NORMALIZERS AND CENTRALIZERS OF
PSEUDO-ANOSOV MAPPING CLASSES

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Let $M$ be an orientable, connected, compact Riemann surface of negative
euler characteristic. Let $\mathcal{M}(M)$ be the mapping class group of $M$, the group
of isotopy classes of orientation preserving self homeomorphisms of $M$. Let $\tau$
be a pseudo-Anosov mapping class belonging to $\mathcal{M}(M)$. We recall that $\tau$
is pseudo-Anosov if it contains a pseudo-Anosov diffeomorphism $t$.

A diffeomorphism, $t$, of $M$ is pseudo-Anosov if there exists a pair of
transverse measured foliations, $(F^s, \mu^s)$, $(F^u, \mu^u)$, and a real number $\lambda > 1$
such that $t(F^s, \mu^s) = (F^s, \lambda \mu^s)$ and $t(F^u, \mu^u) = (F^u, \lambda \mu^u)$. The measured
foliation $(F^s, \mu^s)$ is called the stable foliation for $t$; the measured foliation
$(F^u, \mu^u)$ is called the unstable foliation for $t$; $\lambda$ is the dilatation of $t$.

In this article we prove the following theorem and two corollaries:

**Theorem 1.** The centralizer, $C(\tau)$, of the cyclic subgroup of $\mathcal{M}(M)$
generated by $\tau$ is a finite extension of an infinite cyclic group. The normalizer,
$N(\tau)$, of the cyclic subgroup of $\mathcal{M}(M)$ generated by $\tau$ is either equal to $C(\tau)$
or contains $C(\tau)$ as a normal subgroup of index 2.

**Corollary 2.** $C(\tau)$ and $N(\tau)$ are virtually infinite cyclic (i.e. contain
infinite cyclic subgroups of finite index).

**Corollary 3.** Every torsion free subgroup of $C(\tau)$ or $N(\tau)$ is infinite cyclic.

The main tool for proving these results is given by the following lemma. I thank Albert Fathi of the Universite de Paris-Sud, Orsay, France for showing me how to prove a special case of this lemma.

**Lemma 1.** Suppose $s$ is a diffeomorphism of $M$ and $k$ is a nonzero integer
such that $sts^{-1}$ is isotopic to $t^k$. Then there exists a homeomorphism, $r$, of $M$, isotopic to $s$, and a positive real number, $\rho$, such that the following conditions hold:

1. $rtr^{-1} = t^k$,
2. if $k < 0$, then $r(F^s, \mu^s) = (F^u, \rho^{-1} \mu^u)$ and $r(F^u, \mu^u) = (F^s, \rho \mu^s)$,
3. if $k > 0$, then $r(F^s, \mu^s) = (F^s, \rho^{-1} \mu^s)$ and $r(F^u, \mu^u) = (F^u, \rho \mu^u)$.

Furthermore, $k = -1$, or $+1$.
Proof. Let $t_1 = t^k$ and $t_2 = st^{-1}$. Let $(\mathcal{F}_2^s, \mu_2^s) = s(\mathcal{F}_2^s, \mu^s)$ and $(\mathcal{F}_2^u, \mu_2^u) = s(\mathcal{F}_2^u, \mu^u)$. Then the following equalities hold:

- $t_1(\mathcal{F}_2^s, \mu_2^s) = (\mathcal{F}_2^s, \lambda^{-k}\mu^s)$ and $t_1(\mathcal{F}_2^u, \mu_2^u) = (\mathcal{F}_2^u, \lambda^k\mu^u)$,
- $t_2(\mathcal{F}_2^s, \mu_2^s) = (\mathcal{F}_2^s, \lambda^{-1}\mu_2^s)$ and $t_1(\mathcal{F}_2^u, \mu_2^u) = (\mathcal{F}_2^u, \lambda\mu_2^u)$.

Therefore, $t_1$ and $t_2$ are isotopic pseudo-Anosov diffeomorphisms. By the uniqueness of pseudo-Anosovs, ([FLP], Theorem III, Expose 12), there exists a diffeomorphism, $h$, isotopic to the identity, such that $ht_2h^{-1} = t_1$. Therefore, if we let $r = hs$, then $r$ is isotopic to $s$ and $rtr^{-1} = t^k$. This proves (1).

Following the argument in the proof of Lemma 16, Expose 12, [FLP], we conclude that $r$ sends the stable foliation of $t$ to the stable foliation of $t^k$, and the unstable foliation of $t$ to the unstable foliation of $t^k$.

If $k < 0$, then $r(\mathcal{F}_2^s) = \mathcal{F}_2^u$ and $r(\mathcal{F}_2^u) = \mathcal{F}_2^s$. By the unique ergodicity of the foliations $\mathcal{F}_2^s$ and $\mathcal{F}_2^u$, ([FLP], Theorem I, Expose 12), it follows that there exists positive real numbers $\alpha$ and $\beta$ such that $r(\mathcal{F}_2^s, \mu^s) = (\mathcal{F}_2^u, \alpha\mu^u)$ and $r(\mathcal{F}_2^u, \mu^u) = (\mathcal{F}_2^s, \beta\mu^s)$. Furthermore, we conclude that $\alpha \beta = 1$, since $\mu^s \otimes \mu^u$ gives an area element whose total area must be preserved by any diffeomorphism of $M$. ($M$ has finite area under this form.) This proves (2). (3) follows in a similar manner.

If $k < 0$, then $rtr^{-1}(\mathcal{F}_2^u, \mu^u) = (\mathcal{F}_2^u, \lambda^{-1}\mu^u)$. Since, on the other hand, $t^k(\mathcal{F}_2^u, \mu^u) = (\mathcal{F}_2^u, \lambda^k\mu^u)$, we conclude that $k = -1$. Similarly, if $k > 0$, then $k = 1$. }

From this lemma, we conclude that if $\sigma \in N(\tau)$, then $\sigma$ may be represented by a diffeomorphism preserving the pair of measured foliations for $t$ up to scalar multiplications. Therefore, we now turn our attention to study the group of such diffeomorphisms.

Let $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2\}$ be the pair of foliations for $t$. Let $\mathcal{G}$ be the group of diffeomorphisms, $r$, such that $r(\mathcal{F}) = \mathcal{F}$. Let $\mathcal{G}^*$ be the subgroup of diffeomorphisms, $r$, such that $r(\mathcal{F}_1) = \mathcal{F}_1$ and $r(\mathcal{F}_2) = \mathcal{F}_2$. Clearly, $\mathcal{G}^*$ is a normal subgroup of index 1 or 2 in $\mathcal{G}$. (There may not be any diffeomorphisms of $M$ exchanging the pair of foliations.)

Let $\mu_i$ be a transverse measure on $\mathcal{F}_i$, $i = 1, 2$. Again, by the unique ergodicity of the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, it follows that for each $r \in \mathcal{G}$, there exists a positive real number, $\lambda_r$, such that either:

- $r(\mathcal{F}_1, \mu_1) = (\mathcal{F}_2, \lambda_1^{-1}\mu_2)$ and $r(\mathcal{F}_2, \mu_2) = (\mathcal{F}_1, \lambda_r\mu_1)$

or

- $r(\mathcal{F}_1, \mu_1) = (\mathcal{F}_1, \lambda_1^{-1}\mu_1)$ and $r(\mathcal{F}_2, \mu_2) = (\mathcal{F}_2, \lambda_r\mu_2)$.

In particular, this provides a dilatation homomorphism, $\lambda : \mathcal{G}^* \to \mathbb{R}_+$. Let $\Lambda = \lambda(\mathcal{G}^*)$ and $\mathcal{Sym} = \text{kernel}(\lambda)$. (Note: if $r \in \mathcal{G}$ and $r(\mathcal{F}_1) = \mathcal{F}_2$, then $r^2 \in \mathcal{Sym}$.) If $r \in \mathcal{G}^*$, then $r$ is pseudo-Anosov if and only if $\lambda_r \neq 1$.)

**Lemma 2.** There exists $\lambda_0 > 1$ such that $\Lambda = \{\lambda_0^n | n \in \mathbb{Z}\}$. 
Proof. The set of dilatation factors for pseudo-Anosov maps on a surface of fixed genus is a subset of the algebraic integers. Indeed, it is a discrete subset. This fact was pointed out in [T]. A proof may be found in the paper of Arnoux and Yoccoz [AY]. Their arguments also show that this set is closed. Therefore, $\Lambda$ is a discrete subgroup of algebraic integers. The result follows from standard theorems. □

Lemma 3. $\text{Sym}$ is a finite group.

Proof. Let $\mathcal{L}$ be the collection of separatrices for $\mathcal{F}_1$, (i.e. leaves of $\mathcal{F}_1$ emanating from a singularity of $\mathcal{F}_1$). Since each element in $\mathcal{G}^*$ must permute the leaves of $\mathcal{L}$, we have a natural action of $\mathcal{G}^*$ on $\mathcal{L}$, which restricts to an action of $\text{Sym}$ on $\mathcal{L}$.

Suppose $L \in \mathcal{L}$, $r \in \text{Sym}$ and $r(L) = L$. Since $\lambda_r = 1$, it follows that $r$ fixes $L$ pointwise. Since $L$ is dense in $M$, ([FLP], Expose 9, Lemma 6), $r$ fixes $M$ pointwise. That is, $r$ is the identity. Therefore, the action of $\text{Sym}$ on $\mathcal{L}$ is free and $\text{Sym}$ is a finite group. □

Lemma 4. Let $\pi : \text{Homeo}^+(M) \to \mathcal{M}(M)$ be the natural quotient. The restriction $\pi : \mathcal{G} \to \mathcal{M}(M)$ is injective.

Proof. If $\pi(r) = 1$, then, by definition of $\pi$, $r$ is isotopic to the identity. Therefore, $r^2$ is isotopic to the identity. But $r^2$ is in $\mathcal{G}^*$. Since pseudo-Anosov diffeomorphisms are not isotopic to the identity, we conclude that $r^2 \in \text{Sym}$. By Lemma 3, $r^2$ is finite order, and therefore $r$ is finite order. But a periodic map which is isotopic to the identity is the identity, ([FLP], Expose 12, Lemma 12). □

Proof of Theorem 1. Let $\mathcal{H} = \pi(\mathcal{G})$ and $\mathcal{H}^* = \pi(\mathcal{G}^*)$. By Lemmas 2 and 3, $\mathcal{G}^*$ is a finite extension of an infinite cyclic group. As noted before, either $\mathcal{G} = \mathcal{G}^*$ or $\mathcal{G}^*$ is a normal subgroup of index 2 in $\mathcal{G}$. By Lemma 1, $C(\tau) \subset \mathcal{H}^*$, $N(\tau) \subset \mathcal{H}$ and $N(\tau) \cap \mathcal{H}^* = C(\tau)$. The result follows immediately from Lemma 4. □

Proof of Corollary 2. It suffices to show that $\mathcal{G}^*$ is virtually infinite cyclic. But there is a short exact sequence:

$$1 \to \text{Sym} \to \mathcal{G}^* \xrightarrow{\lambda} \Lambda \to 1$$

with $\Lambda$ infinite cyclic and $\text{Sym}$ finite. Such a sequence always splits, and any splitting determines an infinite cyclic subgroup of finite index in $\mathcal{G}^*$. □

Proof of Corollary 3. Let $G$ be a torsion free subgroup of $N(\tau)$ and $\sigma \in G$. By Lemma 1, if $\sigma$ switches the pair of foliations for $\tau$, then $\sigma^2 \in \text{Sym}$. By Lemma 3, this is impossible. Hence, by Lemma 1, $G \subset C(\tau)$. But then we have a short exact sequence:

$$1 \to G \cap \text{Sym} \to G \xrightarrow{\lambda} \lambda(G) \to 1; \quad \lambda(G) \subset \Lambda.$$
Since $G$ is torsion free, $G \cap \text{Sym} = \{1\}$. (Again, this follows from Lemma 3.) Hence, $G$ is isomorphic to a subgroup of $\Lambda$. The result follows from Lemma 2.

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\section*{References}


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