NORMALIZERS AND CENTRALIZERS OF PSEUDO-ANOSOV MAPPING CLASSES

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Let M be an orientable, connected, compact Riemann surface of negative euler characteristic. Let $\mathcal{M}(M)$ be the mapping class group of M, the group of isotopy classes of orientation preserving self homeomorphisms of M. Let τ be a pseudo-Anosov mapping class belonging to $\mathcal{M}(M)$. We recall that τ is *pseudo-Anosov* if it contains a pseudo-Anosov diffeomorphism t.

A diffeomorphism, t, of M is *pseudo-Anosov* if there exists a pair of transverse measured foliations, $(\mathcal{F}^s, \mu^s)$, $(\mathcal{F}^u, \mu^u)$, and a real number $\lambda > 1$ such that $t(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda^{-1}\mu^s)$ and $t(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda\mu^u)$. The measured foliation $(\mathcal{F}^s, \mu^s)$ is called the *stable foliation for* t; the measured foliation $(\mathcal{F}^u, \mu^u)$ is called the *unstable foliation for* t; λ is the *dilatation of* t.

In this article we prove the following theorem and two corollaries:

Theorem 1 . *The centralizer, $C(\tau)$, of the cyclic subgroup of $\mathcal{M}(M)$ generated by τ is a finite extension of an infinite cyclic group. The normalizer, $N(\tau)$, of the cyclic subgroup of $\mathcal{M}(M)$ generated by τ is either equal to $C(\tau)$ or contains $C(\tau)$ as a normal subgroup of index* 2.

Corollary 2 . *$C(\tau)$ and $N(\tau)$ are virtually infinite cyclic (i.e. contain infinite cyclic subgroups of finite index).*

Corollary 3 . *Every torsion free subgroup of $C(\tau)$ or $N(\tau)$ is infinite cyclic.*

The main tool for proving these results is given by the following lemma. I thank Albert Fathi of the Universite de Paris-Sud, Orsay, France for showing me how to prove a special case of this lemma.

Lemma 1 . *Suppose s is a diffeomorphism of M and k is a nonzero integer such that sts^{-1} is isotopic to t^k. Then there exists a homeomorphism, r, of M, isotopic to s, and a positive real number, ρ, such that the following conditions hold:*

(1) $rtr^{-1} = t^k$,
(2) *if $k < 0$, then $r(\mathcal{F}^s, \mu^s) = (\mathcal{F}^u, \rho^{-1}\mu^u)$ and $r(\mathcal{F}^u, \mu^u) = (\mathcal{F}^s, \rho\mu^s)$,*
(3) *if $k > 0$, then $r(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \rho^{-1}\mu^s)$ and $r(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \rho\mu^u)$.*

Furthermore, $k = -1$, or $+1$.

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Proof. Let $t_1 = t^k$ and $t_2 = sts^{-1}$. Let $(\mathcal{F}^s_2, \mu^s_2) = s(\mathcal{F}^s, \mu^s)$ and $(\mathcal{F}^u_2, \mu^u_2) = s(\mathcal{F}^u, \mu^u)$. Then the following equalities hold:

- $t_1(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda^{-k}\mu^s)$ and $t_1(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda^k\mu^u)$,
- $t_2(\mathcal{F}^s_2, \mu^s_2) = (\mathcal{F}^s_2, \lambda^{-1}\mu^s_2)$ and $t_1(\mathcal{F}^u_2, \mu^u_2) = (\mathcal{F}^u_2, \lambda\mu^u_2)$.

Therefore, t_1 and t_2 are isotopic pseudo-Anosov diffeomorphisms. By the uniqueness of pseudo-Anosovs, ([FLP], Theorem III, Expose 12), there exists a diffeomorphism, h, isotopic to the identity, such that $ht_2h^{-1} = t_1$. Therefore, if we let $r = hs$, then r is isotopic to s and $rtr^{-1} = t^k$. This proves (1).

Following the argument in the proof of Lemma 16, Expose 12, [FLP], we conclude that r sends the stable foliation of t to the stable foliation of t^k, and the unstable foliation of t to the unstable foliation of t^k.

If $k < 0$, then $r(\mathcal{F}^s) = \mathcal{F}^u$ and $r(\mathcal{F}^u) = \mathcal{F}^s$. By the unique ergodicity of the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$, ([FLP], Theorem I, Expose 12), it follows that there exists positive real numbers α and β such that $r(\mathcal{F}^s, \mu^s) = (\mathcal{F}^u, \alpha\mu^u)$ and $r(\mathcal{F}^u, \mu^u) = (\mathcal{F}^s, \beta\mu^s)$. Furthermore, we conclude that $\alpha\beta = 1$, since $\mu^s \bigotimes \mu^u$ gives an area element whose total area must be preserved by any diffeomorphism of M. (M has finite area under this form.) This proves (2). (3) follows in a similar manner.

If $k < 0$, then $rtr^{-1}(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda^{-1}\mu^u)$. Since, on the other hand, $t^k(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda^k\mu^u)$, we conclude that $k = -1$. Similarly, if $k > 0$, then $k = 1$. □

From this lemma, we conclude that if $\sigma \in N(\tau)$, then σ may be represented by a diffeomorphism preserving the pair of measured foliations for t up to scalar multiplications. Therefore, we now turn our attention to study the group of such diffeomorphisms.

Let $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2\}$ be the pair of foliations for t. Let $\mathcal{G}$ be the group of diffeomorphisms, r, such that $r(\mathcal{F}) = \mathcal{F}$. Let $\mathcal{G}^*$ be the subgroup of diffeomorphisms, r, such that $r(\mathcal{F}_1) = \mathcal{F}_1$ and $r(\mathcal{F}_2) = \mathcal{F}_2$. Clearly, $\mathcal{G}^*$ is a normal subgroup of index 1 or 2 in $\mathcal{G}$. (There may not be any diffeomorphisms of M exchanging the pair of foliations.)

Let μ_i be a transverse measure on $\mathcal{F}_i$, $i = 1, 2$. Again, by the unique ergodicity of the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, it follows that for each $r \in \mathcal{G}$, there exists a positive real number, λ_r, such that either:

- $r(\mathcal{F}_1, \mu_1) = (\mathcal{F}_2, \lambda_r^{-1}\mu_2)$ and $r(\mathcal{F}_2, \mu_2) = (\mathcal{F}_1, \lambda_r\mu_1)$

or

- $r(\mathcal{F}_1, \mu_1) = (\mathcal{F}_1, \lambda_r^{-1}\mu_1)$ and $r(\mathcal{F}_2, \mu_2) = (\mathcal{F}_2, \lambda_r\mu_2)$.

In particular, this provides a *dilatation homomorphism*, $\lambda : \mathcal{G}^* \to \mathbb{R}_+$. Let $\Lambda = \lambda(\mathcal{G}^*)$ and $\mathcal{S}ym = kernel(\lambda)$. (Note: if $r \in \mathcal{G}$ and $r(\mathcal{F}_1) = \mathcal{F}_2$, then $r^2 \in \mathcal{S}ym$. If $r \in \mathcal{G}^*$, then r is pseudo-Anosov if and only if $\lambda_r \neq 1$.)

Lemma 2 . *There exists $\lambda_0 > 1$ such that $\Lambda = \{\lambda_0^n | n \in \mathbb{Z}\}$.*

Proof. The set of dilatation factors for pseudo-Anosov maps on a surface of fixed genus is a subset of the algebraic integers. Indeed, it is a discrete subset. This fact was pointed out in [T]. A proof may be found in the paper of Arnoux and Yoccoz [AY]. Their arguments also show that this set is closed. Therefore, Λ is a discrete subgroup of algebraic integers. The result follows from standard theorems. □

Lemma 3 . *$\mathcal{S}ym$ is a finite group.*

Proof. Let $\mathcal{L}$ be the collection of separatrices for $\mathcal{F}_1$, (i.e. leaves of $\mathcal{F}_1$ emanating from a singularity of $\mathcal{F}_1$). Since each element in $\mathcal{G}^*$ must permute the leaves of $\mathcal{L}$, we have a natural action of $\mathcal{G}^*$ on $\mathcal{L}$, which restricts to an action of $\mathcal{S}ym$ on $\mathcal{L}$.

Suppose $L \in \mathcal{L}$, $r \in \mathcal{S}ym$ and $r(L) = L$. Since $\lambda_r = 1$, it follows that r fixes L pointwise. Since L is dense in M, ([FLP], Expose 9, Lemma 6), r fixes M pointwise. That is, r is the identity. Therefore, the action of $\mathcal{S}ym$ on $\mathcal{L}$ is free and $\mathcal{S}ym$ is a finite group. □

Lemma 4 . *Let $\pi : Homeo^+(M) \to \mathcal{M}(M)$ be the natural quotient. The restriction $\pi : \mathcal{G} \to \mathcal{M}(M)$ is injective.*

Proof. If $\pi(r) = 1$, then, by definition of π, r is isotopic to the identity. Therefore, r^2 is isotopic to the identity. But r^2 is in $\mathcal{G}^*$. Since pseudo-Anosov diffeomorphisms are not isotopic to the identity, we conclude that $r^2 \in \mathcal{S}ym$. By Lemma 3, r^2 is finite order, and therefore r is finite order. But a periodic map which is isotopic to the identity is the identity, ([FLP], Expose 12, Lemma 12). □

Proof of Theorem 1. Let $\mathcal{H} = \pi(\mathcal{G})$ and $\mathcal{H}^* = \pi(\mathcal{G}^*)$. By Lemmas 2 and 3, $\mathcal{G}^*$ is a finite extension of an infinite cyclic group. As noted before, either $\mathcal{G} = \mathcal{G}^*$ or $\mathcal{G}^*$ is a normal subgroup of index 2 in $\mathcal{G}$. By Lemma 1, $C(\tau) \subset \mathcal{H}^*$, $N(\tau) \subset \mathcal{H}$ and $N(\tau) \cap \mathcal{H}^* = C(\tau)$. The result follows immediately from Lemma 4. □

Proof of Corollary 2. It suffices to show that $\mathcal{G}^*$ is virtually infinite cyclic. But there is a short exact sequence:

$$1 \to \mathcal{S}ym \to \mathcal{G}^* \xrightarrow{\lambda} \Lambda \to 1$$

with Λ infinite cyclic and $\mathcal{S}ym$ finite. Such a sequence always splits, and any splitting determines an infinite cyclic subgroup of finite index in $\mathcal{G}^*$. □

Proof of Corollary 3. Let G be a torsion free subgroup of $N(\tau)$ and $\sigma \in G$. By Lemma 1, if σ switches the pair of foliations for τ, then $\sigma^2 \in \mathcal{S}ym$. By Lemma 3, this is impossible. Hence, by Lemma 1, $G \subset C(\tau)$. But then we have a short exact sequence:

$$1 \to G \cap \mathcal{S}ym \to G \xrightarrow{\lambda} \lambda(G) \to 1; \qquad \lambda(G) \subset \Lambda.$$

Since G is torsion free, $G \cap \mathcal{S}ym = \{1\}$. (Again, this follows from Lemma 3.) Hence, G is isomorphic to a subgroup of Λ. The result follows from Lemma 2. $\square$

References

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