THE MAPPING CLASS GROUP AND A THEOREM OF MASUR-WOLF

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Abstract. In [M-W], Masur and Wolf proved that the Teichmüller space of genus $g > 1$ surfaces with the Teichmüller metric is not a Gromov hyperbolic space. In this paper, we provide an alternative proof based upon a study of the action of the mapping class group on Teichmüller space.

1. Introduction

As observed in [M-W], the Teichmüller space of surfaces of genus $g > 1$ with the Teichmüller metric shares many properties with spaces of negative curvature. In his study of the geometry of Teichmüller space [Kr], Kravetz claimed that Teichmüller space was negatively curved in the sense of Busemann [Bu]. It was not until about ten years later, that Linch [L] discovered a mistake in Kravetz's arguments. This left open the question of whether or not Teichmüller space was negatively curved in the sense of Busemann. This question was resolved in the negative by Masur in [Ma].

A metric space X is negatively curved, in the sense of Busemann, if the distance between the endpoints of two geodesic segments from a point in X is at least twice the distance between the midpoints of these two segments. An immediate consequence of this definition is that distinct geodesic rays from a point in a Busemann negatively curved metric space must diverge. Masur proved that Teichmüller space is not negatively curved, in the sense of Busemann, by constructing distinct geodesic rays from a point in Teichmüller space which remain a bounded distance away from each other.

In [G], Gromov introduced a notion of negative curvature for metric spaces which, while less restrictive than that of Busemann, implies many of the properties which Teichmüller space shares with spaces of Riemannian negative sectional curvature. This raised the question

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of whether Teichmüller space was negatively curved in the sense of Gromov, (i.e. Gromov hyperbolic). According to one of the definitions of Gromov hyperbolicity, an affirmative answer to this question would rule out so-called "fat" geodesic triangles in Teichmüller space. In [M-W], Masur and Wolf resolved the Gromov hyperbolicity question in the negative by constructing such "fat" geodesic triangles.

As observed in [M-W], the existence of distinct nondivergent rays from a point in Teichmüller space does not preclude Teichmüller space from being Gromov hyperbolic. Apparently for this reason, rather than taking Masur's construction of such rays as the starting point for their proof, Masur and Wolf found their motivation from another source. They observed that the isometry group of the Teichmüller metric is the mapping class group [R], which is not a Gromov hyperbolic group, since it contains a free abelian group of rank 2. This fact, like Masur's result on the existence of distinct nondivergent rays from a point, is insufficient to imply that Teichmüller space is not Gromov hyperbolic. Nevertheless, it served as motivation for Masur and Wolf's construction of "fat" geodesic triangles.

In this paper, we provide an alternative proof of the result of Masur and Wolf. Our proof, like that of Masur and Wolf, is motivated by the fact that the mapping class group is not Gromov hyperbolic. On the other hand, unlike the proof of Masur and Wolf, our proof depends upon one of the deeper consequences of Gromov hyperbolicity. Namely, in order for Teichmüller space to be Gromov hyperbolic, the isometries of Teichmüller space must be governed by Gromov's classification of isometries of Gromov hyperbolic spaces. We show that this classification is incompatible with the structure of the mapping class group.

The outline of the paper is as follows. In section 2, we review the prerequisites for our proof. In section 3, we prove the theorem of Masur and Wolf that Teichmüller space is not Gromov hyperbolic.

2. Preliminaries

2.1. **Teichmüller space.** Let M denote a closed, connected, orientable surface of genus $g \geq 2$. The Teichmüller space T_g of M is the space of equivalence classes of complex structures on M, where two complex structures S_1 and S_2 on M are equivalent if there is a conformal isomorphism $h : S_1 \to S_2$ which is isotopic to the identity map of the underlying topological surface M.

The Teichmüller distance $d([S_1], [S_2])$ between the equivalence classes $[S_1]$ and $[S_2]$ of two complex structures S_1 and S_2 on M is defined as

$\frac{1}{2}\log\inf_h K(h)$, where the infimum is taken over all quasiconformal homeomorphisms $h : S_1 \to S_2$ which are isotopic to the identity map of M and $K(h)$ is the maximal dilitation of h. This infimum is realized by a unique quasiconformal homeomorphism, which homeomorphism is called the Teichmüller map from S_1 to S_2.

As shown by Kravetz [Kr], (T_g, d) is a *straight G-space* in the sense of Busemann ([Bu],[A]). Hence, any two distinct points, x and y, in T_g are joined by a unique geodesic segment (i.e. an isometric image of a Euclidean interval), [x,y], and lie on a unique geodesic line (i.e. an isometric image of $\mathbb{R}$), $\gamma(x, y)$.

Now, fix a conformal structure S on M and let $QD(S)$ be the space of holomorphic quadratic differentials on S. The geodesic rays (i.e. isometric images of $[0,\infty)$) which emanate from the point $[S]$ in T_g are described in terms of $QD(S)$. If q is a holomorphic quadratic differential on S, p is a point on S and z is a local parameter on S defined on a neighborhood U of p, then q may be written in the form $\phi(z)dz^2$ for some holomorphic function ϕ on U. If $\phi(p) \neq 0$ and $z_0 = z(p)$, then on a sufficiently small neighborhood V of p contained in U, we may define a branch $\phi(z)^{1/2}$ of the square root of ϕ. The integral $w = \Phi(z) = \int_{z_0}^{z} \phi(z)^{1/2}dz$ is a conformal function of z and determines a local parameter for S on a sufficiently small neighborhood W of p in V. This parameter w is called a *natural rectangular parameter* for q at the regular point p. In terms of this parameter w, q may be written in the form dw^2. For each nonzero quadratic differential q on S, there is a one-parameter family $\{S_K\}$ of conformal structures on M and quadratic differentials $\{q_K\}$ on S_K obtained by replacing the natural parameters w for q on S by natural parameters w_K for q_K on S_K. The relationship between w_K and w is given by the rule:

$$Rew_K = K^{1/2}Rew \quad Imw_K = K^{-1/2}Imw.$$

The Teichmüller distance from $[S_K]$ to $[S]$ is equal to $\log(K)/2$. The map $t \mapsto [S_{e^{2t}}]$ is a Teichmüller geodesic ray emanating from $[S]$ and every geodesic ray emanating from $[S]$ is of this form. Two nonzero quadratic differentials on S determine the same Teichmüller geodesic ray in T_g emanating from $[S]$ if and only if they are positive multiples of one another.

It is well-known that (T_g, d) is homeomorphic to $\mathbb{R}^{6g-6}$ and closed balls in (T_g, d) are homeomorphic to closed balls in $\mathbb{R}^{6g-6}$. In fact, using the previous description of geodesic rays, an homeomorphism can be constructed from the open unit ball of $QD(S)$ onto T_g. Suppose q is a point in the open unit ball of $QD(S)$. Then $q = kq_1$ where $0 \leq k < 1$ and q_1 is a quadratic differential in the unit sphere of $QD(S)$. Map

q to the point $[S_K]$ on the geodesic ray through $[S]$ in the direction of q_1 where $K = (1+k)/(1-k)$. By the work of Teichmüller, this map is an homeomorphism from the open unit ball of $QD(S)$ onto T_g. Since $QD(S)$ is a complex vector space of dimension $3g-3$, this proves that T_g is homeomorphic to $\mathbb{R}^{6g-6}$. Note also that this homeomorphism maps the closed ball of radius k centered at the origin of $QD(S)$ onto the closed ball of radius $\log(K)/2$ centered at the point $[S]$ in (T_g, d). This proves that closed balls in (T_g, d) are homeomorphic to closed balls in $\mathbb{R}^{6g-6}$.

The mapping class group Γ_g of M is the group of isotopy classes of orientation-preserving homeomorphisms $M \to M$. Γ_g acts on T_g by pulling back conformal structures S on M. In other words, the action of Γ_g on T_g is given by the well-defined rule $[h] \cdot [S] = [h_* S]$, where $h_* S$ is the conformal structure on M determined by compositions of charts of S with restrictions of h^{-1}). Note that, by construction, $h : S \to h_* S$ is a conformal isomorphism. Hence, an homeomorphism $g : S \to S$ is K-quasiconformal if and only if $h \circ g \circ h^{-1} : h_* S \to h_* S$ is K-quasiconformal. It follows that the action of Γ_g on T_g is by isometries of (T_g, d).

It is well-known that Γ_g acts properly discontinuously on T_g (see [M-P2] for a simple proof of this fact).

2.2. **Isometries of Gromov hyperbolic spaces.** Let X be a space equipped with a metric d. X is said to be *proper* if closed balls in X are compact. Since closed balls in (T_g, d) are homeomorphic to closed balls in $\mathbb{R}^{6g-6}$, (T_g, d) is proper. X is said to be *geodesic* if every pair of points $x, y \in X$ can be connected by a *geodesic segment* (i.e. an isometric embedding of an interval). By Kravetz' result that (T_g, d) is a straight G-space in the sense of Busemann discussed in (2.1), (T_g, d) is geodesic.

Gromov ([G], see also [C-D-P], [G-H]) introduced a notion of hyperbolicity for metric spaces which is now called Gromov hyperbolicity. Gromov hyperbolic metric spaces share many of the qualitative properties of hyperbolic space. The notion of Gromov hyperbolicity is defined in terms of the following Gromov product. Let x_0 be a fixed point in X. Denote the distance $d(x, y)$ between two points x and y in X by $|x - y|$. Denote $|x - x_0|$ by $|x|$. The Gromov product (x, y) is defined by the rule $(x, y) = (|x| + |y| - |x - y|)/2$. Note that the triangle inequality implies that $(x, y) \geq 0$ for all x and y in X. X is said to be Gromov hyperbolic if there exists a number $\delta \geq 0$ such that $(x, y) \geq \min((x, z), (y, z)) - \delta$ for all x, y and z in X. If we wish to specify δ, we say that X is Gromov δ-hyperbolic.

A sequence $x_i, i = 1, 2, ...$ of points in X is called *convergent at infinity* if $(x_i, x_j) \to \infty$ for $i, j \to \infty$. We say that two sequences $x_i, i = 1, 2, ...$ and $y_j, j = 1, 2, ...$, each convergent at infinity, are *equivalent* if $(x_i, y_j) \to \infty$ for $i, j \to \infty$. Assuming that X is Gromov hyperbolic, this defines an equivalence relation on the set of sequences in X which are convergent at infinity. The Gromov boundary ∂X of X is defined to be the set of equivalence classes of sequences in X which are convergent at infinity. If a sequence $x_i, i = 1, 2, ...$ is contained in an equivalence class $a \in \partial X$, we write $x_i \to a$ as $i \to \infty$. Every isometry $\phi : X \to X$ of X induces a well defined map $\phi : \partial X \to \partial X$ given by the rule $\phi(a) = b$ if $x_i \to a$ as $i \to \infty$ implies that $\phi(x_i) \to b$ as $i \to \infty$.

Let $\phi : X \to X$ be an isometry of X and $x \in X$. ϕ is said to be *elliptic* if the orbit $\{\phi^n(x) | n \in \mathbb{Z}\}$ of x in X is bounded. ϕ is said to be *hyperbolic* if the map $\phi^* : \mathbb{Z} \to X$ defined by $\phi^*(n) = \phi^n(x)$ is a quasisometry. ϕ is said to be *parabolic* if the orbit of x in X has exactly one point of accumulation in the boundary ∂X of X. [C-D-P]. (Note that the notions of elliptic, hyperbolic and parabolic isometries are well defined independently of x. Note also that the notions of elliptic and hyperbolic isometries make sense for any metric space.)

Remark 2.3. If ϕ is hyperbolic then the quasigeodesic $\phi^* : \mathbb{Z} \to X$ has exactly two limit points on ∂X, $x_+ = \lim_{n\to\infty} \phi^n(x)$ and $x_- = \lim_{n\to\infty} \phi^{-n}(x)$. Each of these points is clearly fixed by ϕ. Moreover, these points x_+ and x_- do not depend upon the choice of x. Hence, the forward orbits $\{\phi^n(y) | n > 0\}$ of each point y in X converge to x_+. The backward orbits $\{\phi^n(y) | n < 0\}$ of each point y in X converge to x_-.

Remark 2.4. The orbit $\{\phi^n(x) | n \in \mathbb{Z}\}$ of an elliptic isometry ϕ, being bounded, has no accumulation points on ∂X. By the previous remark, the orbit $\{\phi^n(x) | n \in \mathbb{Z}\}$ of an hyperbolic isometry ϕ has exactly two accumulation points on ∂X. Finally, by definition, the orbit $\{\phi^n(x) | n \in \mathbb{Z}\}$ of a parabolic isometry ϕ has exactly one accumulation point on ∂X. Hence, the three types of isometries are mutually exclusive.

We shall require the following result ([C-D-P], [G-H]).

Theorem (Gromov [C-D-P]**,** [G-H]**) .** *Let X be a proper, geodesic, Gromov hyperbolic space. Let $\phi : X \to X$ be an isometry of X. Then ϕ is either elliptic, hyperbolic or parabolic. If ϕ is hyperbolic, then ϕ has exactly two fixed points x_+ and x_- in ∂X. The forward orbits $\{\phi^n(x) | n > 0\}$ of each point x in X converge to x_+. The backward orbits $\{\phi^n(x) | n < 0\}$ of each point x in X converge to x_-. If ϕ is* *parabolic, then ϕ has a unique fixed point x on the Gromov boundary ∂X of X.*

If ϕ is hyperbolic, we refer to x_+ as the attracting fixed point of ϕ and to x_- as the repelling fixed point of ϕ.

Remark 2.5. The statement that ϕ is either elliptic, hyperbolic or parabolic is Theorem 2.1 of Chapter 9 of [C-D-P]. The statement that an hyperbolic isometry has exactly two fixed points in ∂X follows from Theorem 16 (i) in Chapter 8 of [G-H]. The convergence properties of these two fixed points x_+ and x_- have already been explained in Remark 2.3. The statement that a parabolic isometry has exactly one fixed point on ∂X is Theorem 17 (i) in Chapter 8 of [G-H].

Remark 2.6. Ghys and de la Harpe use alternative definitions for elliptic, hyperbolic and parabolic isometries than those of [C-D-P]. Their definition of an elliptic isometry is equivalent to that of [C-D-P] by Proposition-Definition 9 of Chapter 8 of [G-H]. Their definition of an hyperbolic isometry is equivalent to that of [C-D-P] by Proposition 21 of Chapter 8 of [G-H]. They define a parabolic isometry to be an isometry which is neither elliptic nor hyperbolic, as defined in [G-H] (see the paragraph before Theorem 17 in Chapter 8 of [G-H]). That this is equivalent to the definition of a parabolic isometry in [C-D-P] follows from the equivalence of the definitions of elliptic and hyperbolic isometries in [C-D-P] and [G-H], Theorem 2.1 of Chapter 9 of [C-D-P] and the mutual exclusivity, as explained in Remark 2.4, of the three types of isometries, as defined in [C-D-P].

3. Isometries of Teichmüller space

In this section, we prove the theorem of Masur and Wolf that Teichmüller space is not Gromov hyperbolic.

Lemma 3.1. *Suppose that (X, d) is a proper, geodesic, Gromov hyperbolic space on which Γ_g acts properly discontinuously by isometries. Let α be an isotopy class of a nonseparating simple closed curve a on M. Let $D_\alpha \in \Gamma_g$ denote the Dehn twist about a. Then D_α is a parabolic isometry of (X, d).*

Proof. Suppose that $\phi \in \Gamma_g$ is of infinite order and $x \in X$. Since Γ_g acts properly discontinuously by isometries on X and closed balls in (X, d) are compact, the orbit $\{\phi^n(x) | n \in \mathbb{Z}\}$ is unbounded. Hence, ϕ is not elliptic. Thus, by Gromov's classification of isometries of proper, geodesic, Gromov hyperbolic spaces discussed in (2.2), ϕ is either parabolic or hyperbolic.

In particular, since D_α has infinite order, D_α is either parabolic or hyperbolic. Suppose that D_α is hyperbolic. Suppose that β is an isotopy class of nonseparating simple closed curves on M. Since any two nonseparating circles on M are topologically equivalent, D_β is conjugate to D_α in Γ_g. Since D_α is hyperbolic, D_β is hyperbolic.

Now suppose that α and β have disjoint representative simple closed curves a and b. Then D_α commutes with D_β. By the usual argument, D_β preserves the fixed point set $\{x_1, x_2\}$ of the hyperbolic isometry D_α of X. We may assume that x_1 is the repelling fixed point of D_α. Then $D_\beta(x_1)$ is the repelling fixed point of $D_\beta \circ D_\alpha \circ D_\beta^{-1}$. Since D_β commutes with D_α, we conclude that $D_\beta(x_1)$ is the repelling fixed point x_1 of D_α. Likewise, $D_\beta(x_2) = x_2$. Thus the fixed point set of the hyperbolic isometry D_β of X is equal to $\{x_1, x_2\}$. (Observe, however, that x_1 need not be the repelling fixed point of D_β. Consider the fact that D_α^{-1} also commutes with D_α, whereas the repelling fixed point of D_α^{-1} is the attracting fixed point of D_α.)

We recall the Lickorish-Humphries generators for Γ_g. Choose a collection of pairwise transitive nonseparating simple closed curves $a_1, ..., a_{2g+1}$ on M such that a_i meets a_{i+1} at exactly 1 point for $1 \leq i \leq 2g$ and a_i is disjoint from a_j if $2 \leq |i-j|$. In other words, $a_1, ..., a_{2g+1}$ is a maximal chain of simple closed curves on M. (It is well-known that any two maximal chains on M are topologically equivalent [Mc].) Let d be a simple closed curve such that d is transverse to a_4, d meets a_4 in exactly one point and d is disjoint from a_i if $i \neq 4$. Let τ_i denote the Dehn twist about a_i and τ denote the Dehn twist about d. The Lickorish-Humphries generators for Γ_g are the mapping classes $\tau_1, ..., \tau_{2g}, \tau$.

Let $\{x_1, x_2\}$ denote the fixed point set of the hyperbolic isometry τ_1 of X. Since a_1 is disjoint from a_i for $3 \leq i$, we conclude that the fixed point set of the hyperbolic isometry τ_i is equal to $\{x_1, x_2\}$ for $3 \leq i$. Likewise, the fixed point set of the hyperbolic isometry τ is equal to $\{x_1, x_2\}$. Finally, since a_2 is disjoint from d, the fixed point set of the hyperbolic isometry τ_2 is equal to the fixed point set $\{x_1, x_2\}$ of the hyperbolic isometry τ. We conclude that the Lickorish-Humphries generators for Γ_g are hyperbolic isometries with a common fixed point set $\{x_1, x_2\}$. Since these generators generate Γ_g, we conclude that each element of Γ_g fixes x_1 and x_2.

Consider a pair a, b of disjoint nonseparating nonisotopic simple closed curves on S (e.g. a_1 and a_3). There exists an homeomorphism $h : S \to S$ which interchanges a and b. Let $\sigma \in \Gamma_g$ denote the isotopy class of h, α denote the isotopy class of a and β denote the isotopy class of b. Then, as is well known, $\sigma \circ D_\alpha \circ \sigma^{-1} = D_\beta$ and $\sigma \circ D_\beta \circ \sigma^{-1} = D_\alpha$. Let η denote the mapping class $D_\alpha \circ D_\beta^{-1}$. By the previous identities, $\sigma \circ \eta \circ \sigma^{-1} = \eta^{-1}$.

The class η has infinite order. Hence, η is either parabolic or hyperbolic. On the other hand, since $\eta \in \Gamma_g$, η has at least two fixed points on ∂X, x_1 and x_2. Hence, η is not parabolic. We conclude that η is hyperbolic and, hence, the fixed point set of η is equal to $\{x_1, x_2\}$. Since σ conjugates η to its inverse, σ must map the repelling fixed point of η to the repelling fixed point of η^{-1}. In other words, σ must map the repelling fixed point of η to the attracting fixed point of η. Likewise, σ must map the attracting fixed point of η to the repelling fixed point of η. We conclude that σ interchanges x_1 and x_2. On the other hand, since $\sigma \in \Gamma_g$, σ fixes x_1 and x_2. This is impossible. Hence, D_α is parabolic.

This completes the proof. $\square$

Theorem 3.2. *Suppose that (X, d) is a proper, geodesic metric space on which Γ_g acts properly discontinuously by isometries. Suppose that pseudo-Anosov mapping classes act hyperbolically on X. Then (X, d) is not Gromov hyperbolic.*

Proof. Suppose that (X, d) is Gromov hyperbolic. By Gromov's classification of isometries of a proper, geodesic, Gromov hyperbolic space discussed in (2.2), it follows that each isometry of (X, d) is either elliptic, hyperbolic or parabolic.

Suppose that $\phi \in \Gamma_g$ is of infinite order. As in the proof of Lemma 3.1, we conclude that ϕ is either parabolic or hyperbolic.

Let α be an isotopy class of a nonseparating simple closed curve a on S. Let $D_\alpha \in \Gamma_g$ denote the Dehn twist about a. By Lemma 3.1, D_α is a parabolic isometry of (X, d).

Now suppose that α and β have disjoint representative simple closed curves a and b. Then D_α commutes with D_β. By the usual argument, D_β preserves the fixed point set $\{x\}$ of the parabolic isometry D_α of X. Thus the fixed point set of the parabolic isometry D_β of X is equal to $\{x\}$.

Consider again the Lickorish-Humphries generators $\tau_1, ..., \tau_{2g}, \tau$ for Γ_g as described in Lemma 3.1. Following the corresponding argument in the proof of Lemma 3.1, we conclude that the Lickorish-Humphries generators for Γ_g are parabolic isometries with a common fixed point set $\{x\}$. Since these generators generate Γ_g, we conclude that each element of Γ_g fixes x.

Now, by Theorem 2 (*iii*) of [M-P1] and the following section on remarks and examples, there exists a pair of involutions σ and ϕ in Γ_g

such that $\sigma \circ \phi$ is a pseudo-Anosov element η of Γ_g. Since σ and ϕ are involutions, $\sigma\eta\sigma^{-1} = \eta^{-1}$.

Since the class η is pseudo-Anosov, η is hyperbolic. On the other hand, since $\eta \in \Gamma_g$, η fixes x. Hence, the fixed point set on ∂X of the hyperbolic isometry η of X consists of x and another point y. Following the corresponding argument in the proof of Lemma 3.1, we conclude that σ interchanges x and y. On the other hand, since $\sigma \in \Gamma_g$, σ fixes x. This is impossible. Hence, (X, d) is not Gromov hyperbolic.

This completes the proof. □

Remark 3.3. Masur and Minsky have shown that the complex of curves $C(M)$ is Gromov-hyperbolic with respect to the natural simplicial metric [M-M]. Since $C(M)$ is equipped with the natural simplicial metric, $C(M)$ is geodesic. The mapping class group Γ_g acts in a natural way on the simplicial complex $C(M)$. Hence, the mapping class group acts by isometries on the geodesic Gromov-hyperbolic space $C(M)$, in contrast to Theorem 3.2. On the other hand, although $C(M)$ is geodesic, $C(M)$ is not proper. Let v be a vertex of $C(M)$ corresponding to the isotopy class of a nontrivial simple closed curve a on M. The vertices w in the unit ball of $C(M)$ centered at v correspond to the isotopy classes of nontrivial simple closed curves b on M which are disjoint from a. There are infinitely many such isotopy classes. Hence, the unit ball of $C(M)$ centered at v contains infinitely many vertices of $C(M)$. These vertices form a discrete closed infinite subset of the unit ball. Hence, the unit ball of $C(M)$ centered at v is not compact.

The Dehn twist about a simple closed curve a on M fixes the vertex of $C(M)$ corresponding to the isotopy class of a. Since Dehn twists are of infinite order, we see that Γ_g does not act properly discontinuously on $C(M)$. Also, since Dehn twists fix a point in $C(M)$, they act elliptically on $C(M)$, in contrast to Lemma 3.1. In fact, every reducible element fixes some point in $C(M)$, (e.g. the barycenter of a simplex corresponding to a reduction family for the element). Hence, every reducible element acts elliptically on $C(M)$.

Interestingly, at least some pseudo-Anosov elements act hyperbolically on $C(M)$. Suppose that $h : M \to M$ is a pseudo-Anosov homeomorphism with stable lamination μ and unstable lamination ν, such that the complementary regions of μ are ideal triangles. Let ϕ be the isotopy class of h in Γ_g. The proof of Proposition 3.6 of [M-M] implies that ϕ^m acts hyperbolically on $C(M)$ for large enough m. In other words, the map $f : \mathbb{Z} \to X$ defined by $f(n) = \phi^{mn}(x)$ is a quasisometry. It follows that the map $\phi^* : \mathbb{Z} \to X$ defined by $f(n) = \phi^n(x)$ is also a quasisometry. Hence, ϕ acts hyperbolically on $C(M)$.

Lemma 3.4. *Let τ be a pseudo-Anosov mapping class in Γ_g. Then τ is an hyperbolic isometry of (T_g, d).*

Proof. We recall that, by definition, τ is represented by an homeomorphism $h : M \to M$ which preserves the projective classes of a pair of transverse measured foliations F_1 and F_2 on M. The pair of measured foliations F_1 and F_2 defines a metric g on M which is locally Euclidean away from the (common) singularities of F_1 and F_2. g determines a Riemmann surface structure S on M. There is a unique quadratic differential q on S such that F_1 is the horizontal measured foliation of q and F_2 is the vertical measured foliation of q. Let x denote the point in Teichmüller space represented by S. The Teichmüller geodesic γ thru x in the direction of q is invariant under τ [B]. Indeed, τ acts on γ by a translation of some positive distance d. Hence, the orbit of x under τ is quasi-isometric to the integers $\mathbb{Z}$. In other words, by definition, τ is an hyperbolic isometry. □

Remark 3.5. The Teichmüller geodesic γ consists of the points $x_t \in T_g$ represented by the Riemann surface structures S_t on M determined by the measured foliations $t^{-1/2}F_1$ and $t^{1/2}F_2$ where $t > 0$. (Note that the transverse measure on the horizontal (resp. vertical) measured foliation determines vertical (resp. horizontal) coordinates.) Note that $x = x_1$. We may assume that F_1 and F_2 are ordered so that $h(F_1) = \lambda^{-1}F_1$ and $h(F_2) = \lambda F_2$, where $\lambda > 1$. Then $\tau \cdot x = x_{\lambda^2}$ and the identity map is the Teichmuller map from S to S_{λ^2}. It follows that the Teichmüller distance from x to $\tau \cdot x$ is equal to $\log(\lambda)$.

Remark 3.6. Pseudo-Anosov elements are hyperbolic isometries of (T_g, d) in the sense of Bers as well as in the sense of Gromov ([B]).

Corollary 3.7. *(Masur-Wolf* [M-W]*) Teichmüller space with the Teichmüller metric is not Gromov hyperbolic.*

Proof. As mentioned above, (T_g, d) is proper and geodesic, and Γ_g acts properly discontinuously by isometries on (T_g, d). The result follows immediately from Theorem 3.2 and Lemma 3.4. □

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