THE VISUAL SPHERE OF TEICHMÜLLER SPACE AND A THEOREM OF MASUR-WOLF

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Abstract. In [M-W], Masur and Wolf proved that the Teichmüller space of genus $g > 1$ surfaces with the Teichmüller metric is not a Gromov hyperbolic space. In this paper, we provide an alternative proof based upon a study of the visual sphere of Teichmüller space.

1. Introduction

As observed in [M-W], the Teichmüller space of surfaces of genus $g > 1$ with the Teichmüller metric shares many properties with spaces of negative curvature. In his study of the geometry of Teichmüller space [Kr], Kravetz claimed that Teichmüller space was negatively curved in the sense of Busemann [Bu]. It was not until about ten years later, that Linch [L] discovered a mistake in Kravetz’s arguments. This left open the question of whether or not Teichmüller space was negatively curved in the sense of Busemann. This question was resolved in the negative by Masur in [Ma].

A metric space $X$ is negatively curved, in the sense of Busemann, if the distance between the endpoints of two geodesic segments from a point in $X$ is at least twice the distance between the midpoints of these two segments. An immediate consequence of this definition is that distinct geodesic rays from a point in a Busemann negatively curved metric space must diverge. Masur proved that Teichmüller space is not negatively curved, in the sense of Busemann, by constructing distinct geodesic rays from a point in Teichmüller space which remain a bounded distance away from each other.

In [G], Gromov introduced a notion of negative curvature for metric spaces which, while less restrictive than that of Busemann, implies many of the properties which Teichmüller space shares with spaces of Riemannian negative sectional curvature. This raised the question of whether Teichmüller space was negatively curved in the sense of Gromov, (i.e. Gromov hyperbolic). According to one of the definitions of Gromov hyperbolicity, an affirmative answer to this question would rule out so-called “fat” geodesic triangles in Teichmüller space. In [M-W], Masur and Wolf resolved the Gromov hyperbolicity question in the negative by constructing such “fat” geodesic triangles.

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As observed in [M-W], the existence of distinct nondivergent rays from a point in Teichmüller space does not preclude Teichmüller space from being Gromov hyperbolic. Apparently for this reason, rather than taking Masur’s construction of such rays as the starting point for their proof, Masur and Wolf found their motivation from another source. They observed that the isometry group of the Teichmüller metric is the mapping class group [R], which is not a Gromov hyperbolic group, since it contains a free abelian group of rank 2. This fact, like Masur’s result on the existence of distinct nondivergent rays from a point, is insufficient to imply that Teichmüller space is not Gromov hyperbolic. Nevertheless, it served as motivation for Masur and Wolf’s construction of “fat” geodesic triangles.

In this paper, we provide an alternative proof of the result of Masur and Wolf. Our proof, unlike that of Masur and Wolf, builds upon Masur’s construction of nondivergent rays from a point in Teichmüller space. On the other hand, unlike the proof of Masur and Wolf, our proof depends upon one of the deeper consequences of Gromov hyperbolicity. Namely, in order for Teichmüller space to be Gromov hyperbolic, the visual sphere of Teichmüller space would have to be Hausdorff. We show that, on the contrary, the visual sphere of Teichmüller space is not Hausdorff. The proof of this fact relies heavily upon the specific nature of Masur’s construction of nondivergent rays. In this way, we show that the result of Masur and Wolf that Teichmüller space is not negatively curved in the sense of Gromov is latent in Masur’s original proof that Teichmüller space is not negatively curved in the sense of Busemann.

The outline of the paper is as follows. In section 2, we review the prerequisites for our proof. In section 3, we prove our main result that the visual sphere of Teichmüller space is not Hausdorff and conclude that Teichmüller space is not Gromov hyperbolic.

2. Preliminaries

2.1. Teichmüller space. Let \( M \) denote a closed, connected, orientable surface of genus \( g \geq 2 \). The Teichmüller space \( T_g \) of \( M \) is the space of equivalence classes of complex structures on \( M \), where two complex structures \( S_1 \) and \( S_2 \) on \( M \) are equivalent if there is a conformal isomorphism \( h : S_1 \rightarrow S_2 \) which is isotopic to the identity map of the underlying topological surface \( M \).

The Teichmüller distance \( d([S_1],[S_2]) \) between the equivalence classes \([S_1],[S_2]\) of two complex structures \( S_1 \) and \( S_2 \) on \( M \) is defined as \( \frac{1}{2} \log \inf_h K(h) \), where the infimum is taken over all quasiconformal homeomorphisms \( h : S_1 \rightarrow S_2 \) which are isotopic to the identity map of \( M \) and \( K(h) \) is the maximal dilatation of \( h \).

As shown by Kravetz [Kr], \((T_g,d)\) is a straight G-space in the sense of Busemann ([Bu],[A]). Hence, any two distinct points, \( x \) and \( y \), in \( T_g \) are joined by a unique geodesic segment (i.e. an isometric image of a Euclidean
geodesic ray through \([S]\) homeomorphism maps the closed ball of radius \(d\) in the unit sphere of \(QD\) unit ball of \(QD\) from the open unit ball of \(QD\). According to the previous description of geodesic rays, a homeomorphism can be constructed in \(QD\). This proves that \(QD\) balls in \((T_g, d)\) are homeomorphic to \(QD\) in \((T_g, d)\). This proves that closed balls in \((T_g, d)\) are homeomorphic to closed balls in \(R^{6g-6}\). It is well-known that \((T_g, d)\) is homeomorphic to \(R^{6g-6}\) and closed balls in \((T_g, d)\) are homeomorphic to closed balls in \(R^{6g-6}\). In fact, using the previous description of geodesic rays, a homeomorphism can be constructed from the open unit ball of \(QD(S)\) onto \(T_g\). Suppose \(q\) is a point in the open unit ball of \(QD(S)\). Then \(q = kq_1\) where \(0 \leq k < 1\) and \(q_1\) is a quadratic differential in the unit sphere of \(QD(S)\). Map \(q\) to the point \([S_K]\) on the geodesic ray through \([S]\) in the direction of \(q_1\) where \(K = (1+k)/(1-k)\). By the work of Teichmüller, this map is a homeomorphism from the open unit ball of \(QD(S)\) onto \(T_g\). Since \(QD(S)\) is a complex vector space of dimension \(3g-3\), this proves that \(T_g\) is homeomorphic to \(R^{6g-6}\). Note also that this homeomorphism maps the closed ball of radius \(k\) centered at the origin of \(QD(S)\) onto the closed ball of radius \(\log(K)/2\) centered at the point \([S]\) in \((T_g, d)\). This proves that closed balls in \((T_g, d)\) are homeomorphic to closed balls in \(R^{6g-6}\).

We shall be particularly interested in the Jenkins-Strebel differentials. These are the quadratic differentials all of whose noncritical horizontal trajectories are closed. Let \(\theta\) be a Jenkins-Strebel differential and \(F\) be the horizontal foliation of \(\theta\). The complement in \(M\) of the critical trajectories of \(F\) consists of \(p\) disjoint open annuli \(A_1, \ldots, A_p\), where \(1 \leq p \leq 3g-3\).
Let $\sigma_i$ be a core curve of the annulus $A_i$. The core curves $\sigma_1, \ldots, \sigma_p$ are distinct, nontrivial, pairwise nonisotopic circles on $M$. Each annulus $A_i$ is foliated by closed leaves of $F$ isotopic to $\sigma_i$. Let $M_i$ be the modulus of the annulus $A_i$. The basic existence and uniqueness theorem of Jenkins-Strebel ([J], [S]) states that there exists a unique quadratic differential $\theta$ in $Q(S)$ with prescribed isotopy classes $\gamma_i = [\sigma_i]$ of core curves and moduli $M_i$ of the corresponding annuli $A_i$. Note that two Jenkins-Strebel differentials on $S$ determine the same Teichmüller geodesic in $T_g$ emanating from $[S]$ if and only if the horizontal foliations of these Jenkins-Strebel differentials are projectively equivalent.

Following Masur [Ma], we define a Strebel ray in $T_g$ emanating from $[S]$ to be a Teichmüller geodesic ray determined by a Jenkins-Strebel differential on $S$. Suppose that $\theta_1$ and $\theta_2$ are Jenkins-Strebel differentials corresponding to the same isotopy classes of core curves, but not necessarily the same moduli, of corresponding annuli. Then, following Masur, we say that the Strebel rays determined by $\theta_1$ and $\theta_2$ are similar. Masur proved that similar Strebel rays emanating from the same point in $T_g$ are nondivergent.

**Theorem (Masur [Ma])**. Let $r$ and $s$ be similar Strebel rays in $T_g$ emanating from a point $x$ in $T_g$. There exists $N < \infty$ such that if $y$ and $z$ are any two points on $r$ and $s$ which are equidistant from $x$, then $d(y, z) \leq N$.

Since $g \geq 2$, there exist distinct similar Strebel rays $r$ and $s$ in $T_g$ emanating from the same point $x = [S]$ in $T_g$. We may construct all such pairs of rays as follows. Choose a collection of disjoint, nontrivial, pairwise nonisotopic circles $\sigma_1, \ldots, \sigma_p$ on $M$, where $2 \leq p \leq 3g - 3$. Let $a = (a_1, a_2, \ldots, a_p)$ and $b = (b_1, \ldots, b_p)$ be $p$-tuples of positive real numbers $a_i$ and $b_i$ such that $a$ and $b$ lie on distinct rays emanating from the origin in $\mathbb{R}^p$. Let $\theta$ be the Jenkins-Strebel differential on $S$ corresponding to the isotopy classes $\gamma_i = [\sigma_i]$ of core curves and moduli $a_i$ of corresponding annuli. Likewise, let $\psi$ be the Jenkins-Strebel differential on $S$ corresponding to the isotopy classes $\gamma_i = [\sigma_i]$ of core curves and moduli $b_i$ of corresponding annuli. Finally, let $r$ and $s$ be the Strebel rays determined by $\theta$ and $\psi$.

Combining the observation of the previous paragraph with his theorem on nondivergence of similar Strebel rays, Masur constructed distinct, nondivergent Teichmüller geodesic rays emanating from the same point in $T_g$. Indeed, any pair of distinct similar Strebel rays emanating from the same point in $T_g$ is such a pair of nondivergent rays. In this way, Masur proved that $T_g$ is not negatively curved in the sense of Busemann [Ma]. The particular nature of Masur’s construction of nondivergent rays will be crucial to our proof that $T_g$ is not negatively curved in the sense of Gromov.

The modulus of a flat cylinder $C$ of circumference $l$ and height $h$ is $\text{Mod}(C) = h/l$. Let $S$ be a conformal structure on $M$. Every cylinder $C$ embedded in $M$ has a conformal structure induced from $S$. $C$ is conformally equivalent to a unique flat cylinder up to change of scale. The modulus of $C$ is the modulus of any such flat cylinder. Let $\gamma$ be an isotopy
class of nontrivial simple closed curves on $M$. The modulus $\text{mod}_S(\gamma)$ of $\gamma$ is defined to be the supremum of the moduli of all cylinders embedded in $M$ with core curve $\sigma \in \gamma$.

For each conformal metric $\rho$ on $S$, let $\ell_\rho(\gamma)$ denote the infimum of the lengths, with respect to $\rho$, of simple closed curves $\sigma \in \gamma$. Let $A_\rho$ denote the area, with respect to $\rho$, of $M$. The extremal length $\text{ext}_S(\gamma)$ of $\gamma$ (with respect to the conformal structure $S$ on $M$) is equal to $\sup_\rho(\ell_\rho(\gamma))^2/A_\rho$.

The extremal length is related to the modulus by the equation $\text{ext}_S(\gamma) = 1/\text{mod}_S(\gamma)$.

According to Kerckhoff [K], the Teichmüller metric $d$ may be expressed in terms of extremal length.

**Theorem (Kerckhoff [K])**. The Teichmüller distance between two points $[S_1]$ and $[S_2]$ in $T_g$ is given by the rule:

$$d([S_1], [S_2]) = \frac{1}{2} \log \sup_\gamma \frac{\text{ext}_{S_1}(\gamma)}{\text{ext}_{S_2}(\gamma)}$$

where the supremum ranges over all isotopy classes $\gamma$ of nontrivial simple closed curves on $M$.

We recall that there is a unique hyperbolic conformal metric $\rho$ on $S$. There exists a unique hyperbolic geodesic in the isotopy class $\gamma$. The hyperbolic length $\ell_\rho(\gamma)$ is the length of this hyperbolic geodesic. Maskit established the following comparisons between the hyperbolic length $\ell_\rho(\gamma)$ and the extremal length $\text{ext}_S(\gamma)$ [M].

**Theorem (Maskit [M])**. Let $\gamma$ be an isotopy class of nontrivial simple closed curves on $M$, $S$ be a conformal structure on $M$ and $\rho$ be the unique hyperbolic conformal metric on $S$. Let $\ell$ be the hyperbolic length $\ell_\rho(\gamma)$ and $m$ be the extremal length $\text{ext}_S(\gamma)$. Then $\ell \leq m\pi$ and $m \leq (1/2)\ell e^{\ell/2}$.

### 2.2. Visual spheres and Gromov hyperbolicity.

Let $X$ be a space equipped with a metric $d$. $X$ is said to be proper if closed balls in $X$ are compact. Since closed balls in $(T_g, d)$ are homeomorphic to closed balls in $\mathbb{R}^{6g-6}$, $(T_g, d)$ is proper. $X$ is said to be geodesic if every pair of points $x, y \in X$ can be connected by a geodesic segment (i.e. an isometric embedding of an interval). By Kravetz’ result that $(T_g, d)$ is a straight $G$-space in the sense of Busemann discussed in (2.1), $(T_g, d)$ is geodesic.

Let $x$ be a point in $X$. A geodesic ray emanating from $x$ is an isometric embedding $r : [0, \infty) \to X$ mapping $0$ to $x$. If $r_1$ and $r_2$ are two geodesic rays in $X$ emanating from $x$ and the function $t \mapsto d(r_1(t), r_2(t))$ is bounded, then we say that $r_1$ and $r_2$ are asymptotic and write $r_1 \sim r_2$. In this way, we define an equivalence relation $\sim$ on the set $R_x$ of geodesic rays in $X$ emanating from $x$. Equip $R_x$ with the topology of uniform convergence on compact sets. The visual sphere of $X$ at $x$ is the quotient space $\partial_{\text{vis}, x} X$ of $R_x$ with respect to the equivalence relation $\sim$. 

Gromov ([G], see also [C-D-P], [G-H]) introduced a notion of hyperbolicity for metric spaces which is now called Gromov hyperbolicity. Gromov hyperbolic metric spaces share many of the qualitative properties of hyperbolic space. We shall not need the precise definition of Gromov hyperbolicity. We shall, however, require the following result.

**Theorem (Gromov [C-D-P])**. Let $X$ be a proper, geodesic, Gromov hyperbolic space and $x$ be a point in $X$. Then the visual sphere $\partial_{vis,x}X$ of $X$ at $x$ is Hausdorff.

**Remark 2.3.** In fact, the visual sphere of a proper, geodesic, Gromov hyperbolic space is metrizable. The visual sphere of such a space does not depend upon the base point $x$ in $X$ and is naturally isomorphic to the Gromov boundary $\partial X$ [C-D-P]. Note that the visual sphere is defined for any metric space. The Gromov boundary, however, is only defined for a restricted class of metric spaces including Gromov hyperbolic spaces.

### 3. The visual sphere of Teichmüller space

In this section, we prove that the visual sphere of Teichmüller space is not Hausdorff and conclude that Teichmüller space is not Gromov hyperbolic.

**Theorem 3.1.** Let $S$ be a conformal structure on $M$ representing a point $x$ in $T_g$. Then the visual sphere $\partial_{vis,x}T_g$ of $T_g$ at $x$, with respect to the Teichmüller metric $d$, is not Hausdorff.

**Proof.** Let $\sigma_0$ and $\sigma_1$ be a pair of disjoint simultaneously nonseparating circles on $M$. For each real number $t$ with $0 < t < 1$, let $\theta_t$ denote the unique Jenkins-Strebel differential on $S$ with core curves $\sigma_0$ and $\sigma_1$ and moduli $M_0 = 1 - t$ and $M_1 = t$. Let $\theta_0$ denote the unique Jenkins-Strebel differential on $S$ with core curve $\sigma_0$ and modulus $M_0 = 1$. Let $\theta_1$ denote the unique Jenkins-Strebel differential on $S$ with core curve $\sigma_1$ and modulus $M_1 = 1$. Let $r_t$ be the geodesic ray in $T_g$ emanating from $x$ corresponding to the nonzero quadratic differential $\theta_t$. The family $\{r_t|0 \leq t \leq 1\}$ is a continuous one-parameter family of geodesic rays in $T_g$ emanating from $x$. Let $[r_t]$ denote the point in $\partial_{vis,x}T_g$ represented by $r_t$.

Note that $r_t$ is similar to $r_{1/2}$ for all $t$ such that $0 < t < 1$. By Masur’s result on nondivergence of similar rays discussed in (2.1), it follows that $r_t$ is asymptotic to $r_{1/2}$ for all $t$ such that $0 < t < 1$. Let $x = [r_{1/2}]$. Then $x = [r_t]$ for all $t$ such that $0 < t < 1$. By continuity of the quotient map from $R_x$ to the visual sphere (recalling that the visual sphere is equipped with the quotient topology), and the convergence of the rays in $R_x$, $[r_0]$ and $[r_1]$ are contained in the closure of $x$ in $\partial_{vis,x}T_g$.

We shall now show, using Maskit’s comparison of extremal and hyperbolic lengths discussed in (2.1), that $[r_0]$ is not equal to $[r_1]$. Since $\sigma_0$ and $\sigma_1$ are simultaneously nonseparating circles on $M$, we may choose a nonseparating circle $\sigma$ on $M$ such that $\sigma$ is disjoint from $\sigma_1$, transverse to $\sigma_0$, and meets $\sigma_0$ in exactly one point. Let $\gamma_i$ denote the isotopy class of $\sigma_i$ and $\gamma$ denote
the isotopy class of \( \sigma \). Let \( \{ S^i_{K} \} \) denote the family of conformal structures on \( M \) determined by \( \theta_i \).

We recall Masur’s description of the surfaces \( \{ S^i_{K} \} \) ([Ma]). The complement of the critical points of \( \theta_i \) and the horizontal leaves of \( \theta_i \) joining critical points of \( \theta_i \) is a single annulus \( R^i \) foliated by closed horizontal leaves of \( \theta_i \) homotopic to \( \sigma_i \). We may assume that \( \sigma_i \) is the central curve of \( R^i \). The surface \( S^i_{K} \) is formed from \( S \) by “fattening” \( R^i \), by cutting \( M \) along \( \sigma_i \) and inserting a standard annulus of appropriate modulus. As \( K \) tends to infinity, the modulus of the inserted annulus tends to infinity. Hence, the modulus of \( \gamma_i \) on \( S^i_{K} \) tends to infinity. In other words, \( ext_{S^i_{K}}(\gamma_i) \) tends to zero.

In particular, \( ext_{S^0_{K}}(\gamma_0) \) tends to zero as \( K \) tends to infinity. Let \( \rho^0_{K} \) denote the unique hyperbolic conformal metric on \( S^0_{K} \). By Maskit’s comparison theorem discussed in (2.1), \( \ell_{\rho^0_{K}}(\gamma_0) \) tends to zero as \( K \) tends to infinity. Since \( \sigma_0 \) meets \( \sigma \) transversely and in a single point, the unique hyperbolic geodesics for the hyperbolic metric \( \rho^0_{K} \) in the isotopy classes of \( \sigma_0 \) and \( \sigma \) also meet transversely and in a single point. \( \ell_{\rho^0_{K}}(\gamma_0) \) and \( \ell_{\rho^0_{K}}(\gamma) \) are the respective lengths of these hyperbolic geodesics. Hence, by Lemma 1 of Chapter 11, Section 3.3 of [A], \( \ell_{\rho^0_{K}}(\gamma) \) tends to infinity as \( K \) tends to infinity. Again, by Maskit’s comparison theorem, \( ext_{S^0_{K}}(\gamma) \) tends to infinity as \( K \) tends to infinity.

On the other hand, note that \( \sigma \) is disjoint from \( \sigma_1 \). Let \( R \) be any annulus on \( S \) disjoint from \( \sigma_1 \) with core curve isotopic to \( \sigma \). By the description of \( S^1_k \) in terms of fattening \( R^1 \) along \( \sigma_1 \), the annulus \( R \) embeds conformally in \( S^1_k \). Hence, the modulus of \( \gamma \) on \( S^1_k \) is bounded below by the constant \( C = mod_S(R) \). In other words, the extremal length of \( \gamma \) on \( S^1_k \) is bounded above by the constant \( 1/C \).

We have shown that \( ext_{S^0_{K}}(\gamma) \) tends to infinity and \( ext_{S^1_{K}}(\gamma) \) remains bounded above as \( K \) tends to infinity. Hence, \( ext_{S^0_{K}}(\gamma)/ext_{S^1_{K}}(\gamma) \) tends to infinity as \( K \) tends to infinity. By Kerckhoff’s description of the Teichmüller metric in terms of extremal length discussed in (2.1), \( d(S^0_{K}, S^1_{K}) \) tends to infinity as \( K \) tends to infinity. We conclude that \( r_0 \) is not asymptotic to \( r_1 \). In other words, \( [r_0] \neq [r_1] \). Hence, we have a pair of distinct points \( [r_0] \) and \( [r_1] \) in the closure of a single point \( [r_{1/2}] \) in the visual sphere \( \partial_{vis,x}T_g \) of \( T_g \) at \( x \). It follows that the visual sphere \( \partial_{vis,x}T_g \) of \( T_g \) at \( x \) is not Hausdorff.

We are now ready to deduce the result of Masur and Wolf.

**Corollary 3.2.** ([Masur-Wolf [M-W]]) Teichmüller space with the Teichmüller metric is not Gromov hyperbolic.

**Proof.** Suppose that \( (T_g, d) \) is Gromov hyperbolic. Closed balls in \( (T_g, d) \) are compact and \( (T_g, d) \) is geodesic. By Gromov’s theorem on the visual sphere of a proper, geodesic, Gromov hyperbolic space discussed in (2.2), it follows that the visual sphere of Teichmüller space is Hausdorff. This contradicts Theorem 3.1. Hence, \( (T_g, d) \) is not Gromov hyperbolic. \( \square \)
References


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