Sym}(n)$, $S(n,p) = \oplus_{i,j} T_j(i)$. Suppose that $m_i \geq p$ for some i. Then $\oplus_{j=1}^{p} T_j(i)$ is a Sylow p-subgroup of $\mathrm{Sym}(\bigcup_{j=1}^{p} \Omega_j(i)) = \mathrm{Sym}(p^{i+1}$, which is not transitive, a contradcition to (b). Hence $m_i < p$ and as $\sum_i m_i p^i = n$, $m_i = n_i$. Thus (c) holds.

(d) Choose k as in (c) with k minimal. Then $n_k \neq 0$ and $k = \lfloor \log_p n \rfloor$. Then by (c), $S_{n,p}$ and $S_{p^k,p}$ have the same derived length. By (b) $S_{p^k,p} \cong S_{p^{k-1},p} \wr C_p$. Also $(S_{p^{k-1},n} \wr C_p)'$ is a subgroup of the base group $S_{p^{k-1}}^{p}$ which projects onto $S_{p^{k-1}}$. Hence $\mathrm{der}(S_{p^{k-1},n} \wr C_p)' \, \mathrm{der}\, S_{p^{k-1},n} = k-1$, by induction. So $\mathrm{der}\, S_{p^k,p} = (k-1)+1 = k$.

(e) Let $\Delta_1, \ldots \Delta_r$ be the non-trivial orbits for y. Then $r = \frac{|\mathrm{Supp}(y)|}{p}$. Pick $x_i \in \Delta_i$ and let $\Delta = \{x_i \mid 1 \leq i \leq r\}$. Let $1 \leq j < p$. Then $x_i \neq x_i^{y^j} \in \Delta_i$ and so $\Delta \cap \Delta^{x^j} = \emptyset$. Thus Δ is block for y. Let T_0 be a Sylow p-subgroup of $\mathrm{Sym}(\Delta)$. Let $T_j = T_0^{x^j}$. Then

$T_j \in \mathrm{Sym}(\Delta^{x^j})$ and $T = \langle T_j \mid o \leq j < p \rangle \cong T_0^p$. Hence T is a p group and y normalizes T. Hence $S = T\langle y \rangle$ is a p-subgroup of $\mathrm{Sym}(n)$ containg y. Moreover $[T, y] \leq \langle y^T \rangle \leq \langle y^S \rangle$. Let $t \in T$. Then $[t^{-1}, y] = tt^{-y}$ acts as t on Δ and so $\mathrm{der}\langle y^S \rangle \geq \mathrm{der}[T, y] \geq \mathrm{der}\, T$. By (d) $\mathrm{der}\, T = \log_p |\Delta| = \log_p r$ and (e) is proved. □

lfsfa **Theorem 18.20** *Let G be a locally finite, simple, finitary group which has a Kegel cover all of whose factors are alternating groups. Then G is an alternating group.*

Proof: Let p be an odd prime so that G has an element x of order p. By 18.17 there exists a postive integer r so that

lfsfa-1 **1.** $\langle x^S \rangle^{(r)} = 1$ for all p subgroups S of G with $x \in S$.

Let $\{(H_i, M_i) \mid i \in I\}$ be a Kegel cover for G with $\bar{H}_i = H_i/M_i \cong \mathrm{Alt}(\Omega_i)$. Let $J = \{i \in I \mid i \in H_i, x \notin M_i\}$.

lfsfa-2 **2.** Let $j \in J$ and $\bar{S}$ a p-subgroup of $\bar{H}_j$ with $\bar{x} \in \bar{S}$. Then $\mathrm{der}\langle \bar{x}^{\bar{S}} \rangle \leq r$.

For this let R be the inverse image of $\bar{S}$ in H_j. Also let S be a Sylow p-subgroup of R with $x \in S$. As $R/M_j = \bar{S}$ is a p-groups, $R = SM_j$. Hence $\mathrm{der}\langle \bar{x}^{\bar{S}} \rangle\, \mathrm{der}\langle x^S \rangle M_j/M_j \leq \mathrm{der}\langle x^S \rangle$ and so 2. follows from **??**.

lfsfa-3 **3.** Let $j \in J$. Then $|\,\mathrm{Supp}_{\Omega_j}|(x) < p^{r+2}$.

Suppose $|\,\mathrm{Supp}_{\Omega_j}|(x) \geq p^{r+2}$. Then by 18.19e there exists a p-subgroup $\bar{S}$ of $\bar{H}_j \cong \mathrm{Alt}(\Omega_j)$ with $\bar{x} \in \bar{S}$ and $\mathrm{der}\langle \bar{x}^{\bar{S}} \rangle \geq r + 1$. But this contradicts 2..

lfsfa-4 **4.** G is a group of finitary permutations.

Let $\mathcal{D}$ be an ultrafilter on I with $J \in \mathcal{D}$. Let $\Omega = \prod_{\mathcal{D}} \Omega_i$. Then G acts on Ω. Since $J \in \mathcal{D}$, x acts non-trivially on Ω. As G is simple, G acts faithfully on Ω. By 3. and as $J \in \mathcal{D}$ we conclude that $|\,\mathrm{Supp}_\Omega(x)| < p^{r+2}$. Hence $x \in \mathrm{FSym}(\Omega)$. Since $G = \langle x^G \rangle$ and $\mathrm{FSym}(\Omega)$ is a normal subgroup of $\mathrm{Sym}(\Omega)$, $G \leq \mathrm{FSym}(\Omega)$.

The theorem now is an immediate consequence of 4. and 9.14. □

Chapter 19

Elementary equivalence

Chapter 20

Exercises

exercises

5. Let M be a minimal normal subgroup of G. If M is locally solvable show that M is elementary abelian (i.e M is abelian and there exists a prime p with $m^p = 1$ for all $m \in M$). elsmn

6. Let G be a universal, locally finite group group and H a countable, locally finite group. Prove that G contains a subgroup isomorphic to H. ehiug

7. Let G be a group and N a normal subgroup of G. If N and G/N are locally finite, prove that G is locally finite. elfblf

8. Let G be an LFS-group such that G has a Kegel cover all of whose factors are alternating groups. Let $x \in G$ such that $C_G(x)$ is abelian. eac

(a) If G is finite determine all possibilities for G and x.

(b) Prove that G is finite.

9. Let U be a P. Hall's universal group. Show that U is simple and explicitly write down a Kegel cover U.

10. Let G be any of non-absolutely simple LFS-groups constructed in section 6. Determine a Kegel-cover for G. ekn

11. Let V be a finite dimensional vector space over the field K and p a prime. Investigate the maximal p-subgroups of $GL_K(V)$. emaxplin

12. Let N be a minimal normal subgroup of G. What can you say about the structure of G? eminnor

13. Construct a locally finite simple group with a maximal subgroup which is a p-group. emaxp

14. Let X be a primitive subgroup of $\mathrm{Sym}(\Omega)$ containing a 3-cycle. Prove that $\mathrm{Alt}(\Omega) \leq$ e3cycle
X.

15. Let K be an infinite, locally finite field. Determine under all pairs (F, K) where F emssl2
is a finite subfield of K so that the subgroup $SL_2(F)$ of $SL_2(K)$ is not contained in any maximal subgroup of $SL_2(K)$.

eplc **16.** Prove that a periodic, linear group with no non-trivial, unipotent, normal subgroups is countable.

euifp **17.** Show that there are uncountably many isomorphism classes of transitive, finitary permutations groups.

euup **18.** Given a set I, a non-principal ultra-filter $\mathcal{D}$ on I and non-empty finite sets $A_i, i \in I$. Suppose that for all positive integers n,

$$\{i \in I \mid |A_i| \leq n\} \notin \mathcal{D}.$$

Also suppose that either all A_i are finite or that I is countable. Show that $\prod_{\mathcal{D}} A_i$ is uncountable.

ecwp **19.** Let A, B be groups such that B acts transitively on the set I. Let $H = A \wr_I B$. Determine H'.

eilsn **20.** Investigate G under the assumption that all infinite subgroups of G are locally subnormal.

ecnas **21.** Let G be any of the non-absolutely simple, locally finite simple group constructed in class and

$$1 = M_0 \trianglelefteq M_1 \trianglelefteq M_2 \trianglelefteq M_3 \ldots \leq G$$

the corresponding series.

(a) Explcitly determine a Kegel coverfor G.

(b) Show that for each $i \geq 0$, M_{i+1}/M_i is residually finite.

(c) Show that G has a composition series all of whose factors are finite.

emnsc **22.** Let N be a minimal normal subgroup of G. Show that N has a sectional cover all of whose factors are characteristically simple.

eqc **23.** Let G be a finite group, $M \trianglelefteq G$ and $x \in G$. Prove that $|C_{G/M}(x)| \leq |C_G(x)|$.

efc **24.** Let G be a locally finite simple group with a Kegel cover all of whose factors are alternating groups. Let $x \in G$ with $C_G(x)$ finite. Prove that G is finite.

djckc **Definition 20.1** *Let K be field, X a group and V an irreducible KX-module. Let $D = \mathrm{End}_{KV}(V)$. Then a JCK-cover for V with respect to X is a set of triples $\{(H_i, V_i, M_i) \mid i \in I\}$ such that for all i in I:*

(a) *H_i is a finitely generated subgroup of X.*

(b) *V_i is a finitely generated DH_i-submodule of V.*

(c) *M_i is a maximal DH_i-submodule of V_i.*

(d) *For each finite dimensional D-subspace W of V, there exists $j \in I$ so that $W \leq V_j$ and $W \cap M_j = 0$.*

25. Let K,X and V as in the above defintion. Show that there exists a JCK-cover for ejckc
V with respect to X. (Hint: Use the usual argument for Kegel covers and the Jacobson-Chevalley Density Theorem.)

26. Let $\mathcal{K}$ be a set of finite sets and $\mathcal{J}$ a partion of $\mathcal{K}$ into finite subsets. For $J \in \mathcal{J}$ put ediag
$T_J = \prod J$. Show that $\mathrm{Diag}(\prod \mathcal{K}) \cong \mathrm{Diag}(\prod_{J \in \mathcal{J}} T_J)$.

27. Let $\mathcal{K}$ and $\mathcal{J}$ be countable sets of finite sets. Determine necessary and sufficient ediis
conditions for $\mathrm{Diag}(\prod \mathcal{K})$ and $\mathrm{Diag}(\prod \mathcal{J})$ to be isomorphic as abstract groups.

28. Let I be a set, $\mathcal{A}$ a subalgebra of $\mathcal{P}(I)$ with $I \in \mathcal{A}$, H a non-abelian simple group erha
and $R = R(H, \mathcal{A})$. Let M be a maximal normal subgroup of R. Show that $R = MR_I$ and conclude that $R/M \cong H$.

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Index