Review Problems for the MTH132 Final/Solutions

#1. Compute the following limits.

(a) $\lim\limits_{x\to -2} \dfrac{x^2-4}{x+2}$

(b) $\lim\limits_{x\to 0} x\cot x$

(c) $\lim\limits_{x\to 1^-} \dfrac{x}{|x-1|}$

(d) $\lim\limits_{x\to 1^+} \dfrac{\sqrt{x^2-2x+1}}{x-1}$

(e) $\lim\limits_{x\to \infty} \dfrac{x^2+2x+7}{2x^2-2x+1}$.

(a) $\lim\limits_{x\to -2} \dfrac{x^2-4}{x+2} = \lim\limits_{x\to -2} \dfrac{(x-2)(x+2)}{x+2} = \lim\limits_{x\to -2} x-2 = -2-2 = \boxed{-4}$.

(b) $\lim\limits_{x\to 0} x\cot x = \lim\limits_{x\to 0} x\dfrac{\cos x}{\sin x} = \dfrac{1}{\lim\limits_{x\to 0}\dfrac{\sin x}{x}} \lim\limits_{x\to 0}\cos x = \dfrac{1}{1}\cdot\cos 0 = 1\cdot 1 = \boxed{1}$.

(c) If $x<1$, then $x-1<0$ and so $|x-1| = -(x-1) = 1-x$. Thus $\lim\limits_{x\to 1^-}\dfrac{x}{|x-1|} = \lim\limits_{x\to 1^-}\dfrac{x}{1-x} = \boxed{+\infty}$, where the last equality holds since for x close to 1 but smaller than one, $1-x$ is a very small positive number and x is close to 1. So $\frac{x}{1-x}$ is a very large positive number.

(d) $\lim\limits_{x\to 1^+} \dfrac{\sqrt{x^2-2x+1}}{x-1} = \lim\limits_{x\to 1^+} \dfrac{\sqrt{(x-1)^2}}{x-1} = \lim\limits_{x\to 1^+}\dfrac{x-1}{x-1} = \lim\limits_{x\to 1^+} 1 = \boxed{1}$.

(e) $\lim\limits_{x\to\infty} \dfrac{x^2+2x+7}{2x^2-2x+1} = \lim\limits_{x\to\infty}\dfrac{1+\frac{2}{x}+\frac{7}{x^2}}{2-\frac{2}{x}+\frac{1}{x^2}} = \dfrac{1+0+0}{2-0+0} = \boxed{\tfrac{1}{2}}$.

#2. Find the equation of the line normal to the graph of $f(x) = \sec x$ at the point whose x coordinate is $\frac{-\pi}{3}$.

Note that $f'(x) = \sec x\tan x = \frac{1}{\cos x}\frac{\sin x}{\cos x} = \frac{\sin x}{\cos^2 x}$. Also $\sin\left(\frac{-\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ and $\cos\left(\frac{-\pi}{3}\right) = \frac{1}{2}$. Thus

$$f'\left(\frac{-\pi}{3}\right) = \frac{\sin(\frac{-\pi}{3})}{\cos^2(\frac{-\pi}{3})} = \frac{-\frac{\sqrt{3}}{2}}{(\frac{1}{2})^2} = -\frac{2^2\sqrt{3}}{2} = -2\sqrt{3}.$$

Hence the tangent line has slope $-2\sqrt{3}$ and so the normal line has slope $-\frac{1}{-2\sqrt{3}} = \frac{1}{2\sqrt{3}}$. Also $f(\frac{-\pi}{3}) = \sec(\frac{-\pi}{3}) = \frac{1}{\frac{1}{2}} = 2$. So the normal lines goes through the point $(-\frac{\pi}{3}, 2)$ and fulfills the equation

$$\frac{y-2}{x-\frac{-\pi}{3}} = \frac{1}{2\sqrt{3}}$$

Hence the equation of the normal line is:

$$\boxed{y = \tfrac{1}{2\sqrt{3}}(x+\tfrac{\pi}{3}) + 2 = \tfrac{1}{2\sqrt{3}}x + (2+\tfrac{\pi}{6\sqrt{3}})}$$

#3. Find the equations of the lines (there are two of them) passing through $(-2,3)$ and tangent to the parabola $y=x^2$.

Let (a,a^2) be a point on the parabola. The line through (a,a^2) and $(-2,3)$ has slope

$$\frac{a^2-3}{a-(-2)} = \frac{a^2-3}{a+2}$$

Note that $(x^2)' = 2x$. So the tangent line to $y=x^2$ at the point (a,a^2) has slope

$$2a$$

So the line through (a, a^2) and $(-2, 3)$ will be tangent to $y = x^2$ if and only if

$$\frac{a^2 - 3}{a + 2} = 2a$$

This holds if and only if $a^2 - 3 = 2a(a+2) = 2a^2 + 4a$, if and only if $a^2 + 4a + 3 = 0$ and if and only if $(a+1)(a+3) = 0$. So $a = -1$ or $a = -3$.

If $a = -1$, the tangent line has slope -2 and goes through $(-1, 1)$. Thus its equation is

$$\frac{y - 1}{x - (-1)} = -2 \text{ and } \boxed{y = -2(x+1) + 1 = -2x - 1}$$

If $a = -3$, the tangent line has slope -6 and goes through $(-3, 9)$. Thus its equation is

$$\frac{y - 9}{x - (-3)} = -6 \text{ and } \boxed{y = -6(x+3) + 9 = -6x - 9}$$

#4. Use the Intermediate Value Theorem to show that $\cos x = x^2$ has a solution.

Let $f(x) = \cos x - x^2$. Then f is a continuous function. We have

$$f(0) = \cos 0 - 0^2 = 1 - 0 = 1$$

and

$$f(2) = \cos 2 - 2^2 \leq 1 - 4 = -3$$

Thus 0 lies between $f(0)$ and $f(2)$. The Intermediate Value Theorem shows that $f(c) = 0$ for some c in $[0, 2]$. Thus $\cos c - c^2 = 0$ and $\cos c = c^2$. So c is a solution of $\cos x = x^2$.

#5. Calculate the derivative of $f(x) = \sqrt{3x+1}$ directly from the definition of derivative.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &&= \lim_{h\to 0} \frac{\sqrt{3(x+h)+1} - \sqrt{3x+1}}{h} \\
&= \lim_{h\to 0} \frac{(\sqrt{3x+3h+1} - \sqrt{3x+1})(\sqrt{3x+3h+1} + \sqrt{3x+1})}{h(\sqrt{3x+3h+1} + \sqrt{3x+1})} &&= \lim_{h\to 0} \frac{\sqrt{3x+3h+1}^2 - \sqrt{3x+1}^2}{h(\sqrt{3x+3h+1} + \sqrt{3x+1})} \\
&= \lim_{h\to 0} \frac{(3x+3h+1) - (3x+1)}{h(\sqrt{3x+3h+1} + \sqrt{3x+1})} &&= \lim_{h\to 0} \frac{3h}{h(\sqrt{3x+3h+1} + \sqrt{3x+1})} \\
&= \lim_{h\to 0} \frac{3}{\sqrt{3x+3h+1} + \sqrt{3x+1}} &&= \frac{3}{\sqrt{3x+1} + \sqrt{3x+1}} \\
&= \boxed{\frac{3}{2\sqrt{3x+1}}}
\end{aligned}$$

#6. Let $f(x) = x^{\frac{1}{3}} + \frac{1}{x-2}$.

(a) Find the domain of f.

(b) For which values of x is f continuous at x.

(c) For which values of x is f differentiable at x.

(a) $x^{\frac{1}{3}} = \sqrt[3]{x}$ is defined for all x, $\frac{1}{x-2}$ is defined for all x with $x - 2 \neq 0$. So the domain of $f(x)$ consists of all real numbers except 2.

(b) $f(x)$ is continuous at all points in its domain, that is at all $x \neq 2$.

(c) $f'(x) = \left(x^{\frac{1}{3}} - (x-2)^{-1}\right)' = \frac{1}{3}x^{-\frac{2}{3}} - (x-2)^{-2} = \frac{1}{\sqrt[3]{x^2}} - \frac{1}{(x-2)^2}$. So $f(x)$ is differentiable at all x except $x = 0$ and $x = 2$.

#7. Let $f(x) = \frac{x^2}{(x-1)^2}$.

(a) Find the horizontal and vertical asymptotes of the graph of f.

(b) Find the critical points and the intervals where f is decreasing and increasing.

(c) Find the intervals where f is concave upward and the intervals where f is concave downward.

(d) Sketch the graph using the information obtained in a)-c).

(a) Since $\lim_{x\to 1} \dfrac{x^2}{(x-1)^2} = +\infty$, $x = 1$ is a vertical asymptote. As $f(x)$ is continuous at all $x \neq 1$ this is the only vertical asymptote.

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^2}{(x-1)^2} = \lim_{x\to\pm\infty} \frac{x^2}{x^2 - 2x + 1} = \lim_{x\to\pm\infty} \frac{1}{1 - \frac{2}{x} + \frac{1}{x^2}} = \frac{1}{1} = 1$$

and so $y = 1$ is a horizontal asymptote.

(b)

$$\begin{aligned} f'(x) &= \tfrac{2x(x-1)^2 - x^2(2(x-1))}{(x-1)^4} \qquad \text{note: common factor } 2x(x-1) \\ &= \tfrac{2x(x-1)(x-1) - 2x(x-1)x}{(x-1)^4} \\ &= \tfrac{2x(x-1)(x-1-x)}{(x-1)^4} \\ &= -\tfrac{2x(x-1)}{(x-1)^4} \\ &= -2\tfrac{x}{(x-1)^3} \end{aligned}$$

$f'(x) = 0$ at $x = 0$

$f'(x)$ is not defined at $x = 1$.

So the critical points are at $x = 0$ and $x = 1$.

$f(0) = 0$, $f(1)$ is not defined.

	$x < 0$	$0 < x < 1$	$1 < x$
-2	$-$	$-$	$-$
x	$-$	$+$	$+$
$(x-1)^3$	$-$	$-$	$+$
f'	$-$	$+$	$-$
f	dec	inc	dec

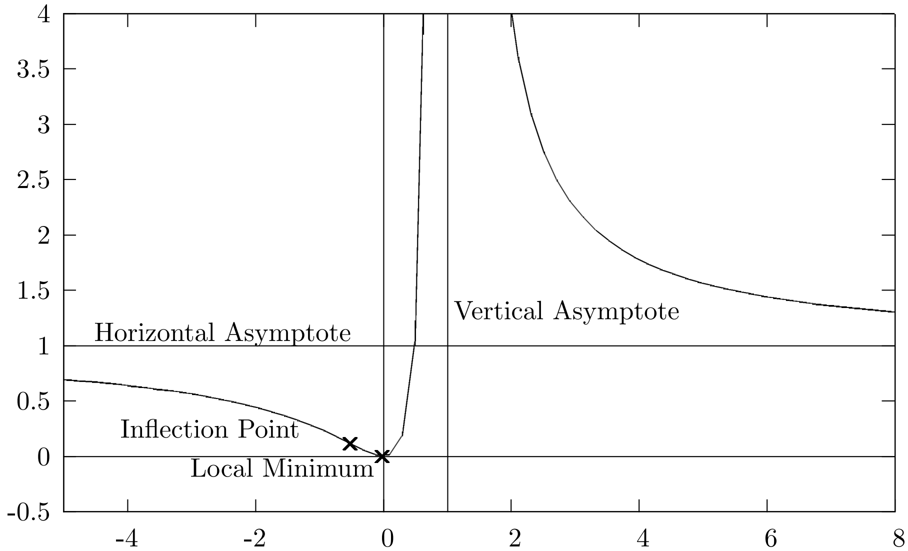
f is decreasing on $(-\infty, 0]$ and $(1, \infty)$

f is increasing on $[0, 1)$.

(c) $f''(x) = -2\frac{1(x-1)^3 - x(3(x-1)^2)}{(x-1)^6} = -2\frac{(x-1)^2(x-1-3x)}{(x-1)^6} = -2\frac{(x-1)^2(-2x-1)}{(x-1)^6} = 4\frac{x+\frac{1}{2}}{(x-1)^4}$.
So $f''(x) = 0$ at $x = -\frac{1}{2}$ and f'' is not defined at $x = 1$.

	$x < -\frac{1}{2}$	$-\frac{1}{2} < x < 1$	$1 < x$
$(x-1)^4$	$+$	$+$	$+$
$x + \frac{1}{2}$	$-$	$+$	$+$
f''	$-$	$+$	$+$
f	con.down	con. up	con. up

Inflection point at $x = -\frac{1}{2}$ and $y = \frac{(-\frac{1}{2})^2}{(-\frac{1}{2}-1)^2} = \frac{\frac{1}{4}}{(-\frac{3}{2})^2} = \frac{\frac{1}{4}}{\frac{9}{4}} = \frac{1}{9}$.
f is concave up on $[-\frac{1}{2}, 1)$ and on $(1, \infty)$.
f is concave down on $(-\infty, -\frac{1}{2}]$.



(d)

#8. State the Mean Value Theorem and use it to show that if $f(0) = 1$ and $f'(x) > 0$ for all x, then $f(x) > 1$ for all $x > 0$.

Mean Value Theorem: Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there exists a point c in (a, b) with

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Suppose now that f is a function with $f(0) = 1$ and $f'(x) > 0$ for all x. Then f is differentiable at all x and so also continuous at all x. Thus the assumptions of the Mean Value Theorem are fulfilled. Let $x > 0$. Then by the Mean Value Theorem (applied with $a = 0$ and $b = x$), there exists c in $(0, x)$ with

$$f'(c) = \frac{f(x) - f(0)}{x - 0}$$

Thus $f(x) - f(0) = xf'(c)$. By assumption $f'(c) > 0$ and $x > 0$. Hence $xf'(c) > 0$.
Thus $f(x) - f(0) > 0$ and so $f(x) > f(0)$. Since $f(0) = 1$ this gives $f(x) > 1$.

#9.

(a) Find $\frac{d^2y}{dx^2}$ if $y = x^{\frac{1}{2}} - \sqrt{x^2+1}$.

(b) Find $f'(x)$ if $f(x) = \frac{x^2+1}{\tan x}$.

(c) Find $\frac{dy}{dx}$ and an equation for the tangent line to the graph of $x^3y + xy^3 = 2$ at $(1,1)$.

(d) Find $\frac{d^2}{dx^2}\left(\int_1^{x^2} \sec t\, dt\right)$.

(a) $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}(x^2+1)^{-\frac{1}{2}}2x = \frac{1}{2}x^{-\frac{1}{2}} - x(x^2+1)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} - \frac{x}{\sqrt{x^2+1}}$
$\frac{d^2y}{dx^2} = \frac{1}{2}\frac{-1}{2}x^{-\frac{3}{2}} - \left(1\cdot(x^2+1)^{-\frac{1}{2}} + x\frac{-1}{2}(x^2+1)^{-\frac{3}{2}}2x\right) = -\frac{1}{4\sqrt{x^3}} - \frac{1}{\sqrt{x^2+1}} + \frac{x^2}{\sqrt{(x^2+1)^3}} = \frac{1}{4\sqrt{x^3}} + \frac{-(x^2+1)+x^2}{\sqrt{x^2+1}^3} =$
$\boxed{\frac{1}{4\sqrt{x^3}} - \frac{1}{\sqrt{x^2+1}^3}}$.

(b) $f'(x) = \boxed{\frac{2x\tan x - (x^2+1)\sec^2 x}{\tan^2 x}}$.

(c) Differentiating both sides of $x^3y + xy^3 = 2$ with respect to x gives:

$$(3x^2y + x^3y') + (1\cdot y^3 + x\cdot 3y^2y') = 0$$

and so

$$(x^3 + 3xy^2)y' = -(3x^2y + y^3)$$

and

$$\frac{dy}{dx} = y' = \boxed{-\frac{3x^2y+y^3}{x^3+3xy^2}}$$

If $x = y = 1$ we get

$$y' = -\frac{3+1}{1+3} = -1$$

Thus the equation of the tangent line at the point $(1,1)$ is

$$\frac{y-1}{x-1} = -1 \text{ and so } y = -(x-1)+1 = \boxed{-x+2}$$

(d) Put $u = x^2$ and $y = \int_1^{x^2} \sec t dt = \int_1^u \sec t dt$. By the Fundamental Theorem of Calculus Part I, $\frac{dy}{du} = \frac{d}{du}\int_1^u \sec t dt = \sec u = \sec x^2$. Also $\frac{du}{dx} = \frac{dx^2}{dx} = 2x$. So by the Chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \sec x^2 \cdot 2x = \boxed{2x\sec x^2}$$

So

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(2x\sec(x^2)) = 2\sec(x^2) + 2x\sec(x^2)\tan(x^2)2x = \boxed{2\sec(x^2) + 4x^2\sec(x^2)\tan(x^2)}$$

#10. The velocity of a particle moving along the x-axis is given by $v(t) = t^2 - 3t + 2$.

(a) Find the acceleration.

(b) Its position is 4 for $t = 0$. Find the position at the time t.

(c) Find the maximum position for t in $[0,2]$.

(a) $a(t) = v'(t) = \boxed{2t-3}$.
(b) $s(t)$ is an antiderivative of $v(t)$ so $s(t) = \frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t + C$. Since $s(0) = 4$, $C = 4$ and so

$$s(t) = \boxed{\frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t + 4}.$$

(c) $v(t) = t^2 - 3t + 2 = (t-1)(t-2)$. So $v(t) = 0$ at $t = 1$ and $t = 2$. Hence $t = 1$ is the only critical point in $(0, 2)$. We have

$$\begin{aligned}
s(0) &= 4 \\
s(1) &= \tfrac{1}{3} - \tfrac{3}{2} + 2 + 4 &&= \tfrac{1}{3} + \tfrac{9}{2} &&= \tfrac{2+27}{6} &&= \tfrac{29}{6} &&= 4\tfrac{5}{6} \\
s(2) &= \tfrac{1}{3}2^3 - \tfrac{3}{2}2^2 + 2\cdot 2 + 4 &&= \tfrac{8}{3} - 6 + 4 &&= \tfrac{8}{3} - 2 &&= \tfrac{8-6}{3} &&= \tfrac{2}{3}
\end{aligned}$$

Thus the maximum position is $\boxed{4\frac{5}{6}}$ at $t = 1$.

#11. Solve the following initial value problems.

(a) $y' = x + 1, y(0) = 1$.

(b) $y' = \sqrt{x}, y(1) = 3$.

(c) $y' = x\sqrt{x^2+1}, y(0) = -1$.

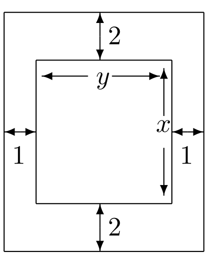
(a) $y = \frac{1}{2}x^2 + x + C$, $1 = y(0) = C$ and so $\boxed{y = \frac{1}{2}x^2 + x + 1}$.
(b) $y = \frac{2}{3}x^{\frac{3}{2}} + C$, $3 = y(1) = \frac{2}{3} + C$, $C = 3 - \frac{2}{3} = \frac{7}{3}$ and so $\boxed{y = \frac{2}{3}x^{\frac{3}{2}} + \frac{7}{3}.}$
(c) We first compute $\int x\sqrt{x^2+1}dx$. Let $u = x^2 + 1$. Then $du = 2xdx$ and $\frac{1}{2}du = xdx$. Thus

$$\int x\sqrt{x^2+1}dx = \int \frac{1}{2}\sqrt{u}du = \frac{1}{2}\frac{2}{3}u^{\frac{3}{2}} + C = \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C$$

Since $-1 = y(0) = \frac{1}{3} + C$ we have $C = -\frac{4}{3}$ and so

$$\boxed{y = \frac{1}{3}(x^2+1)^{\frac{3}{2}} - \frac{4}{3}}$$

#12. A page of a book is to contain a rectangle of printed matter with an area of 30 square inches. If the page is to have a 1-inch margin on the sides and a 2-inch margin at the bottom and the top, find the dimensions of the smallest such page.



Let x be the height and y the width of the printed matter. Then $xy = 30$ and so $y = \frac{30}{x}$. The domain of x is $0 < x < \infty$. The height of the page is $2 + x + 2 = x + 4$ and the width is $1 + y + 1 = y + 2 = \frac{30}{x} + 2$. So the area of the page is

$$A = (x+4)(\frac{30}{x} + 2) = 30 + 2x + \frac{120}{x} + 8 = 2x + \frac{120}{x} + 38$$

We need to minimize A.

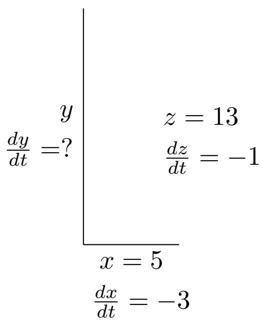
$$\frac{dA}{dx} = 2 - \frac{120}{x^2} = 2\frac{x^2 - 60}{x^2}$$

If $x > \sqrt{60}$, then $\frac{dA}{dx} > 0$ and A is increasing. If $0 < x < \sqrt{60}$, then $\frac{dA}{dx} < 0$ and A is decreasing.
Hence A has absolute minimum at $x = \sqrt{60} = 2\sqrt{15}$. If $x = 2\sqrt{15}$ then $y = \frac{30}{x} = \frac{30}{2\sqrt{15}} = \frac{15}{\sqrt{15}} = \sqrt{15}$. So the dimensions of the page of minimal area are

$$\boxed{(\sqrt{15}+2) \times (2\sqrt{15}+4)}$$

(It might be interesting to note that the height is exactly double the width.)

#13. An extension ladder is leaning against a wall and is collapsing at the rate of 1 foot per sec. The bottom of the ladder is being pushed toward the wall at the rate of 3 feet per sec. How fast and in which direction is the top of the ladder moving up or down the wall when the ladder is 13 feet long and the bottom is 5 feet from the wall?



Let x be the distance of the bottom of the ladder from the wall, y the distance of the top of the ladder from the floor and z the length of the ladder. Since the ladder collapses at a rate of 1 foot per sec, we have $\frac{dz}{dt} = -1$. Since the bottom of the ladder is pushed towards the wall at a rate of 3 ft per sec, $\frac{dx}{dt} = -3$. We need to compute $\frac{dy}{dt}$ when $x = 5$ and $z = 13$. By the Pythagorean Theorem, $x^2 + y^2 = z^2$. Differentiating with respect to t gives

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

and so

$$\frac{dy}{dt} = \frac{z\frac{dz}{dt} - x\frac{dx}{dt}}{y}$$

When $x = 5$ and $z = 13$ we have $y = \sqrt{z^2 - x^2} = \sqrt{13^2 - 5^2} = \sqrt{169 - 25} = \sqrt{144} = 12$ and so

$$\frac{dy}{dt} = \frac{13 \cdot (-1) - 5 \cdot (-3)}{12} = \frac{-13 + 15}{12} = \frac{2}{12} = \frac{1}{6}$$

So the top of the ladder is moving up the wall at rate of $\boxed{\frac{1}{6}\frac{\text{ft}}{\text{sec}}}$.

#14. Make a reasonable first approximation to the root of $x^3 + 12x - 30 = 0$. Use Newton's Method to obtain two successive additional approximation of the root.

$2^3 + 12 \cdot 2 - 30 = 8 + 24 - 30 = 2$. So $x_0 = 2$ is a reasonable first approximation.
Put $f(x) = x^3 + 12x - 30$. Then $f'(x) = 3x^2 + 12$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + 12x_n - 30}{3x_n^2 + 12} = \frac{3x_n^3 + 12x_n - x_n^3 - 12x_n + 30}{3x_n^2 + 12} = \frac{2x_n^3 + 30}{3x_n^2 + 12} = \frac{2}{3}\frac{x_n^3 + 15}{x_n^2 + 4}$$

Hence

$$x_1 = \frac{2}{3}\frac{8 + 15}{4 + 4} = \frac{2}{3}\frac{23}{8} = \frac{23}{12}$$

and

$$x_2 = \frac{2}{3}\frac{(\frac{23}{12})^3 + 15}{(\frac{23}{12})^2 + 4} = \frac{2}{3}\frac{\frac{23^3 + 15 \cdot 12^3}{12^3}}{\frac{23^2 + 4 \cdot 12^2}{12^2}} = \frac{2 \cdot (23^3 + 15 \cdot 12^3)}{3 \cdot 12 \cdot (23^2 + 4 \cdot 12^2)} = \frac{(23^3 + 15 \cdot 12^3)}{18(23^2 + 4 \cdot 12^2)} = \frac{38087}{19890}$$

#15. Let $f(x) = x^2 - 2x$ on $[2, 3]$.

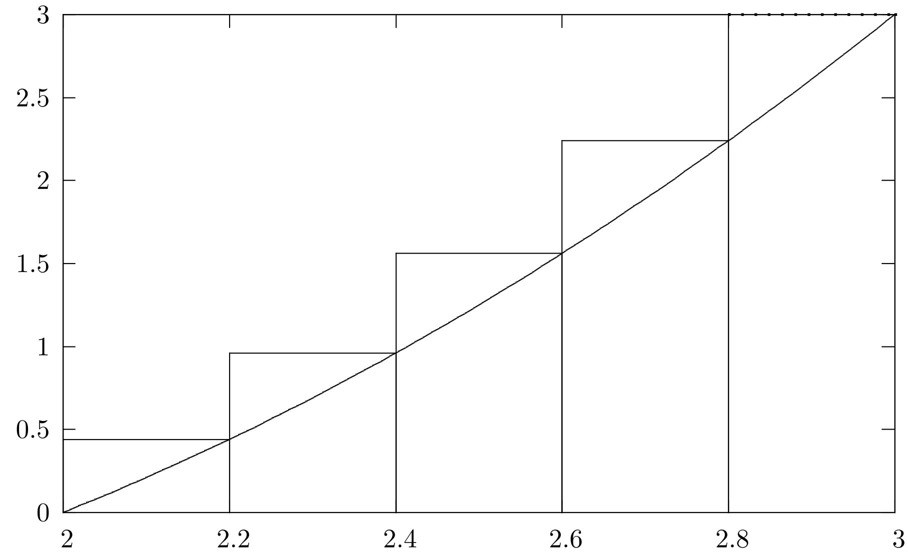
(a) Using 5 intervals of equal length and right endpoints, construct an approximation of the integral $\int_2^3 (x^2 - 2x)dx$. Draw a picture of the rectangles whose area your sum represents.

(b) Using summation notation write an approximation sum for $\int_2^3 (x^2 - 2x)dx$ using n subintervals of equal length and the right endpoints of the subintervals.

(c) Using summation formulas, simplify the sum in b) and take its limit as $n \to \infty$.

(a) The subintervals have length $\frac{3-2}{5} = 0.2$. Thus $x_0 = 2$, $x_1 = 2.2$, $x_2 = 2.4$, $x_3 = 2.6$, $x_4 = 2.8$ and $x_5 = 3$. Since c_k is the right end point of $[x_{k-1}, x_k]$, $c_k = x_k$. Hence

$$\begin{array}{rcccl}
\sum_{k=1}^{5} f(c_k)\Delta x_k & = & (2.2^2 + 2\cdot 2.2) & \cdot & 0.2 \\
& + & (2.4^2 + 2\cdot 2.4) & \cdot & 0.2 \\
& + & (2.6^2 + 2\cdot 2.6) & \cdot & 0.2 \\
& + & (2.8^2 + 2\cdot 2.8) & \cdot & 0.2 \\
& + & (3^2 + 2\cdot 3) & \cdot & 0.2
\end{array}$$



(b) The length of each the intervals is $\Delta x = \frac{1}{n}$ so if c_k is the right end-point the k-interval, then $c_k = 2 + k\frac{1}{n} = \frac{2n+k}{n}$. Thus

$$f(c_k) = c_k^2 - 2c_k = (\frac{2n+k}{n})^2 - 2\frac{2n+k}{n} = \frac{(2n+k)^2 - 2n(2n+k)}{n^2} = \frac{4n^2 + 4nk + k^2 - (4n^2 + 2nk)}{n_2} = \frac{k^2 + 2nk}{n^2}$$

So the Riemann Sum is

$$\sum_{k=1}^{n} f(c_k)\Delta x = \sum_{k=1}^{n} \frac{k^2 + 2nk}{n^2}\frac{1}{n} = \sum_{k=1}^{n} \frac{k^2 + 2nk}{n^3} = \frac{1}{n^3}\sum_{k=1}^{n}(k^2 + 2nk)$$

(c) We have

$$\begin{array}{rcccccl}
\sum_{k=1}^{n}(k^2 + 2nk) & = & \left(\sum_{k=1}^{n} k^2\right) + 2n\left(\sum_{k=1}^{n} k\right) & = & \frac{n(n+1)(2n+1)}{6} + 2n\frac{n(n+1)}{2} & = & n(n+1)(\frac{2n+1}{6} + n) \\
& = & n(n+1)\frac{2n+1+6n}{6} & = & \frac{1}{6}n(n+1)(8n+1) & &
\end{array}$$

Thus

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x = \lim_{n\to\infty} \frac{1}{6}\frac{n(n+1)(8n+1)}{n^3} = \lim_{n\to\infty} \frac{1}{6}(1 + \frac{1}{n})(8 + \frac{1}{n}) = \frac{8}{6} = \boxed{\frac{4}{3}}.$$

#16. Evaluate the following integrals:

(a) $\int_1^2 (x^{\frac{2}{3}} + x^7)dx$.

(b) $\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \sec^2(2x)dx$.

(c) $\int \frac{\sin\sqrt{2x+2}}{\sqrt{2x+2}}dx$.

(d) $\int_0^5 \frac{1}{\sqrt{3x+1}}dx$.

(e) $\int \frac{2-x^2}{(x^3-6x+1)^5}dx$.

(a) $\int_1^2 (x^{\frac{2}{3}} + x^7)dx = \left[\frac{3}{5}x^{\frac{5}{3}} + \frac{1}{8}x^8\right]_1^2 = (\frac{3}{5}2^{\frac{5}{3}} + \frac{1}{8}2^8) - (\frac{3}{5} + \frac{1}{8}) = \frac{3\cdot 2\cdot 2^{\frac{2}{3}}}{5} + 2^5 - \frac{3}{5} - \frac{1}{8} = \frac{48\sqrt[3]{4}+1280-24-5}{40} = \boxed{\frac{48\sqrt[3]{4}+1251}{40}}$

(b) $\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \sec^2(2x)dx = \left[\frac{1}{2}\tan(2x)\right]_{\frac{\pi}{3}}^{\frac{\pi}{6}} = \frac{1}{2}(\tan(\frac{2\pi}{3}) - \tan(\frac{\pi}{3})) = \frac{1}{2}\big(-\sqrt{3}-\sqrt{3}\big) = \boxed{-\sqrt{3}}$.

(c) Put $u = \sqrt{2x+2}$. Then $du = \frac{1}{2\sqrt{2x+2}}2dx = \frac{1}{\sqrt{2x+2}}dx$. Thus

$$\int \frac{\sin\sqrt{2x+2}}{\sqrt{2x+2}}dx = \int \sin u\, du = -\cos u + C = \boxed{-\cos(\sqrt{2x+2}) + C}$$

(d) $\int_0^5 \frac{1}{\sqrt{3x+1}}dx = \left[\frac{2}{3}\sqrt{3x+1}\right]_0^5 = \frac{2}{3}\big(\sqrt{3\cdot 5+1} - \sqrt{1\cdot 0+1}\big) = \frac{2}{3}\big(\sqrt{16}-\sqrt{1}\big) = \frac{2}{3}(4-1) = \frac{2}{3}3 = \boxed{2}$

(e) Put $u = x^3 - 6x + 1$. Then $du = (3x^2-6)dx$ and so $-\frac{1}{3}du = (2-x^2)dx$. Thus
$\int \frac{2-x^2}{(x^3-6x+1)^5}dx = \int -\frac{1}{3}\frac{1}{u^5}du = -\frac{1}{3}\int u^{-5}du = -\frac{1}{3}\frac{1}{-4}u^{-4} + C = \frac{1}{12}\frac{1}{u^4} + C = \boxed{\frac{1}{12(x^3-6x+1)^4} + C}$.

#17. Use linear approximation to approximate $\sqrt[3]{25}$.

Let $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$. Put $x_0 = 27$. Then $y_0 = f(x_0) = \sqrt[3]{27} = 3$. Also $\Delta x = 25 - 27 = -2$. Since $f'(x) = \frac{1}{3}x^{\frac{-2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$ we have $f'(x_0) = \frac{1}{3\sqrt[3]{27^2}} = \frac{1}{3\cdot 3^2} = \frac{1}{27}$.

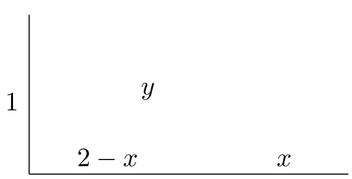
Thus

$$\Delta y \approx dy = f'(x_0)\Delta x = \frac{1}{27}\cdot(-2) = -\frac{2}{27}.$$

So

$$y \approx y_0 + dy = 3 - \frac{2}{27} = \frac{79}{27} = \boxed{2\tfrac{25}{27}}.$$

#18. A woman is 1 mile north of a pavement which runs west to east. She wants to reach a point on the pavement 2 miles east of her current location. The area between the woman and the pavement is grass. She can walk 3 m.p.h. on the grass and 5 m.p.h. on the pavement. She will walk diagonally to a point on the pavement east of her current location and then walk along the pavement the rest of the way. What route takes the least time?



Let y be the number of miles the woman walks on the grass and x the number of miles she walks on the pavement. Since she can walk 3 mph on the grass and 5 mph on the pavement, the total time of her walk will be

$$t = \frac{y}{3} + \frac{x}{5}$$

The domain for x is $0 \le x \le 2$. By the Pythagorean theorem,

$$1 + (2-x)^2 = y^2$$

and so

$$y = \sqrt{1+(2-x)^2} = \sqrt{1+(4-4x+x^2)} = \sqrt{x^2-4x+5}$$

Thus

$$t = \frac{\sqrt{x^2-4x+5}}{3} + \frac{x}{5}$$

and

$$\frac{dt}{dx} = \frac{2x-4}{3\cdot 2\cdot\sqrt{x^2-4x+5}} + \frac{1}{5} = \frac{x-2}{3\sqrt{x^2-4x+5}} + \frac{1}{5}$$

So $\frac{dt}{dx} = 0$ if and only if

$$\begin{aligned}
\tfrac{x-2}{3\sqrt{x^2-4x+5}} &= -\tfrac{1}{5} \\
5(x-2) &= -3\sqrt{x^2-4x+5} \\
(5(x-2))^2 &= 9(x^2-4x+5) \\
25(x^2-4x+4) &= 9x^2-36x+45 \\
25x^2-100x+100 &= 9x^2-36x+45 \\
16x^2-64x+55 &= 0 \\
(4x-5)(4x-11) &= 0
\end{aligned}$$

So $x = \frac{5}{4}$ or $x = \frac{11}{4}$. Since $\frac{11}{4} > 2$, $\frac{11}{4}$ is not in the domain. Also $\frac{dt}{dx}$ is define for all x in $[0,2]$. So $\frac{5}{4}$ is the only critical point.

If $x = \frac{5}{4}$, then $y = \sqrt{(\frac{5}{4})^2 - 4\frac{5}{4} + 5} = \sqrt{(\frac{5}{4})^2 - 5 + 5} = \sqrt{(\frac{5}{4})^2} = \frac{5}{4}$. Thus $t = \frac{1}{3}\frac{5}{4} + \frac{1}{4}\frac{5}{4} = (\frac{1}{3} + \frac{1}{5})\frac{5}{4} = \frac{5}{12} + \frac{1}{4} = \frac{8}{12} = \frac{2}{3}$.

If $x = 0$, then $y = \sqrt{5}$ and $t = \frac{\sqrt{5}}{3}$

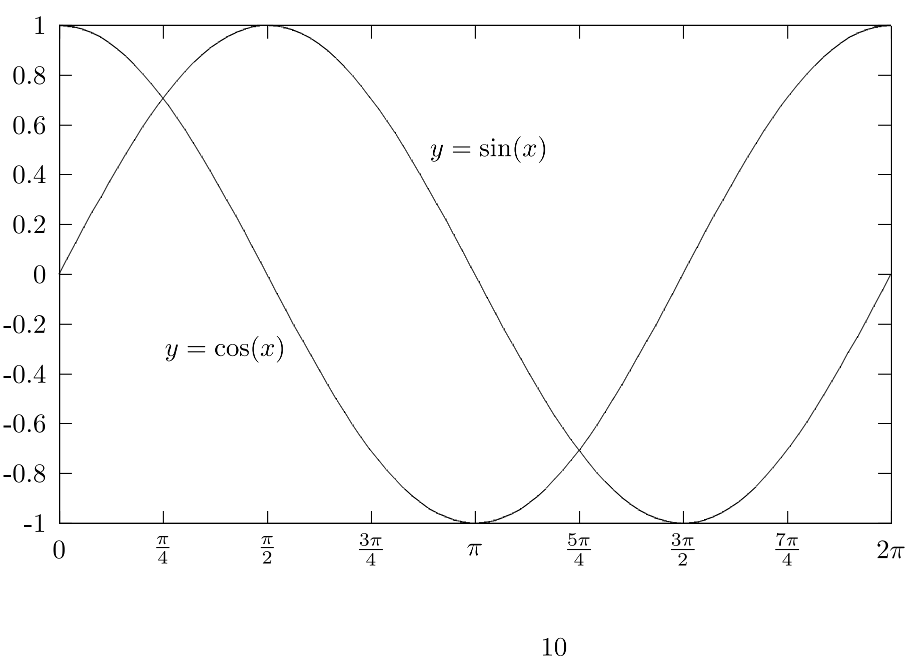
If $x = 2$, then $y = \sqrt{4-8+5} = \sqrt{1} = 1$ and $t = \frac{1}{3} + \frac{2}{5} = \frac{5+6}{15} = \frac{11}{15}$

Since $\frac{2}{3} < \frac{\sqrt{5}}{3}$ and $\frac{2}{3} < \frac{11}{15}$ we conclude that walking for $1\frac{1}{4}$ miles on the grass and for $1\frac{1}{4}$ miles on the pavement produces the shortest time.

#19. Find the average value of $\sin x$ on $[0, \frac{\pi}{2}]$.

$$\mathrm{av}(f) = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{\frac{\pi}{2}-0}\int_0^{\frac{\pi}{2}} \sin x = \frac{2}{\pi}\big[-\cos x\big]_0^{\frac{\pi}{2}} = \frac{2}{\pi}(-\cos(\frac{\pi}{2}) - (-\cos 0)) = \frac{2}{\pi}(-0+1) = \frac{2}{\pi}$$

#20. Find the area bounded by the curves $y = \cos x$ and $y = \sin x$ for $0 \le x \le 2\pi$.



We see that $\sin x \le \cos x$ on $[0, \frac{\pi}{4}]$ and on $[\frac{5\pi}{4}, 2\pi]$. Also $\cos x \le \sin x$ on $[\frac{\pi}{4}, \frac{5\pi}{4}]$.
We compute

$$\begin{array}{rclcl} \int_0^{\frac{\pi}{4}} (\cos x - \sin x)dx & = & \Big[\sin x + \cos x\Big]_0^{\frac{\pi}{4}} & = & (\sin(\frac{\pi}{4}) + \cos(\frac{\pi}{4})) - (\sin 0 + \cos 0) \\ & = & (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}) - (0+1) & = & \sqrt{2} - 1 \end{array},$$

$$\begin{array}{rclcl} \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)dx & = & \Big[-\cos x - \sin x\Big]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} & = & (-\cos(\frac{5\pi}{4}) - \sin(\frac{5\pi}{4})) - (-\cos(\frac{\pi}{4}) - \sin(\frac{\pi}{4})) \\ & = & (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}) + (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}) & = & 2\sqrt{2} \end{array}$$

and

$$\begin{array}{rclcl} \int_{\frac{5\pi}{4}}^{2\pi} (\cos x - \sin x)dx & = & \Big[\sin x + \cos x\Big]_{\frac{5\pi}{4}}^{2\pi} & = & (\sin(2\pi) + \cos(2\pi))) - (\sin(\frac{5\pi}{4}) + \cos(\frac{5\pi}{4})) \\ & = & (0+1) - (-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}) & = & 1 + \sqrt{2} \end{array},$$

The total area is

$$(\sqrt{2} - 1) + 2\sqrt{2} + (1 + \sqrt{2}) = 4\sqrt{2}$$

One could have simplified this calculations by observing that the sum of the left and the right area equals the middle area. So it suffices to calculated the second integral (which is $2\sqrt{2}$) and double it to get the total area of $4\sqrt{2}$.