SYMPLECTIC NORMAL CONNECT SUM

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0. Introduction

Since the publication in 1985 of Gromov's paper [G1] on pseudo-holomorphic curves in symplectic manifolds there has been an increased interest in symplectic manifolds and symplectic topology. In particular, compact symplectic manifolds have become a focus of much study. In 1977 Thurston [T] gave an example of a compact symplectic manifold with first Betti number three, showing that not all compact symplectic manifolds admit a Kähler structure. However, the difference between the family of compact symplectic manifolds and compact Kähler manifolds remains unclear. In fact there are essentially only two general procedures for constructing compact symplectic manifolds: the symplectic fibration construction, originally due to Thurston, and blowing up along symplectic submanifolds, introduced by Gromov [G2]. Recently R. Gompf has introduced a new construction. He considers two symplectic 4-manifolds each containing the compact surface Σ symplectically embedded with trivial normal bundle. By the symplectic neighborhood theorem a tubular neighborhood of Σ in each 4-manifold is symplectomorphic to $\Sigma \times D^2$ equipped with the product symplectic structure. It follows then that the complements of the tubular neighborhoods of Σ in the symplectic 4-manifolds can be symplectically glued together along tubular shell neighborhoods of Σ by the map $Id \times \phi$ where ϕ is an area preserving map of the annulus which interchanges the boundaries. Gompf proceeded by using this construction to show that a compact simply-connected 4-manifold not admitting any complex structure, which he constructed with T. Mrowka [G-M], admits a symplectic structure. He thus produced the first example of a compact simply-connected symplectic 4-manifold not admitting any Kähler structure.

In this paper we introduce a construction of four dimensional symplectic manifolds, that we call *symplectic normal connect sum* which generalizes Gompf's construction. Our procedure constructs a new symplectic 4-manifold $X = X_{-1} \#_{\Psi} X_1$ from pairs (X_i, Σ_i), $i = -1, 1$, where the X_i are symplectic 4-manifolds and the Σ_i are compact embedded symplectic surfaces of genus g and of self-intersection n (for $i = 1$) and $-n$ (for $i = -1$), $n \geq 0$. We symplectically glue the complements of tubular neighborhoods of Σ_{-1} in X_{-1} and Σ_1 in X_1 along tubular shell neighborhoods of Σ_{-1} and

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Σ_1. Changing the gluing map Ψ in general produces different symplectic manifolds X. The details of this construction are given in section 1.

We were led to our construction by the announcement of R. Gompf described above. In fact Gompf's construction is the symplectic normal connect sum for $n = 0$. However, Gompf's procedure relies on the product structure of a neighborhood of Σ to construct the gluing map $Id \times \phi$. Thus, *a priori*, it is not clear that there is a more general version of his construction and, if there is, what form it should take. Moreover to prove the non-zero self-intersection symplectic gluing we use ideas from symplectic reduction, in particular, a result of Duistermaat-Heckman [D-H]. The symplectic normal connect sum when $n \neq 0$ cannot be obtained from the symplectic neighborhood theorem alone. Our aim in this paper is both to describe the symplectic normal connect sum and to provide examples illustrating the full range of the theorem. In particular we give examples of symplectic manifolds which can only be obtained by the symplectic normal connect sum along surfaces of *non-zero* self-intersection.

Independently Gompf has generalized his original result and developed his own version of the symplectic normal connect sum. His result is a symplectic normal connect sum along codimension 2 symplectic submanifolds of $2m$-dimensional symplectic manifolds for $m \geq 2$. His proof does not use symplectic reduction. In the appendix we have given a short and simple proof of this result using our technique. At this time we know of no applications of this generalization when $m > 2$ and the normal bundles of the submanifolds are non-trivial.

The sections of the paper following section 1 are devoted to using the symplectic normal connect sum to construct new examples of compact symplectic four manifolds. The basic building blocks we use are pairs (X_i, Σ_i), $i = -1, 1$, where the X_i are Kähler surfaces and the Σ_i are nonsingular complex curves which satisfy the conditions necessary to build the symplectic normal connect sum. While it is probably the case that, in general, the symplectic form ω that we construct on $X = X_{-1} \#_\Psi X_1$ is not itself Kähler, it is difficult to rule out this possibility. If this occurs then the symplectic form ω cannot be considered *new*. Consequently we construct examples of symplectic manifolds which cannot admit Kähler structures. We use two different invariants to ensure this, namely, b_1, the first betti number and π_1, the fundamental group.

It is well known that, by Hodge theory, the first betti number of a compact Kähler manifold is even. We exploit this by constructing compact symplectic manifolds with odd betti number as follows: Let $\jmath_i : \Sigma_i \hookrightarrow X_i$ be the inclusions and let $(\jmath_i)_* : H_1(\Sigma_i) \to H_1(X_i)$ be the induced maps in homology. We show that if the kernels of both homomorphisms $(\jmath_i)_*$, $i = -1, 1$, are proper then the gluing map Ψ can be chosen so that $b_1(X_{-1} \#_\Psi X_1)$ is odd. We then use this result and the fibered product construction of algebraic geometry to build infinite families of compact symplectic manifolds with odd betti numbers. These examples can only be constructed by the symplectic normal connect sum along surfaces of non-zero self-intersection. Among these examples we have an infinite family built from pairs, X_i, $i = 1, -1$ of minimal Kähler ruled surfaces. In particular, we have

Theorem . *There are pairs (X_i, Σ_i), $i = 1, -1$, where the X_i are minimal ruled surfaces and the Σ_i are embedded holomorphic curves of genus g and self-intersection numbers $4 + 4a$, $(i = 1)$, $-(4 + 4a)$, $(i = -1)$, $a \geq 0$ and gluing maps Φ so that the symplectic normal connect sum $X = X_{-1} \#_\Psi X_1$ along the Σ_i satisfies:*

$$b_1(X) \text{ odd}$$
$$signature(X) = 0$$
$$c_2(X) = \chi(X) = 4 + 12a + 8a^2$$
$$c_1^2(X) = 8 + 24a + 16a^2.$$

The symplectic manifolds X of this theorem cannot be constructed using the self-intersection zero symplectic connect sum. The theorem requires the symplectic normal connect sum for non-zero self-intersection surfaces. Taking $a = 0$ in the theorem we have constructed a compact symplectic manifold with Chern numbers $c_1^2 = 8$ and $c_2 = 4$ and with $b_1 = 1$. For more details see Example 5.3 below.

Next we use the fundamental group as an invariant. Gompf [Go] has shown that any finitely presented group can be realized as the fundamental group of a compact symplectic manifold. However there are still many basic questions about the fundamental group of a compact symplectic manifold. Consider the class of compact Kähler surfaces with fixed Chern numbers $c_1^2 > 0$ and $c_2 > 0$. From the work of Gieseker it follows that there are only finitely many homeomorphism types of such manifolds and, hence, only finitely many fundamental groups. Does this remain true if we consider, instead, the class of compact symplectic manifolds? We show that it is false by proving:

Theorem . *There exists an infinite family $\{Y_\alpha : \alpha \in \mathbf{N}\}$ of symplectic normal connect sums all with the same Chern numbers, $c_1^2 > 0$ and $c_2 > 0$, but each with different fundamental group.*

Moreover, using Gieseker's result, at most only finitely many of these manifolds can be Kähler. In fact, using results of Arapura, Bressler, Ramachandran [A-B-R] and Johnson, Rees [J-R] we show that none of the Y_α are Kähler. For more details see Example 6.2 below.

We are indebted to Dusa McDuff for a simple construction of an S^1-invariant symplectic form on a ruled surface and for pointing out the reference [A].

1. **Symplectic Normal Connect Sum**

Let X_i, $i = -1, 1$, be smooth oriented four manifolds and suppose $\Sigma_i \hookrightarrow X_i$ are embedded oriented surfaces both of genus g with normal bundles ν_i.

Suppose the euler numbers $\chi(\nu_i)$ satisfy:

$$\chi(\nu_1) = +n, \quad \chi(\nu_{-1}) = -n. \tag{1.1}$$

where $n \geq 0$. Let $N(\Sigma_i)$ and $\mathcal{N}(\Sigma_i)$ denote tubular neighborhoods of Σ_i such that the closure $\overline{\mathcal{N}(\Sigma_i)}$ of $\mathcal{N}(\Sigma_i)$ is contained in $N(\Sigma_i)$. Let W_i denote the corresponding *tubular shell neighborhood* $N(\Sigma_i) \setminus \overline{\mathcal{N}(\Sigma_i)}$ of Σ_i in X_i. Suppose that $\Psi : W_{-1} \to W_1$ is an orientation preserving diffeomorphism taking the inside end of the tubular shell neighborhood W_{-1} to the outside end of W_1. We define the *normal connect sum* of X_{-1} and X_1 along Σ_{-1} and Σ_1 *via* Ψ to be the smooth oriented 4-manifold X obtained by gluing $X_{-1} \setminus \overline{\mathcal{N}(\Sigma_{-1})}$ and $X_1 \setminus \overline{\mathcal{N}(\Sigma_1)}$ along the tubular shell neighborhoods W_{-1} and W_1 using Ψ. We will denote X by $X_{-1} \#_\Psi X_1$. Of course the importance of (1.1) in this operation is to insure that the gluing can be done to equip X with the orientation induced from both X_{-1} and X_1.

Suppose now that (X_i, ω_i), $i = -1, 1$ are smooth symplectic four manifolds (compact or not compact, with or without boundary). We have:

Theorem 1.1 (Symplectic normal connect sum)**.** *Suppose that $\Sigma_i \hookrightarrow X_i$ are symplectically imbedded compact surfaces of genus g and that $\chi(\nu_{-1}) = -\chi(\nu_1)$ where ν_i is the normal bundle of Σ_i in X_i. Then after rescaling ω_1 or ω_{-1} there exists a symplectomorphism Ψ of tubular shell neighborhoods of Σ_{-1} and Σ_1 so that the normal connect sum $X = X_{-1} \#_\Psi X_1$ has a symplectic structure ω which agrees with the rescaled ω_i off a neighborhood of Σ_i.*

Remark 1.1. (1) The symplectic form ω can be constructed so that $\omega = \omega_1$ on $X_1 \setminus \overline{\mathcal{N}(\Sigma_1)}$ and $\omega = a\omega_{-1}$ on $X_{-1} \setminus \overline{\mathcal{N}(\Sigma_{-1})}$, where $a \in \mathbf{R}_+$. There is some freedom in the choice of a. However, when $n \neq 0$, it is subject to the following conditions: Let

$$\alpha = \frac{\omega_1[\Sigma_1]}{\omega_{-1}[\Sigma_{-1}]}$$

be the ratio of the symplectic areas of Σ_1 and Σ_{-1}. If $\alpha \leq 1$ then a must be close to α and can be chosen to be arbitrarily close to, but not equal to, α. If $\alpha \geq 1$ then a must be close to $\frac{1}{\alpha}$ and can be chosen to be arbitrarily close to, but not equal to, $\frac{1}{\alpha}$. Here close cannot be made precise because it depends on the size of neighborhoods determined by the symplectic neighborhood theorem. Note that even if $\omega_1[\Sigma_1] = \omega_{-1}[\Sigma_{-1}]$ scaling by $a \neq 1$ is still required.

(2) The case $n = 0$, where the normal bundles ν_i, $i = -1, 1$, are trivial can also be proved directly from the symplectic neighborhood theorem. Using either that technique or the proof below it follows that, in this case, the scaling factor a is exactly α so that if $\omega_1[\Sigma_1] = \omega_{-1}[\Sigma_{-1}]$ then no scaling is necessary.

The proof of Theorem 1.1 uses various normal form results in symplectic geometry to model the shells W_i. From these models it is easy to find a symplectic diffeomorphism to define the necessary symplectic gluing.

Let Σ be a compact surface of genus g. A ruled surface is an S^2-bundle over Σ. Topologically there are two such bundles, the trivial bundle $S^2 \times \Sigma$ and the twisted bundle $S^2 \tilde{\times} \Sigma$. However for our purpose it is more instructive to construct the ruled surfaces from line bundles. Let L_n be the complex line bundle over Σ with Chern class $c_1(L_n) = n$. Denote the trivial bundle by $\mathbf{C}$ and consider the complex two-plane bundle $L_n \oplus \mathbf{C}$. Projectivize each fiber and denote the resulting S^2-bundle over Σ by $S_n = \mathbf{P}(\mathbf{L_n} \oplus \mathbf{C})$. S_n has a natural S^1 action induced by multiplication by $e^{2\pi it}$ on each fiber of $\mathbf{C}$. The image of the section $(0,1)$ in $L_n \oplus \mathbf{C}$ determines an embedded surface Z_0 in S_n, called the *zero section*. If σ is any section of L_n, with isolated zeros, then away from the zeros of σ, $(\sigma, 0)$ determines a surface in S_n. Let Z_∞ denote the closure of this surface. (Z_∞ is clearly independent of the choice of σ.) Z_∞ is called the *infinity section* . It is easy to verify that $Z_0 \cdot Z_0 = n$ so the euler class of the normal bundle of Z_0 is n and that $Z_\infty \cdot Z_\infty = -n$ so the euler class of the normal bundle of Z_∞ is $-n$. We remark that if n is even, S_n is, topologically, $S^2 \times \Sigma$ and if n is odd, S_n is, topologically, $S^2 \tilde{\times} \Sigma$. The above construction can be done in the holomorphic category so that S_n is a complex surface, Z_0 and Z_∞ are holomorphic curves and S_n is fibered by holomorphic lines (see [G-H,p.517]). More generally, if M is a smooth manifold of dimension k and L is a complex line bundle over M with Chern class c then the above construction determines an S^2-bundle over M that we will denote S_c and call a *ruled manifold*. S_c has a zero section, Z_0, and an infinity section, Z_∞. The Chern classes of the normal bundles of Z_0 and Z_∞ are, respectively, c and $-c$. Exactly as above S_c admits a natural S^1 action.

For the proof of Theorem 1.1 we will need to contruct on S_n an S^1-invariant symplectic form τ_n such that Z_0, Z_∞ and the fibers F are symplectic submanifolds. The easiest way to motivate this construction is to assume that such a form exists and to analyse its structure. To this end it is equally easy to work on the ruled manifold S_c. We suppose that S_c admits a symplectic form τ_c and that the S^1 action is hamiltonian. Let $H : S_c \to \mathbf{R}$ be the hamiltonian function (well-defined up to addition of a constant). In this context H is also known as the moment map. Without loss of generality we can suppose that the critical values of H are 0 and 1, corresponding to the critical submanifolds Z_0 and Z_∞, respectively. All other values are regular. Let I be an interval of regular values of H. For each $\lambda \in I$ the level set $H^{-1}(\lambda)$ is a compact $k+1$-dimensional manifold with the structure of a circle bundle $\pi_\lambda : H^{-1}(\lambda) \to M$ over M. The Chern class of this circle bundle is independent of λ and equals c. The restriction of τ_c to $H^{-1}(\lambda)$ is a 2-form invariant under the circle action and so descends to a symplectic form σ_λ on M. (M, σ_λ) is called the symplectic reduction of (S_c, τ_c) at $\lambda \in I$. This gives a family σ_λ, $\lambda \in I$, of symplectic forms on the reduced space M. The work of Duistermaat and Heckman [D-H] shows that for $\lambda, \eta \in I$:

$$[\sigma_\lambda] = [\sigma_\eta] + (\lambda - \eta)c \tag{1.2}$$

where $[\sigma_\lambda]$ denotes the cohomology class of σ_λ. From (1.2) it is easy to see that $[\frac{d}{d\lambda}\sigma_\lambda] = c$. For each λ choose a connection 1-form β_λ on $\pi_\lambda : H^{-1}(\lambda) \to M$ so that $d\beta_\lambda = \pi_\lambda^*(\frac{d}{d\lambda}\sigma_\lambda)$. Now define on $H^{-1}(I)$ the 2-form:

$$\omega = \pi_\lambda^*(\sigma_\lambda) + d\lambda \wedge \beta_\lambda. \tag{1.3}$$

ω is non-degenerate and closed. In fact any S^1 invariant symplectic form on $H^{-1}(I)$ is equivalent to ω up to an S^1-equivariant diffeomorphism preserving the level sets of H (see [McD1]).

Given a principal circle bundle $P \to M$ with Chern class c and a family of symplectic forms $\sigma_\lambda,\ \lambda \in I$ on M satisfying (1.2) it is now easy to construct an S^1-invariant symplectic form on S_c. Let S^1 act on $P \times S^2$ by

$$t \cdot (p, z) = (p \cdot t^{-1}, t \cdot z),\ t \in S^1,$$

where S^1 acts on S^2 by rotation about the north-south axis. The quotient of $P \times S^2$ by this action is S_c. An S^1-invariant height function $h : S^2 \to [0, 1]$ taking the south pole to $\{0\}$ and the north pole to $\{1\}$ induces a map $H : S_c \to [0, 1]$ such that the level sets $H^{-1}(\lambda),\ \lambda \neq 0, 1$ are diffeomorphic to P and $Z_0 = H^{-1}(0),\ Z_\infty = H^{-1}(1)$. Using the family σ_λ of symplectic forms we can construct an S^1-invariant symplectic form ω on $S_c \backslash (Z_0 \cup Z_\infty) = H^{-1}(0, 1)$ using (1.3). Then ω may be smoothly extended over all of S_c so that ω restricts to σ_0 on Z_0 and σ_1 on Z_∞. For more details see [McD-S,Chap.4].

The proof of Theorem 1.1 will also require the following well known theorem whose proof can be found in [W]:

Symplectic Neighborhood Theorem . *Let $(Y_j, \eta_j),\ j = 1, 2$ be symplectic manifolds with symplectic submanifolds Γ_j. Suppose that there is an isomorphism of the symplectic normal bundles of Γ_1 and Γ_2, $\hat{f} : \nu(\Gamma_1) \to \nu(\Gamma_2)$, which covers a symplectic diffeomorphism $f : (\Gamma_1, \omega_1) \to (\Gamma_2, \omega_2)$. Then f may be extended to a symplectic diffeomorphism $F : (N(\Gamma_1), \omega_1) \to (N(\Gamma_2), \omega_2)$ such that $dF = \hat{f} : \nu(\Gamma_1) \to \nu(\Gamma_2)$.*

We are now ready to establish the existence of the operation of symplectic normal connect sum.

Proof of Theorem 1.1. Let S_n denote the ruled surface that is an S^2-bundle over Σ. To construct an S^1-invariant symplectic form on S_n we need only specify a family $\{\sigma_\lambda\}$ of symplectic forms on Σ subject to (1.2). By Moser's theorem, these symplectic forms are determined by their areas $\sigma_\lambda(\Sigma) = \int_\Sigma \sigma_\lambda$, up to a family $\{g_\lambda\}$ of diffeomorphisms of Σ. Thus, an S^1-invariant symplectic form τ_n on S_n is determined up to an S^1 equivariant diffeomorphism by the scalars $\{[\sigma_\lambda](\Sigma)\}$ subject to (1.2). By constructing τ_n to satisfy $\int_{Z_0} \tau_n = \int_{\Sigma_1} \omega_1$, we can suppose that there is a symplectic diffeomorphism $f : (\Sigma_1, \omega_1) \to (Z_0, \tau_n)$. The normal bundles $\nu(\Sigma_1)$ and

$\nu(Z_0)$ both have euler number n and so are isomorphic as symplectic vector bundles. Choose an isomorphism $\hat{f} : \nu(\Sigma_1) \to \nu(Z_0)$ covering f. Then the symplectic neighborhood theorem gives a symplectic diffeomorphism $F : (N(\Sigma_1), \omega_1) \to (N(Z_0), \tau_n)$ with $dF = \hat{f}$. Note that the construction of F involves a choice of a symplectic vector bundle isomorphism. Similarly we can construct a symplectic diffeomorphism $\tilde{F}$ from $(N(\Sigma_{-1}), \omega_{-1})$ to $(N(\tilde{Z}_\infty), \tilde{\tau}_n)$ where $N(\tilde{Z}_\infty)$ is a tubular neighborhood of the infinity section $\tilde{Z}_\infty$ in the ruled surface $\tilde{S}_n$. $(\tilde{S}_n, \tilde{\tau}_n)$ will not, in general, be the same as (S_n, τ_n) since we must insure that $\int_{\Sigma_{-1}} \omega_{-1} = \int_{\tilde{Z}_\infty} \tilde{\tau}_n$.

On the ruled surface (S_n, τ_n) there is a moment map $H : S_n \to [0, 1]$, where 0 and 1 are the critical values corresponding to Z_0 and Z_∞, respectively. Similarly on the ruled surface $(\tilde{S}_n, \tilde{\tau}_n)$ there is a moment map $\tilde{H} : S_n \to [0, 1]$, where 0 and 1 are the critical values corresponding to $\tilde{Z}_0$ and $\tilde{Z}_\infty$, respectively. Choose intervals I and $\tilde{I}$ such that $|I| = |\tilde{I}|$ and so that $H^{-1}(I) \subseteq F(N(\Sigma_1))$ and $\tilde{H}^{-1}(\tilde{I}) \subseteq \tilde{F}(N(\Sigma_{-1}))$. Since τ_n and $\tilde{\tau}_n$ are determined on $H^{-1}(I)$ and $\tilde{H}^{-1}(\tilde{I})$ by a family of scalars satisfying (1.2) we can rescale $\tilde{\tau}_n$ (and consequently, rescale ω_{-1}) so that there is a symplectic diffeomorphism $\varphi : (\tilde{H}^{-1}(\tilde{I}), \tilde{\tau}_n) \to (H^{-1}(I), \tau_n)$. φ takes the inside boundary of $\tilde{H}^{-1}(\tilde{I})$ to the outside boundary of $H^{-1}(I)$ and takes the level sets of $\tilde{H}$ to the level sets of H. The diffeomorphism:

$$\Psi = F^{-1} \circ \varphi \circ \tilde{F} : (\tilde{F}^{-1}(\tilde{H}^{-1}(\tilde{I})), a\omega_{-1}) \to (F^{-1}(H^{-1}(I)), \omega_1)$$

determines the required symplectic gluing map between $(X_{-1}, a\omega_{-1})$ and (X_1, ω_1) where $a \in \mathbf{R}_+$ is the scaling factor. Theorem 1.1 is proved. □

2. Invariants of Symplectic Normal Connect Sums

Let X be a symplectic normal connect sum $X_{-1}\#_\Psi X_1$ as in Theorem 1.1 where X_{-1} and X_1 are both closed. In this section, we shall compute various topological and geometric invariants of X in terms of those of X_{-1} and X_1. These computations will be used in the following sections to discuss a number of examples of symplectic normal connect sums.

Our computations will require various decompositions of X. Let U_i be the complement of $\overline{N(\Sigma_i)}$ in X_i. Then $(U_i, N(\Sigma_i))$ is an open cover of X_i. Note that $U_i \cap N(\Sigma_i)$ is equal to the tubular shell neighborhood W_i of Σ_i along which the normal connect sum X is obtained by gluing with the map Ψ. Thus, we may consider (U_{-1}, U_1) as an open cover of X with $U_{-1} \cap U_1 = W_{-1}$. With this identification, the inclusion of $U_{-1} \cap U_1$ into U_{-1} is just the inclusion of W_{-1} into U_{-1}. The inclusion of $U_{-1} \cap U_1$ into U_1, on the other hand, has been identified with the composition $\jmath \circ \Psi$ of the inclusion map $\jmath : W_1 \to U_1$ and the gluing diffeomorphism $\Psi : W_{-1} \to W_1$. (In the subsequent discussion, we shall encounter a number of inclusion maps. We shall denote all of these maps by $\jmath$.)

The first invariant which we shall discuss is the fundamental group of X, $\pi_1(X)$. By Van Kampen's Theorem, $\pi_1(X)$ is the quotient of the free product of $\pi_1(U_{-1})$ and $\pi_1(U_1)$ by relations arising from $\pi_1(U_{-1}\cap U_1)$. More precisely, from the previous identifications, we see that $\pi_1(X)$ is the universal solution of the following commutative diagram [M]:

$$\begin{array}{ccc} \pi_1(W_{-1}) & \xrightarrow{(\jmath)_*} & \pi_1(U_{-1}) \\ \Big\downarrow{\scriptstyle (\jmath\circ\Psi)_*} & & \Big\downarrow{\scriptstyle (\jmath)_*} \\ \pi_1(U_1) & \xrightarrow{(\jmath)_*} & \pi_1(X). \end{array}$$

The tubular shell neighborhood W_i of Σ_i is an oriented annulus bundle over Σ_i. This annulus bundle retracts onto an oriented circle bundle P_i over Σ_i. Hence, from the discussion in [F], we may construct a presentation for $\pi_1(W_i)$ of the following form:

$$\mathit{Generators:} \quad a_{i,1}, b_{i,1}, ..., a_{i,g}, b_{i,g}, z_i \tag{2.1}$$
$$\mathit{Relations:} \quad \prod_{j=1}^{g}[a_{i,j}, b_{i,j}] = z_i^{in}, \quad a_{i,j}z_i = z_i a_{i,j}, \quad b_{i,j}z_i = z_i b_{i,j}.$$

Note that this presentation is not natural. The "base" classes $a_{i,j}, b_{i,j}$ correspond to arbitrarily chosen lifts to the annulus bundle W_i of loops representing the standard generators of $\pi_1(\Sigma_i)$. The *fiber class* z_i corresponds to the fiber of the associated circle bundle P_i with the appropriate orientation. Note that unlike the "base" classes, $a_{i,j}, b_{i,j}$, the fiber class z_i is natural.

The relations arising from $\pi_1(U_{-1}\cap U_1)$, after making the above identifications, are the following:

$$\begin{gathered} \jmath_*(a_{-1,j}) = (\jmath\circ\Psi)_*(a_{-1,j}), \quad \jmath_*(b_{-1,j}) = (\jmath\circ\Psi)_*(b_{-1,j}), \\ \jmath_*(z_{-1}) = (\jmath\circ\Psi)_*(z_{-1}) \end{gathered} \tag{2.2}$$

Henceforth, we assume, without loss of generality, the following constraints on Ψ_*:

$$\Psi_*(z_{-1}) = z_1^{-1}, \quad \Psi_*(\prod_{j=1}^{g}[a_{-1,j}, b_{-1,j}]) = \prod_{j=1}^{g}[a_{1,j}, b_{1,j}]. \tag{2.3}$$

These constraints arise from the fact that Ψ is an orientation preserving, end reversing diffeomorphism from W_{-1} to W_1 preserving the annuli of these annulus bundles up to isotopy. It can be shown that we can prescribe Ψ to induce any isomorphism from $\pi_1(W_{-1})$ to $\pi_1(W_1)$ which satisfies these two constraints.

Next we wish to discuss the basic homological invariants of X. First of all, there are the Betti numbers of X, b_i, $0\le i\le 4$. Since X is orientable, closed and connected, $b_0 = b_4 = 1$, $b_1 = b_3$. Therefore, the euler characteristic of X satisfies $\chi = 2+2b_1+b_2$. We recall that the euler characteristic is "additive". That is, if (U, U') is an open cover of any topological space T, then:

$$\chi(T) = \chi(U) + \chi(U') - \chi(U\cap U').$$

Applying this identity to the open covers of X_{-1}, X_1 and X introduced above, we obtain the following identities:

$$\chi(X_i) = \chi(U_i) + \chi(N(\Sigma_i)) - \chi(Wi)$$
$$\chi(X) = \chi(U_{-1}) + \chi(U_1) - \chi(W_{-1}).$$

Since the euler characteristic of a closed orientable three manifold is equal to 0, $\chi(W_i) = \chi(P_i) = 0$. Recalling that Σ_i is a compact surface of genus g we have $\chi(N(\Sigma_i)) = \chi(\Sigma_i) = 2 - 2g$. Thus:

$$\chi(X) = \chi(X_{-1}) + \chi(X_1) + 4g - 4. \tag{2.4}$$

If T is a topological space, then $H_j(T)$ and $H^j(T)$ will denote $H_j(T, \mathbf{R})$ and $H^j(T, \mathbf{R})$ respectively. The *integral lattice* in $H_j(T)$ is the image of $H_j(T, \mathbf{Z})$ under the usual homomorphism $H_j(T, \mathbf{Z}) \to \mathbf{H_j(T)}$. A class in $H_j(T)$ is *integral* if it lies in the integral lattice. A subspace of $H_j(T)$ is *rational* if it has a basis consisting of integral classes. The integral lattice in $H_2(X)$ consists precisely of those classes which are represented by smoothly embedded oriented surfaces in X. If F denotes such a surface, then we shall denote the corresponding class in $H_2(X)$ by the same symbol F. The intersection pairing Q is a nondegenerate, symmetric, bilinear form on $H_2(X)$. If $\alpha, \beta \in H_2(X)$, then we shall denote $Q(\alpha, \beta)$ by $\alpha \cdot \beta$. (Q restricts to an integer valued, unimodular, symmetric bilinear form on the integral lattice.) Q is determined by its restriction to the integral lattice. If α_1 and α_2 are integral classes, then $\alpha_1 \cdot \alpha_2$ is equal to the algebraic intersection number of any pair of transverse smoothly embedded oriented surfaces F_i representing α_i. The signature σ of X is equal to $b_2^+ - b_2^-$, where b_2^+ is the rank of a maximal positive definite subspace of $H_2(X)$ and b_2^- is the rank of a maximal negative definite subspace of $H_2(X)$. In order to compute $\sigma(X)$ in terms of $\sigma(X_i)$, we shall appeal to Novikov additivity. The statement of Novikov additivity involves extending the definition of signature to compact oriented four manifolds M with boundary ∂M. The above definition is equally valid in this context. The difference between the closed case and the nonclosed case can be summarized as follows. In the closed case, the intersection pairing is nondegenerate and, hence, $b_2 = b_2^+ + b_2^-$. In the nonclosed case, however, the nullspace of Q is equal to $\jmath_*(H_2(\partial M))$. In the nonclosed case, therefore, we need not have the above relationship between b_2, b_2^+ and b_2^-. (For more details, see [K].)

Let N_i be a tubular neighborhood of Σ_i such that $\overline{\mathcal{N}(\Sigma_i)} \subset N_i$ and $\overline{N_i} \subset N(\Sigma_i)$. Let $M_i = X_i \backslash N_i$. We may assume that $\partial M_i = P_i$ and that $\Psi(P_{-1}) = P_1$. We now have a decomposition of X_i into compact four manifolds with boundary, M_i and $\overline{N_i}$, glued along their common boundary P_i by the identity map. Likewise, we have a decomposition of X into M_{-1} and M_1 glued along their boundaries by the restriction ψ of Ψ to P_{-1} and P_1. We may apply Novikov additivity to these decompositions of X_{-1}, X_1 and X. As a result, we have the following identities:

$$\sigma(X_i) = \sigma(M_i) + \sigma(\overline{N_i})$$

$$\sigma(X) = \sigma(M_{-1}) + \sigma(M_1).$$

In order to express $\sigma(X)$ in terms of $\sigma(X_i)$, we need to compute $\sigma(\overline{N_i})$. Since $\overline{N_i}$ is a closed tubular neighborhood of Σ_i in X_i, $H_2(\overline{N_i})$ is isomorphic to $\mathbf{R}$ with an integral generator Σ_i. Therefore, Σ_i is a basis for $H_2(\overline{N_i})$. Since $\Sigma_i \cdot \Sigma_i = in$, we conclude that:

$$\sigma(\overline{N_i}) = \begin{cases} 0 & \text{if } n = 0, \\ i & \text{otherwise.} \end{cases}$$

It follows immediately from the above identities that:

$$\sigma(X) = \sigma(X_{-1}) + \sigma(X_1). \tag{2.5}$$

Remark 2.1. The computations of $\chi(X)$ and $\sigma(X)$ given above, almost allow us to compute all the Betti numbers of X. If we could compute $b_1(X)$ in terms of $b_1(X_i)$, then we would be able to determine the invariants b_2, b_2^+ and b_2^- for X in terms of those for X_i. The determination of $b_1(X)$, however, involves more information than we have used above. In particular, it depends upon the diffeomorphism Ψ used to construct X.

The second Stiefel-Whitney class of X, w_2, is a characteristic $\mathbf{Z_2}$ cohomology class of dimension 2. It is the obstruction to finding a field of 3-frames over the 2-skeleton of X. As with the Betti numbers of X, the calculation of $w_2(X)$ in terms of $w_2(X_i)$ requires more information than we have used above.

Let ω be a symplectic structure on X. Integrating ω over 2 dimensional homology classes defines a homomorphism:

$$\rho_\omega : H_2(X) \to \mathbf{R}$$

which we shall refer to as the *period map of* ω. The second exterior power of ω is a nowhere zero top dimensional form. Hence, X has a naturally associated volume form ω^2. This volume form determines a natural orientation on X. Given (X, ω), we shall always orient X by this orientation.

Since (X, ω) is a symplectic manifold, it admits a unique homotopy class of compatible almost complex structures. Let J be an almost complex structure in this homotopy class. The orientation on X induced by J depends only upon the homotopy class of J. Indeed, it is easy to see that it agrees with the orientation on X determined by the volume form ω^2. The characteristic classes of (X, J) are the Chern classes, c_1 and c_2. These are integral 2 dimensional, respectively 4 dimensional, cohomology classes of X which depend only upon the homotopy class of J. Hence, they are well defined invariants of (X, ω). These invariants and the topological invariants discussed above bear the following relationship to each other ([B-P-V], chapter IV, section 7):

$$[c_1] = w_2, \quad c_1^2 = 3\sigma + 2\chi, \quad c_2 = \chi. \tag{2.6}$$

Here, $[c_1]$ denotes the reduction of c_1 modulo 2 and c_1^2 denotes the square $c_1 \cup c_1$ of c_1. c_1^2 and c_2 are called the *Chern numbers* of X. From the above identities, it is clear that the Chern numbers of X are topological invariants of X. They satisfy the following congruence ([B-P-V], chapter IV, section 7):

$$c_1^2 + c_2 \equiv 0(12). \tag{2.7}$$

From (2.4), (2.5) and (2.6), we have the following identities:

$$\begin{aligned} c_1^2(X) &= c_1^2(X_{-1}) + c_1^2(X_1) + 8g - 8 \\ c_2(X) &= c_2(X_{-1}) + c_2(X_1) + 4g - 4. \end{aligned} \tag{2.8}$$

3. Complex Surfaces

In this section, we shall discuss restrictions on the invariants of X which arise from the assumptions that X is a complex surface, a Kähler surface or a minimal surface of general type. We shall appeal to these restrictions in order to construct interesting examples of symplectic normal connect sums.

First, suppose that X is a compact complex surface. $h^{(p,q)}$ is the dimension of the Dolbeault cohomology group $H^{(p,q)}(X)$. The geometric genus p_g of X is $h^{(0,2)}$. The irregularity $q(X)$ is $h^{(0,1)}$. These invariants bear the following relationships to the invariants discussed above ([B-P-V], chapter IV, section 2):

$$\text{if } b_1(X) \text{ is even, then } b_1 = 2q \text{ and } b_2^+ = 2p_g + 1,$$
$$\text{if } b_1(X) \text{ is odd, then } b_1 = 2q - 1 \text{ and } b_2^+ = 2p_g.$$

As a consequence of these constraints, it is clear that $q(X)$ and $p_g(X)$ are topological invariants, $q(X)$ of the unoriented, and $p_g(X)$ of the oriented underlying manifold.

Now suppose that X is a Kähler surface with Kähler form ω. In particular, ω is a symplectic form on X. Of course, since X is complex, the invariants of X must satisfy the restrictions discussed above. Further restrictions arise from the Hodge decomposition ([G-H]). The main consequence of this decomposition, for our purposes, is that:

$$b_1 = 2q \text{ and } b_2^+ = 2p_g + 1.$$

In particular, b_1 is even. For complex surfaces, this actually characterizes Kähler manifolds following the work of Kodaira, Miyaoka, Siu and Todorov ([P]):

Theorem . *A compact complex surface is Kähler if and only if its first Betti number is even.*

There are also restrictions on the fundamental group of a compact Kähler manifold. In particular ([J-R]):

Theorem (Johnson-Rees) . *If G_1 and G_2 are two groups which have at least one nontrivial finite quotient each, then the free product $G_1 * G_2$ is not isomorphic to the fundamental group of any compact Kähler manifold. More generally, if H is any group, then the direct product $(G_1 * G_2) \times H$ is not isomorphic to the fundamental group of any compact Kähler manifold.*

A related result is the following ([A-B-R]):

Theorem (Arapura-Bressler-Ramachandran) . *Let G_1 and G_2 be two groups. Let F be a finite group. Let $\phi_j : F \to G_j$ be monomorphisms with $\phi_j(F) \neq G_j$. Then the free product of G_1 and G_2 amalgamated over F via ϕ_1 and ϕ_2 is not isomorphic to the fundamental group of any compact Kähler manifold.*

Note that the result of [A-B-R] implies the first half of the result of [J-R]. It also implies the second half of the result of [J-R] in the case where H is finite. This follows from the observation that $(G_1 * G_2) \times H$ is isomorphic to the free product of $G_1 \times H$ and $G_2 \times H$ amalgamated over H with respect to the obvious monomorphisms of H into $G_i \times H$. Note that the relevant hypotheses imply that G_i is a nontrivial group. Hence, the images of H under these monomorphisms are proper subgroups of $G_{-1} \times H$ and $G_1 \times H$ as required to appeal to the result of [A-B-R].

Finally, suppose that X is a minimal surface of general type. In particular, X is Kähler. The known restrictions on the basic invariants of X, beyond those discussed above, can be summarized as follows ([B-P-V], chapter VII, sections 1 and 3):

$$c_1^2 > 0, \quad c_2 > 0, \quad c_1^2 \leq 3c_2, \quad p_g \leq \frac{1}{2}c_1^2 + 2.$$

4. **Simple Examples**

To construct examples of symplectic normal connect sums we use, as building blocks, pairs (X_i, Σ_i), $i = -1, 1$, where the X_i are compact Kähler surfaces and the Σ_i are nonsingular complex curves of genus g and self-intersection in with $n \geq 0$. The initial step then is to find nonsingular complex curves in Kähler surfaces. There are, of course, many ways of doing this. Among the simplest curves are the hyperplane sections. These are obtained by intersecting a 2-dimensional complex variety in $\mathbf{CP^N}$ (an algebraic surface) with a hyperplane. Such curves, when nonsingular, have positive self-intersection. Curves of negative self-intersection are found by resolving singularities or by simply blowing up positive self-intersection curves sufficiently often. For example, let Σ_1 be a nonsingular curve of genus g and self-intersection $n \geq 0$ in a Kähler surface X_1. Let $\tilde{X_1}$ be the blow-up of X_1 at a point $p \in \Sigma_1$. The proper transform $\tilde{\Sigma_1}$ of Σ_1 is a nonsingular curve in $\tilde{X_1}$ of genus g and self-intersection $n-1$. If $\hat{X_1}$ is the blow-up of X_1 at ℓ

distinct points on Σ_1, where $\ell > n$, then the proper transform $\hat{\Sigma_1}$ of Σ_1 in $\hat{X}_1$ is a nonsingular curve of genus g and self-intersection $n - \ell < 0$. This observation leads to the following simple construction. Let (X_1, Σ_1) be as above and let X_{-1} be the blow-up of X_1 at $2n$ distinct points of Σ_1. The proper transform Σ_{-1} of Σ_1 in X_{-1} is a nonsingular curve of genus g and self-intersection $-n$. The symplectic normal connect sum of (X_i, Σ_i), $i = -1, 1$, determines a symplectic manifold $X = X_{-1}\#_\Psi X_1$ which is perhaps one of the simplest examples of a symplectic normal connect sum.

More generally, let Σ_1 and Σ_1' be nonsingular curves of genus g and self-intersection $n \geq 0$ and $n' \geq 0$, respectively, in Kähler surfaces X_1 and X_1'. Blow up X_1' at $n + n'$ distinct points on Σ_1' to obtain a Kähler surface X'_{-1} and a nonsingular curve Σ'_{-1} of genus g and self-intersection $-n$, (the proper transform of Σ_1' in X'_{-1}). Let $X' = X'_{-1}\#_\Psi X_1$ be the symplectic normal connect sum of (X'_{-1}, Σ'_{-1}) and (X_1, Σ_1). Alternatively, blow up X_1 at n distinct points on Σ_1 and blow up X_1' at n' distinct points on Σ_1' to obtain Kähler surfaces X_0 and X_0' containing nonsingular curves Σ_0 and Σ_0' of genus g and self-intersection zero. Now use the self-intersection zero symplectic normal connect sum to glue (X_0, Σ_0) to (X_0', Σ_0') together to form a symplectic manifold $X'' = X_0\#_{\Psi'}X_0'$. A standard "handle trading" argument shows (for appropriate choice of Ψ') that X'' is diffeomorphic to X'. (In fact, this handle trading argument shows that if we use blowing up to make the self-intersection numbers of a pair of nonsingular curves have opposite sign, the diffeomorphism type of the normal connect sum is insensitive to which curve we blow up.)

Examples, such as those described above, do not require the full range of Theorem 1.1, but as we saw, can be constructed (up to diffeomorphism) with self-intersection zero gluing alone. The crucial point is that each negative curve Σ_{-1} in these examples is obtained by blowing up a nonsingular curve of nonnegative self-intersection. We say that Σ_{-1} is not "genuinely negative". A "genuinely negative" curve is a nonsingular curve of negative self-intersection which cannot be blown down to a nonsingular curve of nonnegative self-intersection. In general, symplectic normal connect sums built using a genuinely negative curve Σ_{-1} cannot be constructed using the self-intersection zero gluing. For this reason, such examples are of particular interest. In the following sections, we will give many examples of this type.

The abundance of examples of nonsingular curves of both positive and negative self-intersection in many different Kähler surfaces shows that the symplectic normal connect sum gives many easily constructed examples of compact symplectic manifolds. Moreover, the Chern numbers and, often, other classical invariants of these examples are easily computed. However, the genus and self-intersection numbers of the curves Σ_i are not themselves sufficient information to determine whether the symplectic manifold $(X_{-1}\#_\Psi X_1, \omega)$ is or is not a Kähler manifold. Thus, to determine if $X_{-1}\#_\Psi X_1$ is a *new* symplectic manifold, further information about the pairs

(X_i, Σ_i) is needed. The required information is often difficult to calculate. This is the subject of the next two sections.

5. Non-Kähler Symplectic Manifolds With b_1 Odd

In this section, we shall give a number of examples of symplectic manifolds whose underlying smooth manifold does not admit any Kähler structure. These manifolds will all be constructed as symplectic normal connect sums of Kähler manifolds. They all have b_1 odd and, hence, are not homeomorphic to any Kähler surface, though some are homeomorphic to complex surfaces.

There are many known examples of symplectic manifolds which are not homeomorphic to Kähler manifolds. In 1976, Thurston gave examples of closed non-Kähler symplectic manifolds by producing closed symplectic manifolds with odd first Betti number. Thurston's examples are surface bundles over symplectic manifolds. His construction of a symplectic structure on these bundles involves the bundle structure. Our first example will be a construction of one of the simplest of Thurston's examples via the operation of symplectic normal connect sum, (or, more precisely, via a slight variation of this operation).

Example 5.1 (Thurston's Torus Bundle Over a Torus)**.** Consider the product $T^2 \times S^2$ equipped with the product Kähler structure. Let x_{-1} and x_1 be a pair of distinct points on S^2. Let Σ_i denote the surface $T^2 \times \{x_i\}$. The surfaces Σ_{-1} and Σ_1 are symplectically embedded surfaces which represent the same homology class in $T^2 \times S^2$ and, hence, have the same symplectic area. The self-intersection of each of these surfaces is equal to 0. Hence, by (2) in remark 1.1, we can glue a tubular shell neighborhood W_{-1} of Σ_{-1} to a tubular shell neighborhood W_1 of Σ_1 by a symplectomorphism Ψ taking the inside end of W_{-1} to the outside end of W_1.

Remark 5.1. Note that in this example we are symplectically gluing the complement of $\overline{\mathcal{N}(\Sigma_{-1})} \cup \overline{\mathcal{N}(\Sigma_1)}$ in $T^2 \times S^2$ to itself along tubular shell neighborhoods of Σ_{-1} and Σ_1. This is possible because Σ_{-1} and Σ_1 are disjoint, have zero self-intersection and the same area. In the case of nonzero self-intersection, the necessity of scaling the symplectic forms in order to glue makes it difficult to perform this type of operation.

Let U be the complement of $\overline{\mathcal{N}(\Sigma_{-1})} \cup \overline{\mathcal{N}(\Sigma_1)}$ in $T^2 \times S^2$. U is the product of T^2 with the two holed sphere $S^2 \setminus (B(x_{-1}) \cup B(x_1))$. Hence, we have the following presentation for $\pi_1(U)$:

$$\textit{Generators}: \quad a, \quad b, \quad z$$

$$\textit{Relations}: \quad [a,b] = 1, \quad az = za, \quad bz = zb.$$

Likewise, we have the following presentations for $\pi_1(W_i)$:

$$\textit{Generators}: \quad a_i, \quad b_i, \quad z_i$$

$$\textit{Relations}: \quad [a_i, b_i] = 1, \quad a_i z_i = z_i a_i, \quad b_i z_i = z_i b_i.$$

In terms of the above presentations, the homomorphism induced by the inclusion of W_i in U can be described as follows:

$$\jmath_*(a_i) = a, \quad \jmath_*(b_i) = b, \quad \jmath_*(z_i) = z.$$

By choosing Ψ appropriately, as explained in section 2, we may prescribe the following restrictions on the isomorphism Ψ_*:

$$\Psi_*(a_{-1}) = a_1 z_1, \quad \Psi_*(b_{-1}) = b_1, \quad \Psi_*(z_{-1}) = z_1^{-1}.$$

From the relations in (2.2), a standard application of Van Kampen's theorem yields a presentation of $\pi_1(X)$ as an HNN extension ([L-S]) of $\pi_1(U)$:

$$\begin{gathered} Generators: \quad a, \quad b, \quad z, \quad t \\ Relations: \quad [a,b]=1, \quad az=za, \quad bz=zb \\ tat^{-1}=az, \quad tbt^{-1}=b, \quad tzt^{-1}=z. \end{gathered}$$

Since $H_1(X, \mathbf{Z})$ is the abelianization of $\pi_1(X)$, we conclude that $H_1(X, \mathbf{Z})$ is a free abelian group of rank 3, with basis given by the homology classes of a, b and t. In particular, $b_1(X)$ is equal to 3. As discussed above, this demonstrates that X is not homeomorphic to any Kähler manifold. (On the other hand, it is easy to see that X is homeomorphic to one of Thurston's examples.)

We now wish to describe a scheme for producing examples of compact symplectic 4-manifolds with b_1 odd. To understand this scheme we need to compute b_1 of a normal connect sum. In order to do this, we shall again use the decomposition of X described in section 2. By applying the Mayer-Vietoris sequence to the covering (U_{-1}, U_1) of X, we obtain the following right exact sequence:

$$H_1(U_{-1} \cap U_1) \xrightarrow{\jmath_* \oplus (\jmath \circ \Psi)_*} H_1(U_{-1}) \oplus H_1(U_1) \to H_1(X) \to 0.$$

From this sequence, we see that:

$$b_1(X) = b_1(U_{-1}) + b_1(U_1) - rank(\jmath_* \oplus (\jmath \circ \Psi)_*).$$

By applying the Mayer-Vietoris sequence to the covering $(U_i, N(\Sigma_i))$ of X_i, on the other hand, it follows that:

$$b_1(X_i) = b_1(U_i) + b_1(N(\Sigma_i)) - rank(\jmath_* \oplus \jmath_*).$$

The intersection term $U_i \cap N(\Sigma_i)$ of this second Mayer-Vietoris sequence is equal to W_i, the annulus bundle corresponding to the disc bundle $N(\Sigma_i)$ over Σ_i. It follows that $\jmath_*$ maps $H_1(U_i \cap N(\Sigma_i))$ onto $H_1(N(\Sigma_i))$. Hence:

$$b_1(N(\Sigma_i)) \le rank(\jmath_* \oplus \jmath_*).$$

Now consider the fiber class z_i of $\pi_1(W_i)$. Since Σ_i is a compact symplectic surface in (X_i, ω_i), it represents a nontrivial class α_i in $H_2(X_i)$. (The cohomology class of the symplectic form ω_i on X_i evaluates nontrivially on α_i.) The intersection pairing Q on $H_2(X_i)$ is nondegenerate and $H_2(X_i)$ has an integral basis. Thus there exists an integral homology class β_i in $H_2(X_i)$

such that $\alpha_i \cdot \beta_i = m_i$ where m_i is a nonzero integer. β_i can be represented by a smoothly embedded, oriented surface F_i. (See [K], chapter II, section 1.) We may assume that F_i is transverse to Σ_i. Thus, we may assume that $F_i \setminus N(\Sigma_i)$ is a smoothly embedded, oriented surface in U_i whose boundary is a disjoint union of circles each representing the homology class $[z_i]$ of z_i or $-[z_i]$. From the definition of the intersection pairing, this boundary represents $m_i[z_i]$. Hence, since $m_i \neq 0$, $[z_i] = 0$ in $H_1(U_i)$. On the other hand, clearly z_i is homologically trivial in $N(\Sigma_i)$. Thus, $[z_i]$ is in the kernel of $\jmath_* \oplus \jmath_*$. From (2.1), it follows that:

$$rank(\jmath_* \oplus \jmath_*) \leq 2g.$$

But $b_1(N(\Sigma_i)) = b_1(\Sigma_i) = 2g$. Hence, we conclude that $b_1(U_i) = b_1(X_i)$. Indeed, we see that the inclusion homomorphism:

$$H_1(U_i) \stackrel{\cong}{\longrightarrow} H_1(X_i)$$

is an isomorphism.

Let $\pi_i : W_i \to \Sigma_i$ denote the projection map of the annulus bundle W_i over Σ_i. For any circle γ in W_i, γ and $\pi_i(\gamma)$ are homologous in $N(\Sigma_i)$ and, hence, in X_i. We may assume, for homological purposes, that Ψ covers an orientation preserving diffeomorphism Ψ_0 from Σ_{-1} to Σ_1. That is, we may assume that we have a commutative diagram as follows:

$$\begin{array}{ccc} W_{-1} & \xrightarrow{\Psi} & W_1 \\ \Big\downarrow{\scriptstyle \pi_{-1}} & & \Big\downarrow{\scriptstyle \pi_1} \\ \Sigma_{-1} & \xrightarrow{\Psi_0} & \Sigma_1. \end{array}$$

From these observations, we see that the following diagram is commutative:

$$\begin{array}{ccc} H_1(W_{-1}) & \xrightarrow{\jmath_*\oplus(\jmath\circ\Psi)_*} & H_1(U_{-1}) \oplus H_1(U_1) \\ \Big\downarrow{\scriptstyle (\pi_{-1})_*} & & \Big\downarrow{\scriptstyle \jmath_*\oplus\jmath_*} \\ H_1(\Sigma_{-1}) & \xrightarrow{\jmath_*\oplus(\jmath\circ\Psi_0)_*} & H_1(X_{-1}) \oplus H_1(X_1). \end{array}$$

Since W_{-1} is an annulus bundle over Σ_{-1} and π_{-1} is the corresponding bundle projection map, $(\pi_{-1})_*$ is surjective. On the other hand, by the previous discussion, $\jmath_* \oplus \jmath_*$ from $H_1(U_{-1}) \oplus H_1(U_1)$ to $H_1(X_{-1}) \oplus H_1(X_1)$ is an isomorphism. Hence, the horizontal homomorphisms in this last diagram have the same rank:

$$rank(\jmath_* \oplus (\jmath \circ \Psi)_*) = rank(\jmath_* \oplus (\jmath \circ \Psi_0)_*).$$

From the above identities, we conclude that:

$$b_1(X) = b_1(X_{-1}) + b_1(X_1) - rank(\jmath_* \oplus (\jmath \circ \Psi_0)_*).$$

On the other hand:

$$2g = b_1(\Sigma_{-1}) = nullity(\jmath_* \oplus (\jmath \circ \Psi_0)_*) + rank(\jmath_* \oplus (\jmath \circ \Psi_0)_*).$$

Therefore:

$$b_1(X) = b_1(X_{-1}) + b_1(X_1) + nullity(\jmath_* \oplus (\jmath \circ \Psi_0)_*) - 2g. \tag{5.1}$$

Of course, since Ψ_0 is a diffeomorphism:

$$kernel[\jmath_* \oplus (\jmath \circ \Psi_0)_*] = kernel[\jmath_*] \cap [((\Psi_0)_*)^{-1}(kernel(\jmath_*))]. \tag{5.2}$$

Let K_i denote the kernel of $\jmath_* : H_1(\Sigma_i) \to H_1(X_i)$. From (5.1) and (5.2), we have the following observations regarding the parity of $b_1(X)$ (when $b_1(X_i)$ is even):

$$b_1(X) \equiv \begin{cases} 0 \pmod 2 & \text{if } K_{-1} = 0 \text{ or } K_1 = 0\ , \\ rank(K_{-1}) \pmod 2 & \text{if } K_1 = H_1(\Sigma_1), \\ rank(K_1) \pmod 2 & \text{if } K_{-1} = H_1(\Sigma_{-1}). \end{cases}$$

Hence, if K_i is not a proper subspace of $H_1(\Sigma_i)$ for either $i = -1$ or $i = 1$, then the parity of $b_1(X)$ is independent of the choice of the symplectic gluing map, Ψ. (Indeed, in these cases, $b_1(X)$ is independent of Ψ.)

As we shall see, the situation is very different when K_i is a proper subspace of $H_1(\Sigma_i)$ for $i = -1, 1$. Henceforth, we assume that we are in this situation. It is easy to see that K_i is actually a rational subspace of $H_1(\Sigma_i)$. We recall that a nonzero integral class is *primitive* if it is not an integer multiple of another integral class by an integer greater than 1. We shall need the following result.

Theorem 5.1. *Let Σ be a closed orientable Riemann surface of genus $g \geq 1$. Let V_{-1} and V_1 be proper rational subspaces of $H_1(\Sigma)$. Then there exists a diffeomorphism $f : \Sigma \to \Sigma$ such that the rank of $f_*(V_{-1}) \cap V_1$ is odd.*

Proof. Let $V_0 = V_{-1} \cap V_1$ and let V_2 denote the subspace of $H_1(\Sigma)$ spanned by V_{-1} and V_1. Let r_i denote the rank of V_i. We begin by reducing the problem to the case where V_0 and V_2 are proper subspaces of $H_1(\Sigma)$.

We recall that we have a $\mathbf{Z}$-valued, nondegenerate, unimodular, antisymmetric pairing J on $H_1(\Sigma, \mathbf{Z})$ defined by algebraic intersection of 1-cycles. Let $V_i^{\perp}$ be the perpendicular subspace of V_i in $H_1(\Sigma, \mathbf{Z})$ with respect to this pairing J. Since J is nondegenerate, the rank of $V_i^{\perp}$ is equal to $2g - r_i$. Since $r_i < 2g$, there is a nonzero class $\alpha_i \in V_i^{\perp}$. Since J is $\mathbf{Z}$-valued, we can assume that α_i is an integral class. In addition, of course, we can assume that α_i is primitive. It is well known that any primitive integral class in $H_1(\Sigma)$ can be represented by a nonseparating simple closed curve. Let γ_i be such a curve representing α_i. It is also well known that any two nonseparating simple closed curves on Σ are equivalent up to a diffeomorphism of Σ. Thus, there exists a diffeomorphism f_0 of Σ such that $f_0(\gamma_{-1}) = \gamma_1$ and, hence, $(f_0)_*(\alpha_{-1}) = \alpha_1$. Thus, without loss of generality, we may assume that $\alpha_{-1} = \alpha_1$. Since $\alpha_1 \in V_i^{\perp}$ for $i = -1, 1$, $V_i \subset \{\alpha_1\}^{\perp}$. It follows that

$V_2 \subset \{\alpha_1\}^\perp$. On the other hand, since $<,>$ is nondegenerate, the rank of $\{\alpha_1\}^\perp$ is equal to $2g-1$. Hence, V_2 is a proper subspace of $H_1(\Sigma)$.

If $V_0 \neq \{0\}$, then we have reached the desired reduction. Suppose, on the other hand that $V_0 = \{0\}$. Let r_i denote the rank of V_i. Then $r_2 = r_{-1}+r_1 \leq 2g$. Since V_i is a nontrivial rational subspace of $H_1(\Sigma)$ for $i = -1, 1$, there exists a nonzero primitive integral class β_i in V_i for $i = -1, 1$. As in the previous paragraph, we may choose a diffeomorphism f_1 of Σ such that $(f_1)_*(\beta_{-1}) = \beta_1$. Let $V'_{-1} = (f_1)_*(V_{-1})$, $V'_1 = V_1$, $V'_0 = V'_{-1} \cap V'_1$ and V'_2 be the subspace of $H_1(\Sigma)$ spanned by V'_{-1} and V'_1. Let r'_i denote the rank of V'_i. By the choice of f_1, $V'_0 \neq \{0\}$. On the other hand, since f_1 is a diffeomorphism, $r'_{-1} = r_{-1}$. Of course, $r'_1 = r_1$. Since $V'_0 \neq \{0\}$, $r'_2 < r'_{-1} + r'_1 \leq 2g$. Hence, V'_0 and V'_2 are proper subspaces of $H_1(\Sigma)$. We have reached the desired reduction.

We may assume, without loss of generality, that V_0 and V_2 are proper subspaces of $H_1(\Sigma)$. Likewise, of course, we may assume that the rank of V_0 is even. Since V_0 is a proper subspace of $H_1(\Sigma)$, $V_0^\perp$ is a proper subspace of $H_1(\Sigma)$. Hence, we can choose a nonseparating simple closed curve c on Σ such that the homology class $[c]$ of c is neither in $V_0^\perp$ nor in V_2. Let f be the Dehn twist about c. This is a diffeomorphism of Σ which is supported in an annular neighborhood of c and twists this neighborhood in a "barber pole" fashion ([B]). The action of f on $H_1(\Sigma)$ is given by the following formula:

$$f_*(\alpha) = \alpha + J([c], \alpha)[c].$$

We shall show that $f_*(V_{-1}) \cap V_1 = V_0 \cap \{c\}^\perp$. Suppose that $\beta \in f_*(V_{-1}) \cap V_1$. In particular, $\beta \in V_1$. Moreover, there exists a class $\alpha \in V_{-1}$ such that $f_*(\alpha) = \beta$. By the previous formula:

$$\beta = \alpha + J([c], \alpha)[c].$$

Suppose that $J([c], \alpha) \neq 0$. Then:

$$[c] = (\beta - \alpha)/J([c], \alpha).$$

Since $\beta \in V_1$ and $\alpha \in V_{-1}$, this implies that $[c] \in V_2$. This contradicts our choice of c. Hence, $J([c], \alpha) = 0$. Hence, from the previous formula, $\beta = \alpha$. This implies that $\beta \in V_{-1}$. Hence, $\beta \in V_0$. Furthermore, it implies that $J([c], \beta) = 0$. Hence, $\beta \in \{c\}^\perp$. Thus, $\beta \in V_0 \cap \{c\}^\perp$.

Suppose, on the other hand, that $\beta \in V_0 \cap \{c\}^\perp$. Then $\beta \in V_{-1}$, $\beta \in V_1$ and $J([c], \beta) = 0$. This last equality implies that $f_*(\beta) = \beta$. Hence, $\beta \in f_*(V_{-1})$. Thus, $\beta \in f_*(V_{-1}) \cap V_1$.

Since J is nondegenerate and $[c] \neq 0$, $\{[c]\}^\perp$ is a subspace of $H_1(\Sigma)$ of codimension 1. Since $[c]$ does not lie in $V_0^\perp$, V_0 is not contained in $\{[c]\}^\perp$. Hence, $V_0 \cap \{[c]\}^\perp$ is a subspace of V_0 of codimension 1. Since, by assumption, the rank of V_0 is even, the rank of $V_0 \cap \{[c]\}^\perp$ is odd. In other words, the rank of $f_*(V_{-1}) \cap V_1$ is odd. This completes the proof of the theorem. □

Remark 5.2. Let Σ, V_{-1} and V_1 be as above. It is clear that the proof of Theorem 5.1 also establishes that there exists a diffeomorphism $f' : \Sigma \to \Sigma$ such that the rank of $f'_*(V_{-1}) \cap V_1$ is even.

We may apply this result to the subspaces $V_{-1} = K_{-1}$ and $V_1 = (\Psi_0)_*^{-1}(K_1)$. Let Ψ' be a symplectic gluing map such that Ψ'_0 is isotopic to $\Psi_0 \circ f$, where f is given by Theorem 5.1. Then the normal connect sum $X_{-1}\#_{\Psi'}X_1$ has odd b_1. Hence, we have proved the following theorem:

Theorem 5.2. *Suppose that $\Sigma_i \hookrightarrow X_i$ are symplectically imbedded compact surfaces of genus g and that $\chi(\nu_{-1}) = -\chi(\nu_1)$ where ν_i is the normal bundle of Σ_i in X_i. Suppose that the kernel K_i of the inclusion homomorphism from $H_1(\Sigma_i)$ to $H_1(X_i)$ is a proper subspace of $H_1(\Sigma_i)$ for $i = -1, 1$. Then there exists a symplectomorphism Ψ of tubular shell neighborhoods of Σ_{-1} and Σ_1 so that $b_1(X_{-1}\#_\Psi X_1)$ is odd.*

Remark 5.3. By remark 5.2, there is also a symplectomorphism Ψ' of tubular shell neighborhoods of Σ_{-1} and Σ_1 so that $b_1(X_{-1}\#_{\Psi'}X_1)$ is even.

We now wish to describe a method for producing symplectically embedded surfaces $\Sigma \hookrightarrow (X,\omega)$ such that the kernel K of $\jmath_* : H_1(\Sigma) \to H_1(X)$ is a proper subspace of $H_1(\Sigma)$. Together with Theorem 5.2, this method provides the scheme promised above. As we shall see, our method has considerable flexibility. Our construction involves the fibered product of two branched covering maps between Riemann surfaces ([H], Chapter II, section 3).

We shall need the following facts about branched coverings. Let $\phi : M \to N$ be a branched covering map of degree d between compact, orientable surfaces M and N. Let Λ_ϕ denote the singular set of ϕ. Let B_ϕ denote the corresponding branch set $B_\phi = \phi(\Lambda_\phi)$. Let $M_0 = M \setminus \phi^{-1}(B_\phi)$ and $N_0 = N \setminus B_\phi$. The restriction of ϕ to M_0 and N_0 is an unbranched covering map ϕ_0 of degree d. As such ϕ_0 is determined by its monodromy representation $\rho(\phi) : \pi_1(N_0) \to \mathcal{S}_d$, where $\pi_1(N_0)$ is the fundamental group of N_0 and $\mathcal{S}_d$ is the symmetric group on d symbols. If p is a point in M, then $deg_p(\phi)$ denotes the degree of ϕ at p. If $deg_p(\phi) > 1$, then p is a branch point of ϕ. The singular set of ϕ consists precisely of the branch points of ϕ. The *total branching number* of ϕ is the sum $\beta(\phi) = \sum_{p\in M}(deg_p(\phi) - 1)$. The Riemann-Hurwitz relation states that $\chi(M) = d\chi(N) - \beta(\phi)$. (For more details, see [B-E], [G-H].)

Let $R_j, j = 1,2,3$ be closed Riemann surfaces of genus g_j. Our idea is to construct a curve C in $R_1 \times R_2$ and obtain the desired surface Σ as a proper transform of C in an appropriate blow up X of $R_1 \times R_2$. There are several requirements which we shall need to meet in order for Σ to satisfy the restrictions imposed by Theorems 1.1 and 5.2. C is constructed as follows. Let $f_j : R_j \to R_3$, $j = 1, 2$, be nonconstant holomorphic maps of degrees d_{ji}. Let C be the following subset of $R_1 \times R_2$:

$$C = \{(x,y) \in R_1 \times R_2 \mid f_1(x) = f_2(y)\}. \tag{5.3}$$

C is a complex curve in $R_1 \times R_2$ ([B-P-V]). This is implicit in the following local description of C, which also gives a complete description of the singularities of C. Let (x, y) be a point in C and let $u = f_1(x) = f_2(y)$ be the corresponding point in R_3. Choose a local coordinate ζ on R_3 vanishing at u. Since f_j is a nonconstant holomorphic map between Riemann surfaces, a standard argument shows that we can choose a local coordinate z on R_1 vanishing at x and a local coordinate w on R_2 vanishing at y such that in terms of these local coordinates we can write f_1 and f_2 as follows:

$$\zeta = f_1(z) = z^p \quad \zeta = f_2(w) = w^q$$

for some positive integers p and q. (See [F-K], chapter I, section 1.) These conclusions imply that f_j is a branched covering map with p the degree of f_1 at x and q the degree of f_2 at y. The pair (z, w) defines a local chart in a neighorhood $\mathcal{U}$ of (x, y) in $R_1 \times R_2$. In this local chart, C has the following description:

$$C \cap \mathcal{U} = \{(z, w) \mid z^p = w^q\}.$$

Note that this description proves that C is a complex curve on $R_1 \times R_2$ ([B-P-V]). It should be noted that C need not be irreducible. In order for Σ to be connected, it is necessary for C to be irreducible. Hence, we shall need to impose further restrictions to ensure that Σ is connected. If either p or q is equal to 1, then this description shows that C is nonsingular at (x, y). (In particular, if f_j is a covering map, then C is nonsingular. Likewise, if the critical values of f_1 are disjoint from those of f_2, then C is nonsingular.) Otherwise, C is singular at (x, y). We shall say that C has a *simple singularity of type* (p, q) at (x, y). Clearly, by this discussion, each singularity of C is a simple singularity of type (p, q) for some (p, q) with $p, q \geq 2$. The singularities of C are isolated and there are only finitely many. Moreover, from this local description of C and the local description of blowing up a surface at a point, it is a simple matter to see that the proper transform Σ of C in an appropriate blow up X of $R_1 \times R_2$ is a smoothly embedded complex curve. For instance, suppose that C has exactly one singularity (x, y) and that (x, y) is a simple singularity of type $(2, 2)$. The local description of C given above shows that this singularity is an ordinary double point. That is, near (x, y) the curve C looks like a pair of distinct complex lines in $\mathbf{C^2}$ passing through the origin, $\{(z, w) \mid w = z\}$ and $\{(z, w) \mid w = -z\}$. If we blow up $R_1 \times R_2$ once at the point (x, y), then the proper transform of C in the blown up surface is a smooth embedded complex curve. (In general, of course, we may have to blow up several times in order to desingularize C. For instance, the proper transform of a simple singularity v of type $(2, 5)$ after blowing up once at v is a simple singularity of type $(2, 3)$. After blowing up once more at the proper transform of v, one has a simple "singularity" of type $(2, 1)$, which is a smooth point.)

Our examples will be built from pairs (X, Σ) where X is a blow-up of $R_1 \times R_2$ and Σ is a desingularization of C. We must ensure that Σ satisfies various properties. In particular, Σ must be connected and the kernel of

$\jmath_* : H_1(\Sigma) \to H_1(X)$ must be proper. In addition, we must find pairs $(X_i, \Sigma_i), i = -1, 1$ to glue. All of these conditions can be met with careful choices of R_1, R_2, R_3, f_1 and f_2. This we do in the following. We remark that while we could proceed in complete generality the necessary computations become very lengthy and tedious. Our intent is to construct interesting examples illustrating Theorem 5.2. Thus, after some general remarks about conditions which guarantee the connectedness of Σ and the properness of K, we will make some simplifying assumptions about R_1, R_2, R_3, f_1 and f_2 and leave the many other cases to the interested reader.

A: Connectedness of Σ

Suppose that Σ is the desingularization of C in an appropriate blow up X of $R_1 \times R_2$. We wish to determine sufficient conditions for Σ to be connected. (In algebraic terms, we wish to ensure that C is irreducible.) Let D be the exceptional divisor in X corresponding to the sequence of blow ups required to obtain X, (where D is the empty divisor if $X = R_1 \times R_2$). Let $\tau : X \to R_1 \times R_2$ denote the holomorphic map obtained by blowing down D in X. There exist holomorphic functions $h_j : \Sigma \to R_j$ such that the restriction $\tau| : \Sigma \to R_1 \times R_2$ is equal to (h_1, h_2). Consider the induced homomorphism of fundamental groups:

$$(f_1)_* : \pi_1(R_1 \setminus f_1^{-1}(B_{f_2})) \to \pi_1(R_3 \setminus B_{f2}).$$

We have the following criterion for Σ to be connected.

Theorem 5.3. *Σ is connected if and only if the restriction of the monodromy representation $\rho(f_2)$ to $(f_1)_*(\pi_1(R_1 \setminus f_1^{-1}(B_{f_2})))$ is transitive.*

Proof. Note that we have the following commutative diagram:

$$\begin{array}{ccc} \Sigma & \xrightarrow{h_2} & R_2 \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle f_2} \\ R_1 & \xrightarrow{f_1} & R_3 \end{array}$$

From the definition of C and the fact that f_1 and f_2 are nonconstant holomorphic maps, it is clear that h_j is a nonconstant holomorphic map. In particular, therefore, $h_1 : \Sigma \to R_1$ is a branched covering map. Hence, $h_1^{-1}(f_1^{-1}(B_{f_2}))$ is a finite set of points in the surface Σ. Thus Σ is connected if and only if $\Sigma \setminus h_1^{-1}(f_1^{-1}(B_{f_2}))$ is connected. It is easy to see from the local descriptions discussed above that $B_{h_1} \subset f_1^{-1}(B_{f_2})$. Since $f_1^{-1}(B_{f_2})$ is a finite set of points in the connected surface R_1, $R_1 \setminus f_1^{-1}(B_{f_2})$ is connected. Hence, the restriction:

$$h_1| : \Sigma \setminus h_1^{-1}(f_1^{-1}(B_{f_2})) \to R_1 \setminus f_1^{-1}(B_{f_2})$$

is an unbranched covering map over a connected base. By standard covering space theory, the total space $\Sigma \setminus h_1^{-1}(f_1^{-1}(B_{f_2}))$ of $h_1|$ is connected if and only if the monodromy representation:

$$\rho(h_1|) : \pi_1(R_1 \setminus f_1^{-1}(B_{f_2})) \to \mathcal{S}_{d_2}$$

is transitive ([B-E]). The proof will follow by comparing $\rho(h_1|)$ with the monodromy representation of the branched covering map f_2:

$$\rho(f_2) : \pi_1(R_3 \setminus B_{f_2}) \to \mathcal{S}_{d_2}.$$

Let x be the basepoint for $\pi_1(R_1 \setminus f_1^{-1}(B_{f_2}))$. We assume that $f_1(x)$ is the basepoint for $\pi_1(R_3 \setminus B_{f_2})$. Since $B_{h_1} \subset f_1^{-1}(B_{f_2})$ and x is not in $f_1^{-1}(B_{f_2})$, the fiber of h_1 over x can be identified, via h_2, with the fiber of f_2 over $f_1(x)$. Since $f_1(x)$ is not a critical value for the degree d_2 map f_2, this fiber $f_2^{-1}(f_1(x))$ consists of d_2 distinct points, $\{y_1, ..., y_{d_2}\}$. Suppose that γ is a loop in $R_1 \setminus f_1^{-1}(B_{f_2})$ based at x. We wish to compute the permutation $\rho(h_1|)(\gamma)$ in the symmetric group $\mathcal{S}_{d_2}$. Let j be an integer with $1 \leq j \leq d_2$. Suppose that $\tilde{\gamma}$ is the unique path in $\Sigma \setminus h_1^{-1}(f_1^{-1}(B_{f_2}))$ such that $h_1 \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = p_j$ where $h_2(p_j) = y_j$. Suppose that $\tilde{\gamma}(1) = p_{j'}$. By the definition of the monodromy representation of a covering map, $(\rho(h_1|)(\gamma))(j) = j'$. Let η be the loop $f_1 \circ \gamma$ in $R_3 \setminus B_{f_2}$ based at $f_1(x)$. By the previous commutative diagram, $h_2 \circ \tilde{\gamma}$ is a path $\tilde{\eta}$ in $R_2 \setminus f_2^{-1}(B_{f_2})$ such that $f_2 \circ \tilde{\eta} = \eta$. Of course, $\tilde{\eta}(0) = y_j$ and $\tilde{\eta}(1) = y_{j'}$. Hence, $(\rho(f_2)(\eta))(j) = j'$. Hence, the following diagram is commutative:

$$\begin{array}{ccc} \pi_1(R_1 \setminus f_1^{-1}(B_{f_2})) & \xrightarrow{\rho(h_1|)} & \mathcal{S}_{d_2} \\ \downarrow (f_1)_* & & \| \\ \pi_1(R_3 \setminus B_{f2})] & \xrightarrow{\rho(f_2)} & \mathcal{S}_{d_2}. \end{array}$$

By this commutative diagram and the previous observations, $\rho(h_1|)$ is transitive if and only if the restriction of $\rho(f_2)$ to $(f_1)_*(\pi_1(R_1 \setminus f_1^{-1}(B_{f_2})))$ is transitive. The proof follows from this equivalence and the previous observations. □

Note that the theorem applies with f_1 and f_2 interchanged. Hence, we can appeal to either criteria to establish the connectedness of Σ. Later, we shall describe a sufficient condition which ensures that at least one (and, hence, both) of these two conditions is satisfied in a rather general context. As indicated above, this context will involve some simplifying assumptions. We stress, however, that these assumptions are not necessary for the general scheme which we are presently discussing.

B: The kernel of $\jmath_*$

Henceforth, we assume that Σ is connected. Let g be the genus of Σ. We wish to ensure that the kernel K of $\jmath_* : H_1(\Sigma) \to H_1(X)$ is a proper subspace of $H_1(\Sigma)$. If $g_1 = 0$ and $g_2 = 0$, then $H_1(X) = \{0\}$. In this case, K is not proper. On the other hand, when $g_1 \geq 1$, we have the following criterion to ensure that K is proper.

Theorem 5.4. *If $g_1 \geq 1$ and $g > g_1 + g_2$, then K is proper.*

Proof. Since $\tau_* : H_1(X) \xrightarrow{\cong} H_1(R_1 \times R_2)$ is an isomorphism, K is equal to the kernel of :

$$(h_1, h_2)_* : H_1(\Sigma) \to H_1(R_1 \times R_2).$$

This homomorphism may be naturally identified with :

$$(h_1)_* \oplus (h_2)_* : H_1(\Sigma) \to H_1(R_1) \oplus H_1(R_2).$$

Thus:

$$K = kernel((h_1)_*) \cap kernel((h_2)_*). \tag{5.4}$$

Since $h_1 : \Sigma \to R_1$ is a holomorphic map between Riemann surfaces, we may compute its degree by computing the number of points in $h_1^{-1}(x)$ where x is a regular value for h_1. Since C has only finitely many singularities, we may choose $x \in R_1$ so that $\pi_1^{-1}(x)$ avoids the singularities of C, where π_1 denotes projection onto the first factor of $R_1 \times R_2$. Since D is the preimage under τ of the singular set of C and since τ is an isomorphism in the complement of D, we can identify $h_1^{-1}(x)$ with $\tau(h_1^{-1}(x))$. But:

$$\tau(h_1^{-1}(x)) = C \cap \pi_1^{-1}(x) = \{x\} \times f_2^{-1}(f_1(x)).$$

Since f_1 is a branched covering map, it is an open map. Hence, we may assume, in addition to the previous assumption on x, that $f_1(x)$ is a regular value for f_2. Since f_2 is a holomorphic map of degree d_2, $\# f_2^{-1}(f_1(x)) = d_2$. Hence, we see that h_1 has degree d_2. Likewise, h_2 has degree d_1.

Since the degree of h_j is nonzero, h_j is a nonconstant holomorphic map. Hence, h_j is a branched covering map between closed, orientable surfaces. It follows that $(h_j)_* : H_1(\Sigma) \to H_1(R_j)$ is surjective. Since $g_1 \geq 1$, $H_1(R_1) \neq \{0\}$. Since $(h_1)_*$ is surjective, it follows from (5.4) that K is not equal to $H_1(\Sigma)$. Suppose that $K = \{0\}$. Then $(h_1)_* \oplus (h_2)_*$ is injective. Hence, $g \leq g_1 + g_2$. This violates our hypothesis and the proof is complete. □

We have the following corollary of Theorem 5.4.

Corollary 5.1. *If $R_2 = S^2$ and $g_1, d_2 \geq 2$, then K is proper.*

Proof. Applying the Riemann-Hurwitz relation to the branched covering map h_1, we conclude that :

$$2 - 2g = d_2(2 - 2g_1) - \beta(h_1).$$

Since $\beta(h_1) \geq 0$ and $g_1, d_2 \geq 2$, this identity implies that $g > g_1$. Since $R_2 = S^2$, $g_2 = 0$. Hence, $g > g_1 + g_2$. Since $g_1 \geq 2$, the result follows immediately from Theorem 5.4. □

C: Simplifying Assumptions

Applying the Riemann-Hurwitz relation to the branched covering maps f_1, f_2, h_1 and h_2, we have the following identities:

$$\begin{aligned} 2-2g_1 &= d_1(2-2g_3) - \beta(f_1) \\ 2-2g_2 &= d_2(2-2g_3) - \beta(f_2) \\ 2-2g &= d_2(2-2g_1) - \beta(h_1) \\ 2-2g &= d_1(2-2g_2) - \beta(h_2). \end{aligned} \tag{5.5}$$

Given $\beta(f_j)$ and $\beta(h_j)$, we can calculate the genus g of Σ from these identities. The calculation of $\beta(f_j)$ and $\beta(h_j)$, however, involves more explicit information regarding the maps f_1 and f_2 and the relationship between their critical values and critical points. To be explicit and for the sake of simplicity, we shall now restrict the discussion. (We continue to stress the fact that these simplifying assumptions are not necessary.) We assume that:

$$R_2 = R_3 = S^2 \quad (g_2 = g_3 = 0). \tag{5.6}$$

We shall also assume that:

$$g_1, d_1, d_2 \geq 2. \tag{5.7}$$

By (5.6), (5.7) and Corollary 5.1, K is proper. (Actually, since $g_1 \geq 2$ and $g_2 = 0$, the hypothesis that $d_1 \geq 2$ is redundant. This follows from the fact that a branched covering map of degree 1 is a homeomorphism. Likewise, the assumption that $g_3 = 0$ is redundant. It follows from the assumption that $g_2 = 0$ and the Riemann-Hurwitz relation for the branched covering map f_2.) In addition, we shall assume that:

$$f_j \text{ is a simple branched covering map.} \tag{5.8}$$

A branched covering $\phi : M \to N$ of degree d between closed, oriented surfaces is *simple* if $\#\phi^{-1}(y) \geq d-1$ for all $y \in N$. Let q be a critical value of ϕ. Then $\phi^{-1}(q) = \{p_1, ..., p_{d-1}\}$ where the degree of ϕ at p_j is equal to 2 if $j = 1$ and 1 if $2 \leq j \leq d-1$. Note that if ϕ is a simple branched covering map, then:

$$\beta(\phi) = \#\Lambda_\phi = \#B_\phi.$$

(Indeed, we can take this as a definition of a simple branched covering map.) Hence, it follows from (5.5), (5.6) and (5.8) that:

$$\begin{aligned} \#\Lambda_{f_1} &= \#B_{f_1} = 2(d_1 + g_1 - 1) \\ \#\Lambda_{f_2} &= \#B_{f_2} = 2(d_2 - 1). \end{aligned} \tag{5.9}$$

Remark 5.4. We have assumed that f_j is a nonconstant holomorphic map between compact Riemann surfaces. Actually, we only need to assume that f_j is a branched covering map between closed, oriented surfaces. Indeed, if $\phi : M \to N$ is any branched covering map between closed, oriented surfaces M and N and N is equipped with a complex structure, then there is a unique complex structure on M such that ϕ is a holomorphic map ([B-G]). We point out that this fact implies that the only freedom one has to realize the branched covering maps f_j as holomorphic maps is in the choice of conformal structure on R_3. If $g_3 = 0$, the Uniformization theorem implies that there is no freedom in prescribing the complex structures.

The following lemma shows that there are branched covers $f_1 : R_1 \to R_3$ and $f_2 : R_2 \to R_3$ satisfying (5.6), (5.7) and (5.8) for any choice of integers $g_1, d_1, d_2 \geq 2$. In addition, it shows that we have complete freedom in prescribing the branch values on S^2 of f_1 and f_2.

Lemma 5.1. *Let g and d be nonnegative integers such that $d \geq 1$. Let $\{x_1, ..., x_r\}$ be a set of r distinct points on S^2 where $r = 2(d + g - 1)$. Then there exists a nonconstant holomorphic branched covering map $f : R \to S^2$ of degree d such that:*

$$(i) B_f = \{x_1, ..., x_r\}$$
$$(ii) genus(R) = g.$$

Proof. Let B denote the chosen set of points $\{x_1, ..., x_r\}$. Let $\mathcal{S}_d$ denote the symmetric group on d symbols. If $1 \leq m, n \leq d$ and $m \neq n$, let (m, n) denote the transposition which interchanges m and n. The fundamental group $\pi_1(S^2 \setminus B)$ has the following presentation:

$$Generators : c_j, \quad 1 \leq j \leq r$$
$$Relations : \quad \prod_{j=1}^{r} c_j = 1,$$

where c_j is represented by a small loop about x_j. Hence, we may define a homomorphism:

$$\rho : \pi_1(S^2 \setminus B) \to \mathcal{S}_d$$

whose values on the sequence of generators:

$$(c_1, ..., c_r)$$

are given by the following sequence of transpositions:

$$(\underbrace{(1,2), ...(1,2)}_{2g+2}, (1,3), (1,3), (1,4), (1,4), ..., (1,d), (1,d)).$$

By the existence theorem of Hurwitz ([B-E]), there is a branched covering map $f : R \to S^2$ of degree d with branch set $B_f = B$ and Hurwitz representation $\rho_f = \rho$. Since the transpositions $\{(1,2), (1,3), ..., (1,d)\}$ form a set of generators of $\mathcal{S}_d$, ρ_f is transitive. Hence, R is connected. It follows from the Riemann-Hurwitz relation that R is a compact Riemann surface of genus g. By remark 5.4, we may assume that f is holomorphic. □

The assumption that f_1 and f_2 are simple branched covering maps has the advantage that all the singularities of C are simple singularities of type $(2, 2)$. In other words, all the singularities of C are ordinary double points. Hence, we can obtain a desingularization of C by blowing up $R_1 \times S^2$ once at each double point of C. Note also that the set of double points on C is in one to one correspondence with the set of common critical values of f_1 and f_2. Let k denote the number of common critical values of f_1 and f_2. By (5.9), it follows that:

$$k \leq min(2(d_1 + g_1 - 1), 2(d_2 - 1)). \tag{5.10}$$

Let ℓ be a nonnegative integer. Let $p_1, ..., p_k$ denote the singular points of C. Let $q_1, ..., q_\ell$ denote ℓ distinct smooth points on C. Let X denote the surface obtained by blowing up $R_1 \times S^2$ exactly once at each point in $\{p_1, ..., p_k, q_1, ..., q_\ell\}$. Let Σ denote the proper transform of C in X. Note that Σ is a desingularization of C. (Of course, there are other possibilities for a desingularization of C. But in any case, we must blow up at least once at each double point of C. Our choice of desingularization is the simplest desingularization which provides sufficient control over the self-intersection of Σ. Once this self-intersection is fixed, the differential topology is insensitive to the choice of desingularization.) The following lemma gives sufficient restrictions upon k to ensure that Σ is connected.

Lemma 5.2. *Let k be any nonnegative integer satisfying (5.10) such that $k \leq max(d_1 + 2g_1 - 1, d_2 - 1)$ where g_1, d_1 and d_2 satisfy (5.7). Then there exist simple branched covering maps $f_1 : R_1 \to S^2$ and $f_2 : S^2 \to S^2$ of degree d_1 and d_2 respectively such that:*

$$\textit{(i)}\#(B_{f_1} \cap B_{f_2}) = k,$$

$$\textit{(ii)}genus(R_1) = g_1,$$

(iii) the curve C is irreducible (i.e. Σ is connected)

where $C = \{(x, y) \in R_1 \times S^2 \mid f_1(x) = f_2(y)\}$.

Proof. Let $r_1 = 2(d_1 + g_1 - 1)$ and $r_2 = 2(d_2 - 1)$. Since $k \leq min(r_1, r_2)$, we may choose subsets B_1 and B_2 of S^2 for which B_j consists of r_j distinct points and $B_1 \cap B_2$ consists of k distinct points. For any choice of such subsets, by Lemma 5.1, there exist simple branched covering maps $f_1 : R_1 \to S^2$ and $f_2 : S^2 \to S^2$ of degree d_1 and d_2 respectively such that $B_{f_j} = B_j$. For any such pair of branched covering maps, we have condition (i) of the lemma satisfied. It remains to choose f_1 and f_2 appropriately to ensure that Σ is connected. This we do by appealing to Theorem 5.3.

We may assume that the monodromy representations of f_1 and f_2 are as described in the proof of Lemma 5.1. By assumption, $k \leq d_2 - 1$ or $k \leq d_1 + 2g_1 - 1$. Suppose that $k \leq d_2 - 1$. Since $B_{f_2} \setminus B_{f_1}$ consists of $r_2 - k$ points, where $r_2 = 2(d_2 - 1)$, there are at least $d_2 - 1$ distinct points in $B_{f_2} \setminus B_{f_1}$. Given the freedom we have to prescribe the branch loci, we can assume that $z_1, ..., z_{d_2-1}$ lie in $B_{f_2} \setminus B_{f_1}$. Let c'_j be the generator of $\pi_1(S^2 \setminus B_{f_2})$ corresponding to the point z_j. From our assumption regarding the monodromy representations, we conclude that $\rho(f_2)(c'_j) = (1, j)$. The generator c'_j is represented by a small loop γ'_j around z_j. Since z_j is not in B_{f_1}, we may assume that γ'_j bounds a small disc D_j in $S^2 \setminus B_{f_1}$. Since f_1

is a covering map over $S^2 \setminus B_{f_1}$, D_j lifts to a small disc $\tilde{D}_j$ in R_1. Hence, γ'_j lifts to a loop $\tilde{\gamma}'_j$ in $R_1 \setminus f_1^{-1}(B_{f_2})$. This implies that c'_j is in the image L_1 of $\pi_1(R_1 \setminus f_1^{-1}(B_{f_2}))$ under $(f_1)_*$. Hence, $(1, j)$ is in the image of L_1 under $\rho(f_2)$. By Theorem 5.3, we conclude that Σ is connected. Likewise, if $k \leq d_1 + 2g_1 - 1$, then we can choose f_1 and f_2 so that Σ is connected. □

In light of Lemma 5.2, we assume, in addition to (5.10), that:

$$k \leq max(d_1 + 2g_1 - 1, d_2 - 1). \tag{5.11}$$

D: The genus and self-intersection of Σ

We shall construct our examples by appealing to Lemma 5.2. The following lemma gives the genus and self-intersection of the resulting Riemann surfaces Σ.

Lemma 5.3. *Suppose that f_1 and f_2 are simple branched covering maps as in Lemma 5.2. Let Σ be the proper transform of C obtained by blowing up C at the k double points and ℓ distinct nonsingular points of C. Then:*

$$genus(\Sigma) = 1 + d_1 d_2 + d_2(g_1 - 1) - k - d_1$$
$$\Sigma \cdot \Sigma = 2d_1 d_2 - 4k - \ell.$$

Proof. Since Σ is a smoothly embedded complex curve in the complex surface X, we have the following well-known consequence of the adjunction formula ([B-P-V], chapter I, section 6):

$$c_1(X)(\Sigma) = \chi(\Sigma) + \Sigma \cdot \Sigma. \tag{5.12}$$

X is obtained by blowing up $R_1 \times S^2$ at $k+\ell$ distinct points, $\{p_1, ..., p_k, q_1, ..., q_\ell\}$ as chosen above. Let E_j be the (-1)-curve corresponding to p_j, $\tau^{-1}(p_j)$. Let F_j be the (-1)-curve corresponding to q_j, $\tau^{-1}(q_j)$. It follows from Theorem 9.1(vii) in chapter I of [B-P-V], that:

$$c_1(X) = \tau^*(c_1(R_1 \times S^2)) - E_1^* - ... - E_k^* - F_1^* - ... - F_\ell^*$$

where E_j^* is the Poincare Dual of E_j and F_j^* is the Poincare Dual of F_j. Hence:

$$c_1(X)(\Sigma) = c_1(R_1 \times S^2)(\tau_*(\Sigma)) - E_1 \cdot \Sigma - ... - E_k \cdot \Sigma - F_1 \cdot \Sigma - ... - F_\ell \cdot \Sigma.$$

Since p_j is an ordinary double point of C and Σ is the proper transform of C with respect to blowing up once at each point in $\{p_1, ..., p_k, q_1, ..., q_\ell\}$, $E_j \cdot \Sigma = 2$. Since q_j is a smooth point of C, $F_j \cdot \Sigma = 1$. The divisor $D = E_1 + ... + E_k + F_1 + ...F_\ell$ meets Σ at exactly $2k + \ell$ points, 2 on each component E_j of D and 1 on each component F_j. All of these points of intersection are smooth points of transverse intersection. Since τ is an isomorphism off D, $\tau_*(\Sigma) = C$ in $H_2(R_1 \times S^2)$. Hence, from the previous equation:

$$c_1(X)(\Sigma) = c_1(R_1 \times S^2)(C) - 2k - \ell. \tag{5.14}$$

By the Kunneth formula and the fact that $H_1(S^2) = \{0\}$, $H_2(R_1 \times S^2)$ has a basis $\{B, F\}$ where B is represented by a smoothly embedded complex curve $R_1 \times \{y\}$ for any y in S^2 and F is represented by a smoothly embedded complex curve $\{x\} \times S^2$ for any x in R_1. So $C = bB + fF$ for some real numbers b and f. We can compute b and f by considering the intersection form Q on $H_2(R_1 \times S^2)$. Clearly, $B \cdot B = 0$, $B \cdot F = F \cdot B = 1$ and $F \cdot F = 0$. Hence, $C \cdot B = f$ and $C \cdot F = b$. On the other hand, we can compute $C \cdot B$ and $C \cdot F$ directly as follows. We may choose x such that $f_1(x)$ is neither a critical value of f_1 nor of f_2. Likewise, we may assume that $f_2(y)$ is not a critical value of f_1 or f_2. (Furthermore, we may choose x and y such that B and F avoid the blow-up locus $\{p_1, ..., p_k, q_1, ..., q_\ell\}$. We shall appeal to this assumption in the discussion below.) With these assumptions, all the pairwise intersection points of C, $R_1 \times \{y\}$ and $\{x\} \times S^2$ are smooth points of transverse intersection. Hence, since all these curves are complex, we can compute their intersection numbers by counting. Thus, for instance, $C \cdot B = \#(C \cap (R_1 \times \{y\}))$. By the definition of C:

$$C \cap (R_1 \times \{y\}) = f_1^{-1}(f_2(y)) \times \{y\}.$$

Since $f_2(y)$ is not a critical value of the degree d_1 map f_1, $\#(C \cap (R_1 \times \{y\})) = d_1$. Hence, $C \cdot B = d_1$. Likewise, $C \cdot F = d_2$. Thus:

$$C = d_2 B + d_1 F. \tag{5.15}$$

Since B is a smoothly embedded complex curve of genus g_1 and self-intersection 0, it follows from the adjunction formula that:

$$c_1(R_1 \times S^2)(B) = \chi(B) + B \cdot B = 2 - 2g_1.$$

Likewise, since F is a smoothly embedded complex curve of genus 0:

$$c_1(R_1 \times S^2)(F) = \chi(F) + F \cdot F = 2.$$

Combining these observations, we conclude that:

$$c_1(R_1 \times S^2)(C) = d_2(2 - 2g_1) + d_1 2.$$

From (5.14), it follows that:

$$c_1(X)(\Sigma) = d_2(2 - 2g_1) + d_1 2 - 2k - \ell \tag{5.16}$$

Since Σ is a compact Riemann surface of genus g, it follows from (5.12) and (5.16) that:

$$2g = 2 + \Sigma \cdot \Sigma + d_2(2g_1 - 2) + 2k + \ell - 2d_1. \tag{5.17}$$

It remains to compute $\Sigma \cdot \Sigma$. Topologically, blowing up corresponds to a connect sum with $\overline{\mathbf{CP}}^2$. The associated (-1)-curve corresponds to $\overline{\mathbf{CP}}^1 \subset \overline{\mathbf{CP}}^2$. It follows from this description and the Mayer-Vietoris sequence, that $H_2(X)$ has a basis $\{B', F', E_1, ..., E_k, F_1, ..., F_\ell\}$, where B' is the proper transform of B and F' is the proper transform of F. Therefore, we may write:

$$\Sigma = bB' + fF' + a_1E_1 + ... + a_kE_k + b_1F_1 + ... + b_\ell F_\ell$$

for some coefficients $b, f, a_1, ..., a_k, b_1, ..., b_\ell$. As before, we can compute the coefficients by appealing to the intersection form Q. Since we have chosen B and F to avoid the blow-up locus, the previous observations imply that $B' \cdot F' = B \cdot F = 1$, $B' \cdot E_j = 0$, $B' \cdot F_j = 0$, $F' \cdot E_j = 0$ and $F' \cdot F_j = 0$. Since $E_1, ..., E_k, F_1, ..., F_\ell$ are disjoint (-1)-curves, $E_j \cdot E_j = -1$, $E_j \cdot E_{j'} = 0$ if $j \neq j'$, $E_j \cdot F_{j''} = 0$, $F_{j''} \cdot F_{j''} = -1$ and $F_{j''} \cdot F_{j'''} = 0$ if $j'' \neq j'''$. Again, since B and F avoid the blow-up locus, $\Sigma \cdot B' = C \cdot B = d_1$ and $\Sigma \cdot F' = C \cdot F = d_2$. On the other hand, since p_j is an ordinary double point of C, $\Sigma \cdot E_j = 2$. Since q_j is a smooth point of C, $\Sigma \cdot F_j = 1$. These facts imply that:

$$\Sigma = d_2B' + d_1F' - 2E_1 - ... - 2E_k - F_1 - ... - F_\ell.$$

From this identity, we compute that:

$$\Sigma \cdot \Sigma = 2d_1d_2 - 4k - \ell. \tag{5.18}$$

Together with (5.17), this implies that:

$$g = 1 + d_1d_2 + d_2(g_1 - 1) - k - d_1. \tag{5.19}$$

□

E: Examples

Now we are ready to construct our examples. To each example there is an associated set of nonnegative integers g_1^-, d_1^-, d_2^-, k^-, ℓ^-, g_1^+, d_1^+, d_2^+, k^+, ℓ^+, g and n satisfying the following constraints:

$$g_1^\pm, d_1^\pm, d_2^\pm \geq 2 \tag{5.20}$$

$$k^\pm \leq min(2(d_1^\pm + g_1^\pm - 1), 2(d_2^\pm - 1)) \tag{5.21}$$

$$k^\pm \leq max(d_1^\pm + 2g_1^\pm - 1, d_2^\pm - 1). \tag{5.22}$$

$$g = d_1^\pm d_2^\pm + d_2^\pm(g_1^\pm - 1) - k^\pm - d_1^\pm \tag{5.23}$$

$$\pm n = 2d_1^\pm d_2^\pm - 4k^\pm - \ell^\pm. \tag{5.24}$$

Any such set of integers will be called an *admissible set of parameters*. The following theorem shows that we have complete freedom in prescribing an admissible set of parameters.

Theorem 5.5. *If $g_1^\pm$, $d_1^\pm$, $d_2^\pm$, $k^\pm$, $\ell^\pm$, g and n are admissible parameters, then there exists a symplectic normal connect sum $X = X^- \#_\Psi X^+$ along surfaces Σ^- and Σ^+ of genus g such that $b_1(X)$ is odd and $X^\pm$ is obtained by blowing up the ruled surface $R_1^\pm \times S^2$ exactly $k^\pm + \ell^\pm$ times, where $R_1^\pm$ is a compact Riemann surface of genus $g_1^\pm$*

Proof. Let $R_1^{\pm}$ be a compact Riemann surface of genus $g_1^{\pm}$. By (5.20), (5.21), (5.22), Lemma 5.2, (5.23), (5.24) and Lemma 5.3, we conclude that there exist smooth symplectically embedded surfaces $\Sigma^{\pm}$ of genus g and self-intersection $\pm n$ in surfaces $X^{\pm}$ obtained by blowing up $R_1^{\pm} \times S^2$ exactly $k^{\pm} + \ell^{\pm}$ times. By (5.20) and Corollary 5.1, it follows that the kernel $K^{\pm}$ of :

$$\jmath_* : H_1(\Sigma^{\pm}) \to H_1(X^{\pm})$$

is a proper subspace of $H_1(\Sigma^{\pm})$. Hence, by Theorem 5.2, there exists a symplectomorphism Ψ of tubular shell neighborhoods of Σ^- and Σ^+ so that $b_1(X^- \#_{\Psi} X^+)$ is odd. □

Remark 5.5. Since g and n are determined by the other parameters, they are not effective parameters for this construction. We include them only for the sake of convenience. It can be seen that varying ℓ^- and ℓ^+ does not affect the differential topology of the normal connect sums X in Theorem 5.5. Hence, in some sense, ℓ_- and ℓ_+ are also not effective parameters for this construction.

Actually, there may be several examples corresponding to a given admissible set of parameters. There are several choices involved in the construction. It seems possible that the choice of the gluing symplectomorphism Ψ in Theorem 1.1 may effect the symplectomorphism type of X, even if we stay within a fixed isotopy class. Changing the isotopy class of Ψ may lead to non-diffeomorphic or non-homeomorphic manifolds. (It is not even clear how the geometry and topology depend upon the branched covering maps $f_j^{\pm}$. The geometry may change under perturbations of $f_j^{\pm}$. The topology may depend upon the interplay between the critical values and monodromy representations of $f_j^{\pm}$.) All of our examples are constructed by appealing to Theorem 5.2. The proof of that theorem shows that it is always possible to change the parity of b_1 by changing the symplectic gluing map Ψ to some symplectic gluing map Ψ' (remark 5.3). But it may also be possible to have different odd (or even) values for b_1. It is not our purpose here to address any of these issues, though we hope to address some of them in future work.

The following corollary describes the invariants of the symplectic normal connect sums in Theorem 5.5.

Corollary 5.2. *Let $X = X^- \#_{\Psi} X^+$ with b_1 odd be a symplectic normal connect sum as in Theorem 5.5. Then:*

$$\sigma(X) = -2d_1^- d_2^- - 2d_1^+ d_2^+ + 3k^- + 3k^+$$

$$\begin{aligned} c_2(X) = \chi(X) = &(d_1^- - 2)(d_2^- - 2) + (d_1^+ - 2)(d_2^+ - 2) \\ &+ 3(d_1^- d_2^- + d_1^+ d_2^+ - 2k^- - 2k^+) + 2g_1^-(d_2^- - 2) \\ &+ 2g_1^+(d_2^+ - 2) + k^- + k^+ \end{aligned}$$

$$\begin{aligned} c_1^2(X) =& 2(d_1^- - 2)(d_2^- - 2) + 2(d_1^+ - 2)(d_2^+ - 2) \\ &+ 4g_1^-(d_2^- - 2) + 4g_1^+(d_2^+ - 2) - k^- - k^+. \end{aligned}$$

In particular, $\sigma(X) < 0$ *and, hence,* $c_1^2(X) < 2c_2(X)$.

Proof. By the multiplicativity of Euler characteristics for product spaces:

$$\chi(R_1^\pm \times S^2) = 4 - 4g_1^\pm.$$

By previous remarks concerning the intersection form Q on $H_2(R_1^\pm \times S^2)$, it is clear that:

$$\sigma(R_1^\pm \times S^2) = 0.$$

From (2.6), we conclude that:

$$\begin{aligned} c_1^2(R_1^\pm \times S^2) &= 8 - 8g_1^\pm \\ c_2(R_1^\pm \times S^2) &= 4 - 4g_1^\pm. \end{aligned}$$

Since $X^\pm$ is obtained by blowing up $R_1^\pm \times S^2$ exactly $k^\pm + \ell^\pm$ times:

$$\begin{aligned} \chi(X^\pm) &= 4 - 4g_1^\pm + k^\pm + \ell^\pm \\ \sigma(X^\pm) &= -k^\pm - \ell^\pm \\ c_1^2(X^\pm) &= 8 - 8g_1^\pm - k^\pm - \ell^\pm \\ c_2(X^\pm) &= 4 - 4g_1^\pm + k^\pm + \ell^\pm. \end{aligned} \tag{5.25}$$

The formulas for $\sigma(X)$, $\chi(X)$, $c_1^2(X)$ and $c_2(X)$ follow immediately from (2.4), (2.5), (2.8) and (5.25). From the formula for $\sigma(X)$ and (5.24), we find that:

$$4\sigma(X) = -2(d_1^- d_2^- + d_1^+ d_2^+) - 3(\ell^- + \ell^+).$$

Since $d_j^\pm > 0$, we conclude that $\sigma(X) < 0$. Hence, by (2.6), $c_1^2(X) < 2c_2(X)$. □

Since $b_1(X)$ is odd, none of these examples are homeomorphic to any Kähler surface. On the other hand, by remark 5.3, there exists a symplectomorphism Ψ' of tubular shell neighborhoods of Σ_- and Σ_+ such that $b_1(X_- \#_{\Psi'} X_+)$ is even. Let X' denote this normal connect sum $X_- \#_{\Psi'} X_+$. By (2.4), (2.5) and (2.8), X' has the same euler characteristic, signature and Chern numbers as X.

Question . Is X' homeomorphic to any Kähler surface? Or indeed, for suitable Ψ', is X' itself a Kähler surface?

Example 5.2 (A Family with b_1 Odd and c_1^2 Unbounded)**.** We shall now give a "1-parameter" family of examples of the preceding type for which

$c_1^2 > 0$ and c_1^2 is unbounded. Let a be a nonnegative integer. Consider the following choice of parameters:

$$\begin{gathered} g_1^- = g_1^+ = 2+2a, \quad d_1^- = d_1^+ = 2 \\ d_2^- = 3+a, \quad d_2^+ = 2+a \\ k^- = 4+2a \quad k^+ = 1, \quad \ell^- = \ell^+ = 0 \\ g = 4+7a+2a^2, \quad n = 4+4a. \end{gathered} \tag{5.26}$$

These choices give an admissible set of parameters $g_1^\pm$, $d_1^\pm$, $d_2^\pm$, $k^\pm$, $\ell^\pm$, g and n. Let X be the normal connect sum with b_1 odd which was constructed in Theorem 5.5. By our choice of parameters and Corollary 5.2, we have the following identities:

$$\begin{gathered} \sigma(X) = -(5+2a) \\ \chi(X) = c_2(X) = 9+14a+8a^2 \\ c_1^2(X) = 3+22a+16a^2. \end{gathered} \tag{5.27}$$

The above identities and the fact that $b_1(X)$ is odd imply that X is not homeomorphic to any complex surface. For suppose that X is a complex surface. Let Z be a minimal model for X. Since $b_1(Z) = b_1(X)$, $b_1(Z)$ is odd. Since c_1^2 increases under blow downs, $c_1^2(Z) \geq c_1^2(X)$. By (5.27), therefore, $c_1^2(Z) \geq 3$. On the other hand, since Z is a minimal surface with b_1 odd, the table in chapter VI, section 1 of [B-P-V] implies that $c_1^2(Z) \leq 0$. Hence, X is not homeomorphic to any complex surface.

Let X' be the normal connect sum with b_1 even obtained by changing the gluing map Ψ to a map Ψ' as explained above. As we have previously observed, the euler characteristic, signature and Chern numbers of X' agree with those of X. Suppose that $a > 0$ and X' is complex. By (5.27) and the table in chapter VI, section 1 of [B-P-V], X' is a surface of general type.

Recall that a negative self-intersection nonsingular curve in a Kähler surface can always be constructed by blowing up any nonsingular nonnegative self-intersection curve sufficiently often. By our terminology this curve is not genuinely negative. (See section 4.) Of course, such curves cannot exist on minimal Kähler surfaces. Thus, to form a symplectic normal connect sum of two minimal Kähler surfaces we must find genuinely negative curves. The next proposition describes a family of such curves.

Proposition 5.1. *The desingularization Σ^- of the fibered product of two branched covering maps $f_1^- : R_1^- \to S^2$ and $f_2^- : S^2 \to S^2$ corresponding to the parameters g_1^-, d_1^-, d_2^-, k^- and ℓ^- in (5.26) is genuinely negative.*

Proof. Suppose, on the contrary, that Σ^- is the proper transform of a smoothly embedded curve C' of nonnegative self-intersection in a surface Y. Y is obtained from X^- by blowing down some exceptional divisor D' in X^-. C' is the image of Σ^- under the corresponding blow down map τ^-. Since C' is smoothly embedded, no component of D' can meet Σ^- with multiplicity greater than 1. Each component E' of D' is a rational curve in X^- of negative self-intersection. Consider the restriction h of $\pi_1 \circ \tau^-$ to E', where π_1 is projection onto the first factor of $R_1^- \times S^2$. h is a holomorphic map from E' to R_1^-. Since E' is a rational curve and $g_1^- \geq 2$, h must be constant. Hence E' lies in the preimage of a curve G of the form $\{x'\} \times S^2$. Thus E' is either the proper transform G' of G or an exceptional curve of τ^-. In either case, E' is a (-1)-curve. Suppose that E' is an exceptional curve of τ^-. Since every point in the blow up locus is a double point of the fibered product C^-, $\Sigma^- \cdot E = 2$. If we blow down E, then the image of Σ^- will be singular. Since this contradicts our smoothness assumption on C', we conclude that $E = G'$. Homological considerations as employed in our previous discussion imply that $C^- \cdot G = d_1^- = 2$. Let m be the number of singular points of C^- contained in G. Since G is smooth, further arguments as employed above imply that $G' \cdot G' = -m$ and $\Sigma^- \cdot G' = 2 - 2m$. Since G' is equal to the (-1)-curve E', $m = 1$. Hence, $\Sigma^- \cdot E' = 0$. Hence, D' does not intersect Σ^-. Thus, $C' \cdot C' = \Sigma^- \cdot \Sigma^-$. By (5.26), $\Sigma^- \cdot \Sigma^- = -(4 + 4a)$. Hence, $C' \cdot C' < 0$. This contradicts the assumption that C' is nonnegative. Hence, Σ^- is genuinely negative. Indeed, any smooth blow down of Σ^- has the same self-intersection as Σ^-. □

Example 5.3 (A Family with b_1 Odd, c_1^2 Unbounded, X_j Minimal)**.** We now wish to modify the previous example to obtain examples of symplectic normal connect sums which are constructed as normal connect sums of minimal surfaces. These examples will be normal connect sums of ruled surfaces of positive genus. They will in fact be diffeomorphic to blow downs of the examples just constructed. Let $k = k^-$ and $C = C^-$ as in the previous example. Let $\{p_1, ..., p_k\}$ be the k distinct double points of C. Let x_j be the first coordinate of p_j. Let $G_j = \{x_j\} \times S^2$. By the homological considerations above, $G_j \cdot C = 2$. G_j and C are complex curves meeting at p_j with multiplicity 2. Hence, $G_j \cap C = \{p_j\}$. It follows that $\{x_1, ..., x_k\}$ is a set of k distinct points on R_1^-. Hence, $\{G_1, ..., G_k\}$ is a collection of k distinct fibers of the ruled surface $R_1^- \times S^2$. Let E_j' be the proper transform of the fiber G_j. Consider the divisor $D' = E_1' + ... + E_k'$. By the previous discussion, D' is an exceptional divisor. Indeed, D' is a union of k disjoint (-1)-curves. Each of these curves is disjoint from Σ^-. Let S_{-1} be the surface obtained by blowing down D'. Since S_{-1} is obtained from $R_1^- \times S^2$ by blowing up at k^- points on distinct fibers of the ruled surface $R_1^- \times S^2$ and then blowing down the corresponding proper transforms of these fibers, S_{-1} is a ruled surface of genus g_1^-. By the table in chapter VI, section 1 of [B-P-V], S_{-1} is a minimal surface. Let Σ_{-1} be the image of Σ^- in S_{-1}. Σ_{-1} is a smoothly embedded curve of genus g and self-intersection $-(4+4a)$ in S_{-1}. Since S_{-1} is a minimal surface, Σ_{-1} is a genuinely negative curve.

For the same reasons, we may blow down a (-1)-curve in the complement of Σ^+ to obtain a smoothly embedded curve Σ_1 of genus g and self-intersection $4 + 4a$ in a ruled surface S_1 of genus g_1^+. The kernels $K_{\pm 1}$

corresponding to the inclusions of $\Sigma_{\pm 1}$ in $S_{\pm 1}$ are isomorphic to the kernels $K^{\pm}$ and, hence, are proper. Therefore, we can apply Theorem 5.2 to form a normal connect sum $S_{-1}\#_{\Phi}S_1$ with b_1 odd. Let S denote this normal connect sum. Appealing to the previous formulas for the basic invariants, we have the following identities:

$$\begin{aligned} \sigma(S) &= 0 \\ c_2(S) = \chi(S) &= 4 + 12a + 8a^2 \\ c_1^2(S) &= 8 + 24a + 16a^2. \end{aligned} \tag{5.28}$$

The example S^0 of a compact symplectic manifold with $c_1^2 = 8$, $c_2 = 4$ and $b_1 = 1$ which was discussed in the introduction corresponds to $a = 0$. By (5.26) S^0 is a normal connect sum of two ruled surfaces S_{-1} and S_1 of genus 2 along surfaces $\Sigma_{\pm}$ of genus 4 and self-intersection ± 1. The proof of Theorem 5.2 shows that we may choose the gluing map Φ so that $b_1(S^0) = 1$.

Note that the (-1)-curves in $X^{\pm}$ which were blown down to obtain $S_{\pm 1}$ are disjoint from $\Sigma^{\pm}$. Hence, these (-1)-curves embed in X. Indeed, they form a family of disjoint (-1)-curves in X. If we blow down this family of (-1)-curves in X, we obtain a symplectic manifold diffeomorphic to S, (assuming that we have the appropriate correspondence between the gluing maps Ψ and Φ used to construct X and S).

Question . Are these symplectic manifolds $S_{-1}\#_{\Phi}S_1$ minimal?

6. Non-Kähler Symplectic Manifolds With b_1 Even

In this section, we shall give further examples of symplectic manifolds whose underlying smooth manifold does not admit any Kähler structure. Again, these manifolds will all be constructed as symplectic normal connect sums of Kähler manifolds. Since all of the examples of this section have b_1 even, the proof that these examples do not admit any Kähler structure involves more than the calculation of b_1. On the other hand, since b_1 is even, our argument actually demonstrates that none of these are homeomorphic to any complex surface. (See the result quoted from [P] in section 3.)

Example 6.1 (Nontrivial Free Products)**.** This example will be obtained by a two stage construction. First, using the operation of symplectic normal connect sum, we shall construct a compact symplectic four manifold X with an embedded symplectic surface Σ of self-intersection 0 such that $\pi_1(X \setminus \Sigma)$ is a free group on two generators c and d. X has the additional property that the fiber class z of Σ is represented by the commutator $[d, c]$. Secondly, we shall form the symplectic normal connect sum $Y = X^{-1}\#_{\Phi}X^1$ of two copies of X, X^{-1} and X^1, along the respective copies of Σ, Γ_{-1} and Γ_1. With an appropriate choice of Φ, Y is a symplectic manifold whose fundamental group is isomorphic to $G_1 * G_2$ where G_1 and G_2 are two groups which have at least one nontrivial finite quotient each. By the result of [J-R] mentioned above, the fundamental group of Y is not the fundamental group of any compact Kähler manifold. This conclusion also follows from the result of [A-B-R].

Remark 6.1. Of course, this implies that Y is not homeomorphic to any Kähler manifold. Actually, by the same result of [J-R], we can deduce that Y is non-Kähler in a stable sense. Suppose that S is any compact symplectic manifold. The product $Y \times S$ is, of course, a compact symplectic manifold. Let H be the fundamental group of S. $\pi_1(Y \times S)$ is isomorphic to $(G_1 * G_2) \times H$. Hence, $Y \times S$ is also not homeomorphic to any Kähler manifold. Note that this conclusion can also be deduced from the result of [A-B-R] provided we restrict to compact Kähler manifolds S with finite fundamental groups.

For the first stage of the construction, let $X_1 = \mathbf{CP^2}$. Fix a positive integer k with $k \geq 3$ and let Σ_1 be a nonsingular curve of degree k in X_1. Σ_1 is a smooth complex curve in X_1 whose genus g and self-intersection n are given by:

$$g = (k-1)(k-2)/2, \quad n = k^2. \tag{6.1}$$

Since $k \geq 3$, $g \geq 1$. Let R be a compact Riemann surface of genus g. Choose two distinct points x_{-1} and x_0 on T^2. Let X_{-1} denote the surface obtained by blowing up $R \times T^2$ at k^2 distinct points on $R \times \{x_{-1}\}$ and let τ denote the natural projection from X_{-1} to $R \times T^2$. Let Σ_{-1} be the proper transform of $R \times \{x_{-1}\}$ and Σ_0 be the proper transform of $R \times \{x_0\}$. Proper transformation decreases the self-intersection of a surface by 1 for each point of blowing up lying on the surface. Hence, Σ_{-1} has self-intersection $-k^2$, though Σ_0 has self-intersection 0. On the other hand, since proper transformation does not effect the genera of embedded surfaces, Σ_{-1} and Σ_0 both have genus equal to g. By Theorem 1.1, we can form the symplectic normal connect sum $X = X_{-1} \#_\Psi X_1$ along Σ_{-1} and Σ_1. Denote this symplectic manifold by X.

By Theorem 1.1, X admits a symplectic structure which agrees with the symplectic structure on X_{-1} off a tubular neighborhood of Σ_{-1}. We may assume that this tubular neighborhood avoids Σ_0. Hence, Σ_0 naturally embeds in X as a symplectic surface Σ of genus g and self-intersection 0. Let X^i be a copy of X and let Γ_i be a copy of Σ in X^i. By Theorem 1.1, we can form the symplectic normal connect sum $Y = X^{-1} \#_\Phi X^1$ along Γ_{-1} and Γ_1.

In order to calculate the fundamental group of Y, we need to compute the fundamental group of V_i, the complement of a closed tubular neighborhood of Γ_i in X^i. V_i is, of course, isomorphic to the complement X^* of a closed tubular neighborhood of Σ in X. To understand X^* we reconsider the construction of X as a normal connect sum. Consider the cover (U_{-1}, U_1) of X as described in section 2. U_i is the complement of a closed tubular neighborhood of Σ_i in X_i. As in previous discussions, we identify the intersection $U_{-1} \cap U_1$ with the tubular shell neighborhood W_{-1} of Σ_{-1} in U_{-1} so that the inclusion of $U_{-1} \cap U_1$ in U_1 is identified with the composition $\Psi \circ \jmath$. By the previous assumption on Σ_0, Σ avoids U_1. Hence, this cover of X restricts to a cover (U^*_{-1}, U_1) of X^*, where U^*_{-1} corresponds to the complement of $\overline{\mathcal{N}(\Sigma)}$ in U_{-1}. We may identify U^*_{-1} with the complement of $\overline{\mathcal{N}(\Sigma_0)} \cup \overline{\mathcal{N}(\Sigma_{-1})}$ in X_{-1}. The projection $\tau : X_{-1} \to R \times T^2$, thereby, restricts to a projection:

$$\rho : U^*_{-1} \to R \times (T^2 \setminus (B(x_0) \cup B(x_{-1}))),$$

where $B(x_i)$ is a disc neighborhood of x_i in T^2. By examining this blow down map ρ we can see that $R \times (T^2 \setminus (B(x_0) \cup B(x_{-1})))$ is isomorphic to the complement of a closed tubular neighborhood of a codimension 2-submanifold D^* of U^*_{-1}. (Indeed D^* is the intersection of the exceptional divisor for τ with U^*_{-1}. The assertion follows from the fact that τ is an isomorphism in the complement of the exceptional divisor.)

It is a well-known fact that if N is a codimension 2 submanifold of a manifold M, the fundamental group $\pi_1(M)$ is the quotient of $\pi_1(N)$ obtained by adding, for each component N_i of N, the relation $\eta_i = 1$, where η_i is the fiber class of N_i in M, (i.e. the class represented by the linking circle of N_i in M). Hence, it follows that $\pi_1(U^*_{-1})$ has the following presentation::

$$\begin{aligned} &\mathit{Generators}: \quad a_1, \quad b_1, ..., a_g, \quad b_g, \quad c, \quad d, \quad e \\ &\mathit{Relations}: \quad \prod_{j=1}^{g}[a_j, b_j] = 1, \quad a_j c = c a_j, \quad a_j d = d a_j, \quad a_j e = e a_j \\ &\qquad b_j c = c b_j, \quad b_j d = d b_j, \quad dce = 1. \end{aligned}$$

In addition, it can be shown that the fiber class of Σ_{-1} in X_{-1} is trivial in U^*_{-1}. (This follows from the fact that E_j meets Σ_{-1} at exactly one point and this point is a transverse point of intersection. Hence, we can represent the fiber class by a small loop γ on E_j encircling this point of intersection. Since the point of intersection is a transverse point of intersection, we may assume that the intersection of E_j with U^*_{-1} is a disc, the complement of a closed disc in the 2-sphere E_j. The existence of this disc proves that the fiber class is trivial in U^*_{-1}.) In this presentation, c, d and e correspond to a free basis for $\pi_1(T^2 \setminus (B(x_0) \cup B(x_{-1})))$. These generators are chosen so that the circle surrounding x_{-1} corresponds to dce and the circle surrounding x_0 corresponds to $e^{-1}d^{-1}c^{-1}$. As a consequence of the last relation, the puncture x_0 corresponds to the class $[d, c]$ in $\pi_1(U^*_{-1})$. On the other hand, by a result of Zariski, we have the following presentation for $\pi_1(U_1)$:

$$\mathit{Generator}: \quad z, \qquad \mathit{Relation}: \quad z^k = 1.$$

Here z is the fiber class of the nonsingular curve Σ_1 of degree k in $X_1 = \mathbf{CP^2}$.

Let W_{-1} be the tubular shell neighborhood of Σ_{-1} in X_{-1}. As in (2.1), we have the following presentation for $\pi_1(W_{-1})$:

$$Generators: \quad a_{-1,1}, \quad b_{-1,1}, ..., a_{-1,g}, \quad b_{-1,g}, \quad z_{-1}$$

$$Relations: \quad \prod_{j=1}^{g} [a_{-1,j}, b_{-1,j}] = z_{-1}^{-k^2}, \quad a_{-1,j} z_{-1} = z_{-1} a_{-1,j}, \quad b_{-1,j} z_{-1} = z_{-1} b_{-1,j}.$$

The homomorphism $\jmath_* : \pi_1(W_{-1}) \to \pi_1(U^*_{-1})$ can be described as follows:

$$\jmath_*(a_{-1,j}) = a_j, \quad \jmath_*(b_{-1,j}) = b_j, \quad \jmath_*(z_{-1}) = 1.$$

The last identity is simply the previous observation regarding the fiber class of Σ_{-1} in U^*_{-1}. The homomorphism $(\jmath \circ \Psi)_* : \pi_1(W_{-1}) \to \pi_1(U_1)$ can be described as follows:

$$(\jmath \circ \Psi)_*(a_{-1,j}) = 1, \quad (\jmath \circ \Psi)_*(b_{-1,j}) = 1, \quad (\jmath \circ \Psi)_*(z_{-1}) = z.$$

Hence, by Van Kampen's Theorem, we conclude that $\pi_1(X^*)$ is a free group on two generators, c and d. In addition, we see that the fiber class of Σ in X^* corresponds to the commutator $[d, c]$. (This follows from the previous remarks regarding the punctures x_{-1} and x_0.)

Now to calculate the fundamental group of Y we apply Van Kampen's Theorem to the covering (V_{-1}, V_1) of Y. Since V_i is a copy of X^*, we see that $\pi_1(V_i)$ is a free group on two generators, c_i and d_i. Let $W'_{-1} = V_{-1} \cap V_1$, the tubular shell neighborhood of Γ_{-1} in X^{-1}. Using a presentation for $\pi_1(W'_{-1})$ as in (2.1), the homomorphism $\jmath_* : \pi_1(W'_{-1}) \to \pi_1(V_{-1})$ can be described as follows:

$$\jmath_*(a_{-1,j}) = 1, \quad \jmath_*(b_{-1,j}) = 1, \quad \jmath_*(z_{-1}) = [d_{-1}, c_{-1}].$$

The last identity corresponds to the previous observation regarding the fiber class z of Σ in X^*. We may prescribe the following restrictions on the isomorphism Φ_*:

$$\Phi_*(a_{-1,j}) = a_{1,j} z_1, \quad \Phi_*(b_{-1,j}) = b_{1,j}, \quad \Phi_*(z_{-1}) = z_1^{-1}. \qquad (6.2)$$

Again, by Van Kampen's theorem, we obtain the following presentation of $\pi_1(Y)$:

$$Generators: \quad c_{-1}, \quad d_{-1}, \quad c_1, \quad d_1$$

$$Relations: \quad [d_1, c_1] = 1, \quad [d_{-1}, c_{-1}] = [d_1, c_1]^{-1}.$$

As a consequence of these relations, we have the relation $[d_{-1}, c_{-1}] = 1$. Let G_i be the group corresponding to the presentation:

$$Generators: \quad c_i, d_i \quad Relations: \quad [d_i, c_i] = 1.$$

Clearly, $\pi_1(Y)$ is isomorphic to the free product $G_{-1} * G_1$. On the other hand, G_i is a free abelian group of rank 2. In particular, G_i has at least one nontrivial finite quotient. Hence, by the result of [J-R] or the result of [A-B-R], $\pi_1(Y)$ is not the fundamental group of any compact Kähler manifold. As a consequence, Y is not homeomorphic to any compact Kähler manifold.

Remark 6.2. If G is a group, let $b_1(G)$ denote $b_1(S)$ where S is any space with $\pi_1(S)$ isomorphic to G. As is well known, $b_1(G)$ is well-defined independently of S. We have the following alternative argument for the above conclusion based on the fact that $\pi_1(Y)$ contains a subgroup G of index two such that $b_1(G) = 5$. We shall establish this fact below. Suppose that $\pi_1(Y)$ is isomorphic to $\pi_1(W)$ for some compact Kähler manifold W. The subgroup G is isomorphic to $\pi_1(S)$ for some 2-fold cover S of W. Since S is a cover (unbranched) of W, we can pull back the Kähler structure on W to obtain a Kähler structure on S. Since S is a finite cover of the compact space W, S is compact. Hence, $b_1(S)$ is even. But $b_1(S) = b_1(G) = 5$. This is impossible. This gives an alternative argument that $\pi_1(Y)$ is not isomorphic to the fundamental group of any compact Kähler manifold.

An example of a subgroup G of $\pi_1(Y)$ as above can be exhibited as follows. Let Σ be a surface of genus 2. Let a_1, b_1, a_2, b_2 be a standard set of generators for $\pi_1(\Sigma)$. Let γ be a simple closed curve representing the commutator $[a_1, b_1]$ in $\pi_1(\Sigma)$. Let C be the 2 dimensional CW complex obtained by attaching a disc D^2 to Σ by a homeomorphism from S^1 to γ. Note that $\pi_1(Y)$ is isomorphic to $\pi_1(C)$. Consider the 2-fold cover Γ of Σ given by the monodromy representation:

$$\rho : \pi_1(\Sigma) \to \mathbf{Z_2}$$
$$\rho(a_1) = \rho(a_2) = 1 \quad \rho(b_1) = \rho(b_2) = 0$$

Γ is a surface of genus 3. Of course, $b_1(\Gamma) = 6$. The preimage γ' of γ in Γ is a disjoint union of simple closed curves γ_1 and γ_2. γ_j represents a nontrivial homology class in $H_1(\Gamma)$ and $\gamma_1 + \gamma_2 = 0$. Hence, γ_1 and γ_2 span a 1 dimensional subspace of $H_1(\Gamma)$. Let $D^2_j, j = 1, 2$ be a pair of discs with boundaries S^1_j. We can extend Γ to a 2-fold cover C' of C by attaching each D^2_j to Γ by a homeomorphism from S^1_j to γ_j. The fundamental group of C' is of course isomorphic to a subgroup G of index 2 in $\pi_1(Y)$. Hence, $b_1(G) = b_1(C')$. $H_1(C')$ is isomorphic to the quotient of $H_1(\Gamma)$ by the subspace of $H_1(\Gamma)$ which is spanned by γ_1 and γ_2. Hence, by the previous observations, $b_1(C') = b_1(\Gamma) - 1 = 5$. Thus, $b_1(G)$ is equal to 5.

Example 6.2 (Infinite Families With Fixed Chern Numbers)**.** We shall modify the construction of the previous example to obtain the following theorem.

Theorem 6.1. *There exists an infinite family of compact symplectic* 4-*manifolds with fixed Chern numbers, no two of which are homeomorphic.*

Proof. We construct such a family as a variation on the previous example. The description of $\pi_1(V_i)$, $\pi_1(W'_i)$ and the homomorphisms induced by inclusion are as given in the previous example. The variation is in the prescription of Φ_* given in (6.2). Let α be a positive integer. We vary the prescription of Φ_* as follows:

$$\Phi_*(a_{-1,j}) = a_{1,j} z_1^{\alpha}, \quad \Phi_*(b_{-1,j}) = b_{1,j}, \quad \Phi_*(z_{-1}) = z_1^{-1}.$$

Let Y_α be the corresponding symplectic normal connect sum. (Note that Y_1 is the manifold of the previous example.) As in the previous example, we obtain the following presentation of $\pi_1(Y_\alpha)$:

$$\textit{Generators}: \quad c_{-1}, \quad d_{-1}, \quad c_1, \quad d_1$$

$$\textit{Relations}: \quad 1 = [d_1, c_1]^{\alpha} \quad [d_{-1}, c_{-1}] = [d_1, c_1]^{-1}.$$

As a consequence of these relations, we have the relation:

$$[d_{-1}, c_{-1}]^{\alpha} = 1.$$

Let G_i^{α} be the group corresponding to the presentation:

$$\textit{Generators}: \quad c_i, \quad d_i \quad \textit{Relations}: \quad [d_i, c_i]^{\alpha} = 1.$$

G_i^{α} is a one relator group with relator $r_i = u_i^{\alpha}$ where u_i is the simple commutator $[d_i, c_i]$. Hence, the abelianization of G_i^{α} is a free abelian group of rank 2. Therefore, in particular, G_i^{α} is an infinite group. By Proposition 5.17 in chapter II of [L-S], u_i is not a proper power in the free group on the generators c_i and d_i. Hence, by Theorem 5.2 in chapter IV of [L-S], u_i has order α in G_i^{α} and all elements of finite order in G_i^{α} are conjugates of powers of u_i. In particular, the orders of torsion elements in G_i^{α} are precisely the divisors of α. Let F_i^{α} be the cyclic subgroup of G_i^{α} of order α generated by u_i. Since G_i^{α} is an infinite group, the finite group F_i^{α} is a proper subgroup of G_i^{α}. Let ϕ be the isomorphism from F_{-1}^{α} to F_1^{α} which sends u_{-1} to u_1^{-1}. Clearly, by the above presentation, $\pi_1(Y_\alpha)$ is isomorphic to the free product with amalgamation $G_{-1}^{\alpha} *_\phi G_1^{\alpha}$ over the proper finite subgroups F_{-1}^{α} and F_1^{α}. In particular, by Theorem 2.6 in chapter IV of [L-S], G_i^{α} embeds in $\pi_1(Y_\alpha)$. Furthermore, by Theorem 2.7 in chapter IV of [L-S], every element of finite order in $\pi_1(Y_\alpha)$ is conjugate to an element of G_{-1}^{m} or G_1^{m}. In particular, the orders of torsion elements in $\pi_1(Y_\alpha)$ are precisely the divisors of α. Thus, $\pi_1(Y_\alpha)$ is isomorphic to $\pi_1(Y_\beta)$ if and only if $\alpha = \beta$. Hence, Y_α is homeomorphic to Y_β if and only if $\alpha = \beta$.

By the discussion in section 2, the Chern numbers of Y_α are independent of α. These numbers can be calculated from (2.8) and (6.1) and the following facts:

$$\chi(X_{-1}) = k^2, \quad \sigma(X_{-1}) = -k^2, \quad \chi(X_1) = 3, \quad \sigma(X_1) = 1,$$

where X_{-1} is the blow up of $R \times T^2$ at k^2 points as above and $X_1 = \mathbf{CP^2}$. The result is that:

$$c_1^2(Y_\alpha) = 2(5k-3)(k-3), \qquad c_2(Y_\alpha) = 2(4k^2 - 9k + 3). \tag{6.3}$$

Fixing k, the compact symplectic manifolds Y_α have fixed Chern numbers. On the other hand, the groups $\pi_1(Y_\alpha)$ are distinct and, hence, no two of the Y_α are homeomorphic.

□

Remark 6.3. By varying the degree k of the curve Σ_1 in $\mathbf{CP^2}$, the above construction yields a "2-parameter" family of compact symplectic 4-manifolds:

$$\{Y_\alpha(k) | k \geq 3, \alpha > 0\}.$$

Note that $\pi_1(Y_\alpha(k))$ is independent of k and $\pi_1(Y_\alpha)$ is isomorphic to $\pi_1(Y_\beta)$ if and only if $\alpha = \beta$. On the other hand, the Chern numbers of $Y_\alpha(k)$ are independent of α and the Chern numbers of $Y_\alpha(k)$ are equal to those of $Y_\beta(k')$ if and only if $k = k'$. Hence, $Y_\alpha(k)$ is homeomorphic to $Y_\beta(k')$ if and only if $(k, \alpha) = (k', \beta)$. In particular, by fixing an integer $k \geq 3$, we obtain an infinite family of compact symplectic 4 manifolds with the same Chern numbers but distinct fundamental groups. On the other hand by fixing the integer $\alpha > 0$, we obtain an infinite family of compact symplectic 4 manifolds with the same fundamental groups but distinct Chern numbers. Indeed, the Chern numbers strictly increase with respect to k. Hence, these manifolds cannot even be homeomorphic to blow ups or blow downs of one another.

When $k > 3$, using the examples in Theorem 5.1 and results of Gieseker, we can exhibit a striking contrast between compact symplectic 4-manifolds and compact Kähler surfaces. Gieseker's results show that there are only finitely many diffeomorphism types among all surfaces of general type with given Chern numbers ([B-P-V], chapter VII, section 1). This, of course, implies that there are only finitely many homeomorphism types among all surfaces of general type with given Chern numbers. (Note that these results do not assume minimality.) Hence, by the classification of complex surfaces ([B-P-V]) there are only finitely many homeomorphism types among all complex surfaces with fixed Chern numbers satisfying $c_1^2 > 0$ and $c_2 > 0$. When $k > 3$, the infinite family Y_α shows that the analogous statement is false for compact symplectic 4-manifolds. Indeed, when $k > 3$, (6.3) implies that $c_1^2(Y_\alpha) > 0$ and $c_2(Y_\alpha) > 0$. (Indeed, if $k > 3$ and Y_α is complex, then Y_α is necessarily of general type.)

These compact symplectic manifolds formally resemble surfaces of general type. This observation motivates the following questions:

Question . What is the geography of compact symplectic 4-manifolds?

Question . What is the geography of minimal compact symplectic 4-manifolds?

By geography we mean: what values in $\mathbf{Z} \times \mathbf{Z}$ are of the form $(c_1^2(X), c_2(X))$, for some (minimal) compact symplectic 4-manifold X? By the work of Van de Ven [V] there are no restrictions on (c_1^2, c_2) for compact almost complex 4-manifolds. However, there are well-known strict constraints on (c_1^2, c_2) for compact Kähler surfaces [B-P-V]. The geography problem for compact minimal surfaces of general type remains open and is a subject of current research. The above questions ask where symplectic manifolds lie between almost complex manifolds and complex Kähler manifolds. We hope to return to this problem in future papers.

We can show that $Y_\alpha(k)$ is not homeomorphic to a complex surface for any $k \geq 3$ and $\alpha > 0$. Suppose, on the contrary, that $Y_\alpha(k)$ is homeomorphic to a complex surface W for some $k \geq 3$ and $\alpha > 0$. From the computation of $\pi_1(Y_\alpha)$ given above, we see that $b_1(W) = b_1(Y_\alpha(k)) = 4$. Since W is a complex surface with even b_1, W is Kähler. Thus $\pi_1(Y_\alpha)$ is isomorphic to the fundamental group of some compact Kähler manifold. From the description of $\pi_1(Y_\alpha)$ given above, we see that this contradicts the result of [A-B-R].

Alternatively, we could appeal to the fact that $\pi_1(Y_\alpha)$ has a subgroup G of index 2 with $b_1(G) = 5$ as in remark 6.2. The existence of G is established as in remark 6.2 with the following modifications. The attaching homeomorphism from S^1 to γ must be replaced by a covering map of degree m from S^1 to γ. The attaching maps from S^1_j to γ_j must be similarly modified.

One advantage of the second argument is that it allows us to conclude that Y_α is stably non-Kähler as in remark 6.1. For if S is any compact Kähler manifold, then $b_1(S)$ is even. On the other hand, $G \times \pi_1(S)$ is a subgroup of index 2 in $\pi_1(Y_\alpha \times S)$ with:

$$b_1(G \times \pi_1(S)) = 5 + b_1(S).$$

Hence, $Y_\alpha \times S$ has a subgroup of index 2 with odd b_1.

In light of these examples and Gieseker's results, it is interesting to ask:

Question . Which groups occur as the fundamental groups of compact symplectic 4-manifolds with fixed Chern numbers $c_1^2 > 0$ and $c_2 > 0$?

7. **Blowing Down**

In [G2] Gromov introduced the operations of symplectic blowing up and symplectic blowing down. Let Σ_{-1} be a symplectically embedded surface of genus 0 and self-intersection -1 (a $(-1)-curve$) in a symplectic 4-manifold $(\tilde{X}, \tilde{\omega})$. Suppose that $\int_{\Sigma_{-1}} \tilde{\omega} = \lambda^2 \pi$. Then Σ_{-1} has a tubular neighborhood $N_\epsilon(\Sigma_{-1})$ so that the tubular shell $(W_{-1}, \tilde{\omega})$ is symplectically diffeomorphic to $(B_{\lambda+\epsilon}(0) \setminus \overline{B_\lambda(0)}, \Omega)$, where $B_r(0)$ is the ball of center 0, radius r in $\mathbf{R^4}$, and where Ω is the standard symplectic form on $\mathbf{R^4}$. Recall that to blow down Σ_{-1} we delete $N_\epsilon(\Sigma_{-1})$ and using the symplectic diffeomorphism glue in $B_{\lambda+\epsilon}(0)$. The resulting symplectic manifold (X, ω) is, up to symplectic isotopy, independent of ϵ. For more details, see [McD2]. It is not difficult to verify that the Chern numbers of $\tilde{X}$ and X are related by:

$$\begin{aligned} c_1^2(X) &= c_1^2(\tilde{X}) + 1 \\ c_2(X) &= c_2(\tilde{X}) - 1 \end{aligned} \tag{7.1}$$

It is interesting to notice that this blowing down operation can be considered as a special case of the symplectic normal connect sum. Let Σ_{-1} be a $(-1)-curve$ in a symplectic 4-manifold (X_{-1}, ω_{-1}) with $\int_{\Sigma_{-1}} \omega_{-1} = \lambda^2 \pi$. Let (X_1, ω_1) be $(\mathbf{CP^2}, \omega_\mathbf{0})$ where ω_0 is the Fubini-Study 2-form normalized such that $\int_{\mathbf{CP^1}} \omega_0 = \lambda^2\pi$ and let $\Sigma_1 = \mathbf{CP^1} \hookrightarrow \mathbf{CP^2}$. Then the symplectic normal connect sum of X_{-1} and X_1 along Σ_{-1} and Σ_1 is diffeomorphic to the blow down of X_{-1}. Since $c_1^2(\mathbf{CP^2}) = \mathbf{9}$ and $c_2(\mathbf{CP^2}) = \mathbf{3}$, (2.4) agrees with (7.1).

More generally, let Σ_{-1} be a symplectically embedded surface of genus 0 and self-intersection -4 (a $(-4) - $ *curve*) in a symplectic 4-manifold (X_{-1}, ω_{-1}). Let (X_1, ω_1) be $\mathbf{CP^2}$ with the Fubini-Study 2-form and let Σ_1 be a nonsingular quadric curve in $\mathbf{CP^2}$. (Σ_1 is an embedded holomorphic curve of genus zero and self-intersection 4). Let $X = X_{-1}\#_\Psi X_1$ be the symplectic normal connect sum along Σ_{-1} and Σ_1. We remark that the topology of X is independent of the choice of gluing map Ψ. Also by (2.4),

$$c_1^2(X) = c_1^2(X_{-1}) + 1$$
$$c_2(X) = c_2(X_{-1}) - 1.$$

Thus X is a smooth symplectic manifold which has had the (-4)-curve Σ_{-1} *blown down*. In algebraic geometry the blowing down or collapsing of negative self-intersection curves is defined, though the resulting complex surface has an isolated singularity. In the symplectic category we can, using the symplectic normal connect sum, define blowing down (-4)-curves smoothly.

As for blowing down of (-1)-curves there is a simple topological interpretation of blowing down (-4)-curves. In the case of a (-1)-curve, one replaces the tubular neighborhood of the (-1)-curve with a standard ball. The (-1)-curve is replaced by a point. This corresponds to the drop in b_2. (Since the tubular neighborhood of a (-1)-curve is diffeomorphic to the complement of a ball in $\overline{\mathbf{CP^2}}$, one obtains the corresponding topological interpretation of blowing up as connect sum with $\overline{\mathbf{CP^2}}$.) In the case of a (-4)-curve, the tubular neighborhood of the (-4)-curve is replaced by the tangent bundle of $\mathbf{RP^2}$ (appropriately oriented). (This follows from the observation that the complement of a tubular neighborhood of a nonsingular quadric curve in $\mathbf{CP^2}$ is a tubular neighborhood of a complementary $\mathbf{RP^2} \subset \mathbf{CP^2}$.) The (-4)-curve C is replaced by $\mathbf{RP^2}$. Again, b_2 drops by 1. This corresponds to the fact that C is orientable whereas $\mathbf{RP^2}$ is nonorientable. On the other hand, $\mathbf{RP^2}$ represents a nontrivial $\mathbf{Z_2}$ homology class in the blown down manifold. (This follows from the fact that $\mathbf{RP^2} \subset \mathbf{CP^2}$ has odd self-intersection). Hence blowing down of (-4)-curves does not "collapse" the (-4)-curve, not even on the level of homotopy. (It does "collapse" the (-4)-curve on the level of homology with real coefficients.)

Appendix

Independently of our work, R. Gompf obtained a version of Theorem 1.1 in arbitrary dimensions. His proof is different from ours, relying on a flow argument rather than symplectic reduction. In this section, we give a simple proof of Gompf's generalization using the symplectic reduction argument of section 1.

Let (X_i, ω_i), $\ i = -1, 1,\ $ be symplectic manifolds of dimension $2n$ and (N, η) be a closed symplectic manifold of dimension $2n-2$. Let $\jmath_i : N \hookrightarrow X_i$ be symplectic embeddings with normal bundles ν_i. Suppose that $c_1(\nu_{-1}) = -c_1(\nu_1)$. Let ρ_N be a closed 2-form on N so that $[\rho_N] = c_1(\nu_1)$ and let ρ be a closed 2-form on X_{-1} so that $\jmath_{-1}^* \rho = \rho_N$. Then for sufficiently small t the 2-form:

$$\tilde{\omega}_{-1} = \omega_{-1} + t\rho \tag{A.1}$$

is a symplectic form on X_{-1}.

Theorem A.1 (Gompf)**.** *For each $i = 1, -1$ there exist pairs of tubular neighborhoods V_i, U_i with $\jmath_i(N) \subset V_i$ and $\overline{V_i} \subset U_i$ and a symplectomorphism*

$$\Psi : (U_{-1} \setminus \overline{V_{-1}}, \tilde{\omega}_{-1}) \to (U_1 \setminus \overline{V_1}, \omega_1)$$

such that the normal connect sum $X = X_{-1} \#_\Psi X_1$ has a symplectic form ω which agrees with ω_1 and $\tilde{\omega}_{-1}$ off a neighborhood of the gluing locus.

Proof. Following the discussion of section 1 we can construct an S^1-invariant symplectic form τ on an S^2 bundle over N by specifying a family of symplectic forms $\{\sigma_t\}$ on N satisfying (1.2). The family we choose is:

$$\sigma_t = \omega_1 + t\rho_N, \quad 0 \le t \le t_0,$$

where t_0 is chosen so that the forms $\{\sigma_t\}$ are symplectic for t satisfying $0 \le t \le t_0$. The resulting S^2 bundle, S, has symplectic form τ and moment map $H_\tau : S \to [0, t_0]$ such that τ restricts to ω_1 on the zero section $Z_0 = H_\tau^{-1}(0)$ and to $\omega_1 + t_0\rho_N$ on the infinity section $H_\tau^{-1}(t_0)$. Moreover, Z_0 has normal bundle with Chern class $[\rho_N] = c_1(\nu_1)$.

By the symplectic neighborhood theorem there are tubular neighborhoods W_0 of Z_0 in S and W_1 of N in X_1 such that:

$$F : (W_0, \tau) \to (W_1, \omega_1)$$

is a symplectomorphism which restricts to the identity map on Z_0. Since $H_\tau^{-1}[0, t_1]$ is contained in W_0 for some $t_1 < t_0$, we can suppose that $W_0 = H_\tau^{-1}[0, t_1]$. Then F symplectically identifies W_1 with $H_\tau^{-1}[0, t_1]$.

Choose $t < t_1$ sufficiently small so that the 2-form $\tilde{\omega}_{-1}$ defined in (A.1) is symplectic. Now construct an S^1-invariant symplectic form υ on S using the family of symplectic forms on N:

$$\sigma_s = \omega_1 + s\rho_N, \quad 0 \le s \le t.$$

(S, υ) has moment map $H_\upsilon : S \to [0, t]$. Denote the infinity section, $H_\upsilon^{-1}(t)$, by Z_∞. The normal bundle of Z_∞ has Chern class $-[\rho_N] = -c_1(\nu_1) = c_1(\nu_{-1})$ and υ restricts on Z_∞ to the form:

$$\omega_1 + t\rho_N = \omega_{-1} + t\rho_N = \tilde{\omega}_{-1_{|_N}}.$$

Hence by the symplectic neighborhood theorem there are tubular neighborhoods W_∞ of Z_∞ in S and W_{-1} of N in X_{-1} such that there is a symplectomorphism:

$$\tilde{F} : (W_\infty, \upsilon) \to (W_{-1}, \tilde{\omega}_{-1})$$

which restricts to the identity map on Z_∞. By choosing W_∞ smaller, if necessary, we can suppose that $W_\infty \setminus Z_\infty = H_\upsilon^{-1}(s_0, t)$ for some $s_0 > 0$. Thus on $W_\infty \setminus Z_\infty$ the symplectic form υ is determined by the family of forms $\sigma_s = \omega_1 + s\rho_N, \quad s_0 < s < t$. Recall that on $H^{-1}(s_0, t) \subset H^{-1}(0, t_0)$ the symplectic form τ is determined by the same family of forms. Hence there is a symplectomorphism:

$$\psi : (H^{-1}(s_0, t), \tau) \to (W_\infty \setminus Z_\infty, \upsilon)$$

The composition of the three symplectomorphisms: $F, \ \tilde{F}, \ \psi$ defines the required symplectic gluing. □

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