#1. (24 pts) Find the following limits:

(a) (8 pts) $\lim_{x\to 0}\frac{\sin 2x}{x}+\lim_{x\to\frac{\pi}{4}}\frac{\sin 2x}{x}$

(b) (8 pts) $\lim_{x\to 1^-}(8x+3)\frac{|x-1|}{x-1}$.

(c) (8 pts) $\lim_{x\to 0}\frac{x(\cos x-1)}{\sin x-x}$(**Ignore this part. It requires L'Hospital's Rule, which was not covered this year.**)

(a)

$$\lim_{x\to 0}\frac{\sin 2x}{x}+\lim_{x\to\frac{\pi}{4}}\frac{\sin 2x}{x}=\lim_{x\to 0}\left(\frac{\sin 2x}{2x}\frac{2x}{x}\right)+\frac{\sin(2\cdot\frac{\pi}{4})}{\frac{\pi}{4}}=\left(\lim_{y\to 0}\frac{\sin y}{y}\right)\cdot 2+\frac{4\sin(\frac{\pi}{2})}{\pi}=1\cdot 2+\frac{4\cdot 1}{\pi}=\boxed{\frac{2\pi+4}{\pi}}$$

(b) If $x<1$, then $x-1<0$ and so $|x-1|=-(x-1)$. Hence

$$\lim_{x\to 1^-}(8x+3)\frac{|x-1|}{x-1}=\lim_{x\to 1^-}(8x+3)\frac{-(x-1)}{x-1}=\lim_{x\to 1^-}(8x+3)\cdot(-1)=\lim_{x\to 1^-}-(8x+3)=-(8\cdot 1+3)=\boxed{-11}$$

(c) (**Ignore this solution, unless you happen to know L'Hospital's Rule**)

$$\begin{aligned}\lim_{x\to 0}\frac{x(\cos x-1)}{\sin x-x}&=\lim_{x\to 0}\frac{(x(\cos x-1))'}{(\sin x-x)'}\\&=\lim_{x\to 0}\frac{1(\cos x-1)+x(-\sin x-0)}{(\cos x-1)}\\&=\lim_{x\to 0}\frac{\cos x-1-x\sin x}{\cos x-1}\\&=\lim_{x\to 0}\frac{(\cos x-1-x\sin x)'}{(\cos x-1)'}\\&=\lim_{x\to 0}\frac{-\sin x-0-(1\cdot\sin x+x\cos x)}{-\sin x}\\&=\lim_{x\to 0}\frac{2\sin x+x\cos x}{\sin x}\\&=\lim_{x\to 0}\frac{(2\sin x+x\cos x)'}{(\sin x)'}\\&=\lim_{x\to 0}\frac{(2\cos x+1\cdot\cos x-x\sin x)'}{\cos x}\\&=\lim_{x\to 0}\frac{3\cos x-x\sin x}{\cos x}\\&=\frac{3\cos(0)-0\cdot\sin(0)}{\cos(0)}\\&=\frac{3\cdot 1}{1}\\&=\boxed{3}\end{aligned}$$

#2. (24 pts) Find the derivative of the following functions (**do not simplify**):

(a) (8 pts) $f(x)=\sin^2 x+\sin(x^2)+(x+1)\cos x+\frac{\cos x}{x+1}$.

(b) (8 pts) $f(x) = \tan\left(\frac{1}{x} + \sqrt{1-\sqrt{x}}\right)$

(c) (8 pts) $f(x) = \int_x^{x^2} \sqrt{1+t^3}\,dt + \int (x^{2008}+1)^{88} dx$

(a)

$$\left(\sin^2 x + \sin(x^2) + (x+1)\cos x + \frac{\cos x}{x+1}\right)'$$
$$= 2\sin x\cos x + \cos(x^2)\cdot 2x + 1\cdot\cos x + (x+1)(-\sin x) + \frac{(-\sin x)(x+1) - \cos x\cdot 1}{(x+1)^2}$$

(b)

$$\left(\tan\left(\frac{1}{x} + \sqrt{1-\sqrt{x}}\right)\right)' = \sec^2\left(\frac{1}{x} + \sqrt{1-\sqrt{x}}\right)\left(-\frac{1}{x^2} + \frac{1}{2\sqrt{1-\sqrt{x}}}\cdot -\frac{1}{2\sqrt{x}}\right)$$

(c)

$$\left(\int_x^{x^2} \sqrt{1+t^3}\,dt + \int (x^{2008}+1)^{88} dx\right)' = \sqrt{1+(x^2)^3}\cdot 2x - \sqrt{1+x^3} + (x^{2008}+1)^{88}$$

#3. (12 pts) Use the *definition* of the derivative as a *limit* to calculate $f'(x)$ for $f(x) = \frac{1}{x}$. (There will be no credit for other methods).

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
&= \lim_{h\to 0} \frac{x-(x+h)}{h(x+h)x} \\
&= \lim_{h\to 0} \frac{-h}{h(x+h)x} \\
&= \lim_{h\to 0} \frac{-1}{(x+h)x} \\
&= \frac{-1}{(x+0)x} \\
&= -\frac{1}{x^2}
\end{aligned}$$

#4. (15 pts)

(a) (10 pts) Let y be the implicit function of x determined by $y^2 - 2x - 4y - 1 = 0$. Find $\frac{dy}{dx}$.

(b) (5 pts) Using part (a), find the equation of the line tangent to the curve $y^2 - 2x - 4y - 1 = 0$ at the point $P(-2, 1)$.

(a)

$$\begin{aligned}
(y^2 - 2x - 4y - 1)' &= 0' \\
2yy' - 2 - 4y' &= 0 \\
2yy' - 4y' &= 2 \\
2(y-2)y' &= 2 \\
(y-2)y' &= 1 \\
y' &= \boxed{\frac{1}{y-2}}
\end{aligned}$$

(b) For $x = -2$ and $y = 1$ we get

$$y' = \frac{1}{y-2} = \frac{1}{1-2} = \frac{1}{-1} = -1.$$

So the tangent line has slope one and goes through the point $P(-2, 1)$. Hence the equation of the tangent line is

$$y - 1 = (-1)(x - (-2))$$
$$y - 1 = -x - 2$$
$$\boxed{y = -x - 1}$$

#5. (12 pts) When a circular plate of metal is heated in an oven, its radius increased at the rate of 0.01 cm/min. At what rate is the plate's area increasing when the radius is 10 cm?

Let A be the area of the plate and r the radius. We know that $\frac{\mathrm{d}r}{\mathrm{d}t} = 0.01$ and need to compute $\frac{\mathrm{d}A}{\mathrm{d}t}$ when $r = 10$.

$$A = \pi r^2$$
$$\frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi r \frac{\mathrm{d}r}{\mathrm{d}t}$$

With $r = 10$ and $\frac{\mathrm{d}r}{\mathrm{d}t} = 0.01$ we get

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi r \frac{\mathrm{d}r}{\mathrm{d}t} = 2\pi \cdot 10 \cdot 0.01 = \boxed{0.2\pi \frac{\mathrm{cm}^2}{\mathrm{min}}}$$

#6. (12 pts) A rock thrown vertically upward reaches a height of $s = 24t - 0.8t^2$ meters in t seconds. How long does it take the rock to reach the highest point? And how high does the rock go?

$$v = s' = (24t - 0.8t^2)' = 24 - 1.6t$$

The rock reaches maximum height when $v = 0$:

$$0 = 24 - 1.6t$$
$$t = \frac{24}{1.6} = \frac{240}{16} = \frac{60}{4} = \boxed{15\mathrm{sec}}$$

At $t = 15$ the height is

$$s = 24t - 0.8t^2 = 24 \cdot 15 - 0.8 \cdot 15^2 = (24 - 0.8 \cdot 15) \cdot 15 = (24 - 12) \cdot 15 = 12 \cdot 15 = \boxed{180\mathrm{m}}.$$

#7. (32 pts) Evaluate the following integrals:

(a) (8 pts) $\int_1^4 \left(2x - \frac{1}{\sqrt{x}}\right) \mathrm{d}x$.

(b) (8 pts) $\int_0^{\frac{\pi^2}{4}} \frac{\sin\sqrt{x}\cos\sqrt{x}}{2\sqrt{x}} \mathrm{d}x$

(c) (8 pts) $\int_0^{\pi} \frac{1}{2}(\cos x + |\cos x|)\mathrm{d}x$

(d) (8 pts) $\int t^{-2} \sin\left(1 + \frac{1}{t}\right) \mathrm{d}t$

(a)

$$\int_1^4 \left(2x - \frac{1}{\sqrt{x}}\right) \mathrm{d}x = \left[x^2 - 2\sqrt{x}\right]_1^4 = (4^2 - 2\sqrt{4}) - (1^2 - 2\sqrt{1}) = (16 - 2\cdot 2) - (1-2) = 12 - (-1) = \boxed{13}$$

(b) $\int_0^{\frac{\pi^2}{4}} \frac{\sin\sqrt{x}\cos\sqrt{x}}{2\sqrt{x}} \mathrm{d}x = ?$

$u = \sin(\sqrt{x})$

$\mathrm{d}u = (\sin(\sqrt{x}))' \,\mathrm{d}x = \cos(\sqrt{x})\frac{1}{2\sqrt{x}}\mathrm{d}x = \frac{\cos(\sqrt{x})}{2\sqrt{x}}$

$x = \frac{\pi^2}{4}$: $u = \sin(\sqrt{\frac{\pi^2}{4}}) = \sin(\frac{\pi}{2}) = 1$

$x = 0$: $u = \sin(\sqrt{0}) = \sin 0 = 0.$

$$\begin{aligned}
\int_0^{\frac{\pi^2}{4}} \frac{\sin\sqrt{x}\cos\sqrt{x}}{2\sqrt{x}} \mathrm{d}x &= \int_0^{\frac{\pi^2}{4}} \sin\sqrt{x}\frac{\cos\sqrt{x}}{2\sqrt{x}} \mathrm{d}x \\
&= \int_0^1 u \mathrm{d}u \\
&= \left[\frac{1}{2}u^2\right]_0^1 \\
&= \frac{1}{2}1^2 - \frac{1}{2}0^2 \\
&= \boxed{\frac{1}{2}}
\end{aligned}$$

(c) If x is in $[0, \frac{\pi}{2}]$, then $\cos x \geq 0$ and so $|\cos x| = \cos x$.
If x is in $[\frac{\pi}{2}, \pi]$, then $\cos x \leq 0$ and so $|\cos x| = -\cos x$. Thus

$$\begin{aligned}
\int_0^\pi \frac{1}{2}(\cos x + |\cos x|)\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \frac{1}{2}(\cos x + |\cos x|)\mathrm{d}x \;+\; \int_{\frac{\pi}{2}}^\pi \frac{1}{2}(\cos x + |\cos x|)\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{2}(\cos x + \cos x)\mathrm{d}x \;+\; \int_{\frac{\pi}{2}}^\pi \frac{1}{2}(\cos x - \cos x|)\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \cos x \mathrm{d}x \;+\; \int_{\frac{\pi}{2}}^\pi 0 \mathrm{d}x \\
&= [\sin x]_0^{\frac{\pi}{2}} + 0 \\
&= \sin(\frac{\pi}{2}) - \sin 0 \\
&= 1 - 0 \\
&= \boxed{1}
\end{aligned}$$

#8. (10 pts) Find the linearization of the function $f(x) = \sqrt{x}$ at the point $a = 25$. Then, using this linearization, estimate $\sqrt{25.2}$. (No credit will be given for any other methods.) Recall that

$$L(x) = f(a) + f'(a)(x - a).$$

We compute

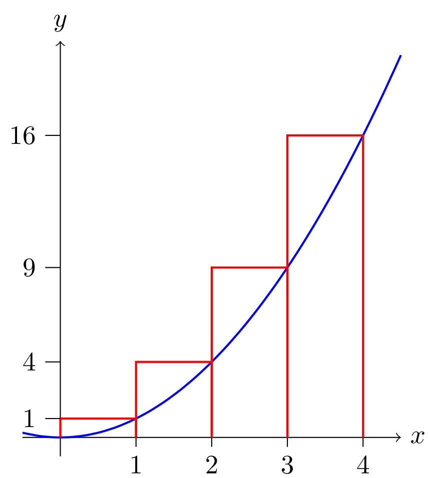
$$f(a) = \sqrt{25} = 5.$$
$$f'(x) = (\sqrt{x})' = \frac{1}{2\sqrt{x}}$$
$$f'(a) = \frac{1}{2\sqrt{25}} = \frac{1}{10}$$

and so

$$L(x) = f(a) + f'(a)(x-a) = \boxed{5 + \frac{1}{10}(x-25)}.$$

Thus
$$\sqrt{25.2} \approx L(25.2) = 5 + \frac{1}{10}(25.2 - 25) = 5 + \frac{1}{10} \cdot 0.2 = 5 + 0.02 = \boxed{5.02}.$$

#9. (10 pts) Consider $f(x) = x^2$ for $x \in [0, 4]$. Sketch the rectangles associated with the upper sum of $f(x)$ over $[0, 4]$ by dividing the interval into four sub-intervals with equal length, and find this upper sum.



$$\text{Upper sum} = 1^2 \cdot 1 + 2^2 \cdot 1 + 3^2 \cdot 1 + 4^2 \cdot 1 = 1 + 4 + 9 + 16 = 5 + 25 = \boxed{30}$$

Alternatively, we could use the formula $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$:

$$\text{Upper sum} = \sum_{i=1}^{4} i^2 \cdot 1 = \frac{4 \cdot (4+1) \cdot (2 \cdot 4 + 1)}{6} = \frac{4 \cdot 5 \cdot 9}{6} = 2 \cdot 5 \cdot 3 = 30.$$

#10. (24 pts) Let $y = f(x) = \frac{x^2+3x-3}{x-1}$.

(a) (2 pts) Given that we can write the function as $y = f(x) = x+4+\frac{1}{x-1}$. Indicate the vertical asymptote of $y = f(x)$, and the oblique asymptote of $y = f(x)$ as well. (**No** technical details are needed to justify your answer.)

(b) (6 pts) Find the intervals where $f(x)$ is increasing and the intervals where $f(x)$ is decreasing. Identify the points where $f(x)$ has a local maximum and the points where $f(x)$ has a local minimum. (It is helpful to use the form $y = f(x) = x + 4 + \frac{1}{x-1}$ to evaluate $f'(x)$.)

(c) (10 pts) Find the intervals on which $f(x)$ is concave-up and the intervals on which $f(x)$ is concave-down.

(d) (6 pts) Sketch the graph $y = f(x)$ using the information from parts (a)-(c).

(a) From $f(x) = x + 4 + \frac{1}{x-1}$ we see that $\boxed{\text{x=1}}$ is a vertical asymptote and $\boxed{\text{y=x+4}}$ is an oblique asymptote.

(b)

$$f'(x) = (x+4+\frac{1}{x-1})' = 1 - \frac{1}{(x-1)^2} = \frac{(x-1)^2-1}{(x-1)^2} = \frac{x^2-2x+1-1}{(x-1)^2} = \frac{x^2-2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}.$$

Hence $f'(x) = 0$ for $x = 0$ and $x = 2$. Also $f'(x)$ is not defined at $x = 1$.

	$(-\infty, 0)$	$(0, 1)$	$(1, 2)$	$(2, \infty)$
x	$-$	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$
$(x-1)^2$	$+$	$+$	$+$	$+$
f'	$+$	$-$	$-$	$+$
f	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

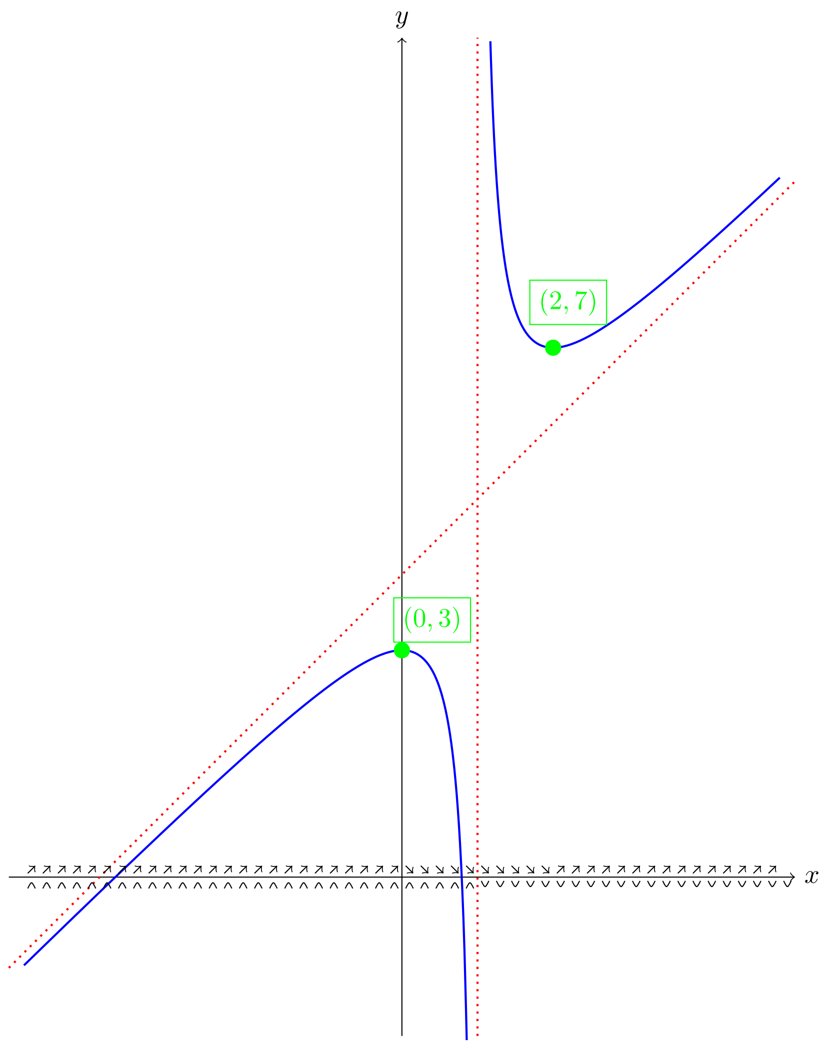
Hence f is increasing on $(\infty, 0]$ and on $[2, \infty)$; and f is decreasing on $[0, 1)$ and on $(1, 2]$.
Moreover f has a local maximum at $x = 0$ (with $f(0) = \frac{-3}{-1} = 3$) and at $x = 2$ (with $f(2) = 2+4+\frac{1}{2-1} = 6 + 1 = 7$.

(c)

$$f''(x) = \left(1 - \frac{1}{(x-1)^2}\right)' = 0 - (-2)\frac{1}{(x-1)^3} = \frac{2}{(x-1)^3}.$$

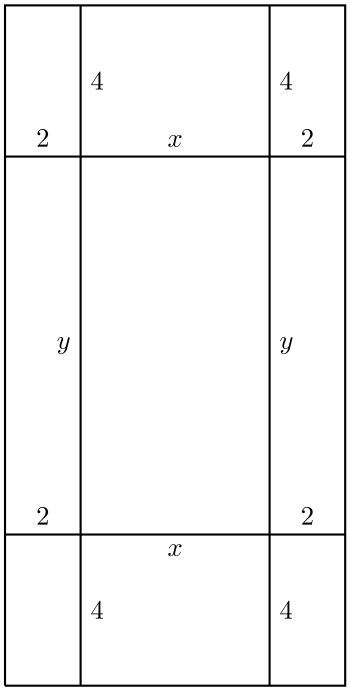
	$(-\infty, 1)$	$(1, \infty)$
$(x-1)^3$	$-$	$+$
f''	$-$	$+$
f	$\cap$	$\cup$

Thus f is concave up on $(1, \infty)$ and concave down on $(-\infty, 1)$.



y
x
(2, 7)
(0, 3)

#11. (15 pts) You are designing a rectangular poster to contain 50 in.^{2} of printing with a 4-in. margin at the top and bottom and a 2-in. margin on each side. What overall dimension will minimize the amount of paper used. **Give an argument to show that your answer does give a minimum value.**



We need to minimize the amount of paper $A = (x+4)(y+8)$.
We know that 50= Area of the printing= xy. So $y = \frac{50}{x}$ and

$$A = (x+4)(\frac{50}{x} + 8) = 50 + \frac{200}{x} + 8x + 32 = \frac{200}{x} + 8x + 82.$$

The domain for x is $(0, \infty)$. We compute

$$A' = \left(\frac{200}{x} + 8x + 82\right)' = -\frac{200}{x^2} + 8 = \frac{8x^2 - 200}{x^2} = 8\frac{x^2 - 25}{x^2}$$

A' is defined for all x in the domain of $(0, \infty)$ of x. $A' = 0$ at $x = 5$ (note that -5 is not in the domain).

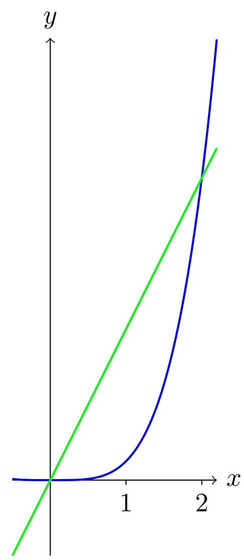
	$(0,5)$	$(5,\infty)$
$x^2 - 25$	$-$	$+$
x^2	$+$	$+$
f'	$-$	$+$
f	$\searrow$	$\nearrow$

Hence A is decreasing on $(0,5)$ and increasing on $(5,\infty)$. Thus A has an absolute minimum at $x = 5$. If $x = 5$, then $y = \frac{50}{x} = 10$ and the dimensions of the paper are

$$(5+4) \times (10+8) = \boxed{9\text{in} \times 18\text{in}}$$

#12. (10 pts) Find the area of the region enclosed by the curve $y = x^4$ and the line $y = 8x$.

We will first sketch the graph of the two curves:



To find the intersection points:

$$\begin{aligned} x^4 &= 8x \\ x^4 - 8x &= 0 \\ x(x^3 - 8) &= 0 \end{aligned}$$

So the intersection points are at $x = 0$ and $x = 2$. From the picture we see that the larger function on the interval $(0, 2)$ is $8x$. Thus

$$\begin{aligned} \text{Area} &= \int_0^2 (8x - x^4)\mathrm{d}x \\ &= \left[4x^2 - \frac{1}{5}x^5\right]_0^2 \\ &= \left[\frac{20x^2 - x^5}{5}\right]_0^2 \\ &= \frac{20 \cdot 2^2 - 2^5}{5} - 0 \\ &= \frac{80 - 32}{5} \\ &= \boxed{\frac{48}{5}} \end{aligned}$$