ON INJECTIVE HOMOMORPHISMS BETWEEN TEICHMÜLLER MODULAR GROUPS I

NIKOLAI V. IVANOV AND JOHN D. MCCARTHY

Abstract. In this paper and its sequel, we prove that injective homomorphisms between Teichmüller modular groups of compact orientable surfaces are necessarily isomorphisms, if an appropriately measured "size" of the surfaces in question differs by at most one. In particular, we establish the co-Hopfian property for modular groups of surfaces of positive genus.

1. Introduction

Let S be a compact orientable surface. The *Teichmüller modular group* Mod_S of the surface S, also known as the *mapping class group* of S, is the group of isotopy classes of orientation-preserving diffeomorphisms $S \to S$. The *pure modular group* PMod_S is the subgroup of Mod_S consisting of isotopy classes of diffeomorphisms which preserve each component of ∂S. The *extended modular group* Mod_S^* of S is the group of isotopy classes of *all* (including orientation-reversing) diffeomorphisms $S \to S$.

Before turning to the main results of the paper, we would like to point out the following two corollaries.

Theorem 1 . *Let S be a compact connected orientable surface of positive genus. Suppose that S is not a torus with at most two holes. Then* Mod_S *is co-Hopfian, (i.e. every injective homomorphism* $\mathrm{Mod}_S \to \mathrm{Mod}_S$ *is an isomorphism).*

Note that Mod_S is also a Hopfian group, (i.e. every surjective homomorphism $\mathrm{Mod}_S \to \mathrm{Mod}_S$ is an isomorphism). As is well known, a finitely generated group is Hopfian if it is residually finite. The last

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property was proved for modular groups by E. Grossman [G]. See also [I3], Exercise 1.

Theorem 1 provides an affirmative answer to a question communicated by D. D. Long to the first author: "Is every injective homomorphism $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_S$ an isomorphism provided S is a closed surface of genus greater than 1?"

Note that it is usually quite nontrivial to establish the co-Hopfian property for a group of geometric interest. G. Prasad [P] proved that irreducible lattices in linear analytic semisimple groups are co-Hopfian if the dimension of the associated symmetric space is $\neq 2$ (cf. [P], Proposition, p. 242). After the initial results of M. Gromov [GG] (cf. [GG], 5.4.B), E. Rips and Z. Sela [RS] and Z. Sela [S] proved recently the co-Hopfian property for a wide class of hyperbolic groups (cf. [RS], Section 3). So, Theorem 1 turns out to be a new instance of the well-known, but still mysterious, analogy between modular groups and lattices and between Teichmüller spaces and hyperbolic spaces.

Theorem 2 . *Let S be a closed orientable surface of genus at least 2. Then there is no injective homomorphism $\mathrm{Out}(\pi_1(S)) \to \mathrm{Aut}(\pi_1(S))$. In particular, the natural epimorphism $\mathrm{Aut}(\pi_1(S)) \to \mathrm{Out}(\pi_1(S))$ is nonsplit.*

The second statement of Theorem 2 answers a question of J. S. Birman, stated as a part of Problem 8 of S. M. Gersten's list of Selected Problems in [GS]. For genus at least 3, this nonsplitting result follows also from Proposition 2 of a paper of G. Mess's [Me]. (The relation between Mess's paper [Me] and Birman's question apparently went unnoticed.) Birman asked whether the natural homomorphism $\mathrm{Aut}(\pi_1(S)) \to \mathrm{Out}(\pi_1(S))$ splits when S is a closed surface of genus greater than 1. Note that this homomorphism is an isomorphism when S is a sphere or a torus. Birman suggested this question as an algebraic variation of the generalized Nielsen realization problem, which also has an affirmative solution when S is a sphere or a torus. Contrary to the statement in Problem 8 in [GS], it is not equivalent to this realization problem. The realization problem seems to be of a different nature than Birman's question.

Theorems 1 and 2 are deduced from our main results, concerned with injective homomorphisms between Teichmüller modular groups. The relationship between Long's question and injective homomorphisms between modular groups needs no explanation. The relationship between Theorem 2 and injective homomorphisms between modular groups may be summarized as follows. Let S be a closed surface of genus greater than 1 and S' be the surface obtained from S by deleting the interior of a disc containing the basepoint x for the fundamental group $\pi_1(S)$. As is well known, there are natural isomorphisms $\mathrm{Mod}^*_{S'} \to \mathrm{Aut}(\pi_1(S))$ and $\mathrm{Mod}^*_S \to \mathrm{Out}(\pi_1(S))$. Given these identifications, the above answer to Birman's question, as well as the more general result of Theorem 2 concerning the nonexistence of injective homomorphisms $\mathrm{Out}(\pi_1(S)) \to \mathrm{Aut}(\pi_1(S))$, follows from the nonexistence of injective homomorphisms $\mathrm{Mod}_S \to \mathrm{Mod}_{S'}$.

The results of [I2] and [M] provide a complete description of automorphisms of modular groups. Roughly speaking, essentially all automorphisms of modular groups are geometric. Based upon these results on automorphisms, one might expect that essentially all injective homomorphisms between modular groups are geometric. Our main results, stated below as Theorems 3–6, verify this expectation for a large class of pairs (S, S').

Theorem 3 . *Let S and S' be compact connected orientable surfaces. Suppose that the genus of S is at least 2 and S' is not a closed surface of genus 2. Suppose that the maxima of ranks of abelian subgroups of Mod_S and $\mathrm{Mod}_{S'}$ differ by at most one. If $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ is an injective homomorphism, then ρ is induced by a diffeomorphism $H : S \to S'$, (i.e. $\rho([G]) = [HGH^{-1}]$ for every orientation-preserving diffeomorphism $G : S \to S$, where we denote by $[F]$ the isotopy class of a diffeomorphism F). In particular, ρ is an isomorphism.*

As is well known, the maxima of ranks of abelian subgroups of Mod_S is equal to $3\mathbf{g} - 3 + \mathbf{b}$, where $\mathbf{g}$ is the genus and $\mathbf{b}$ is the number of boundary components of S [BLM]. If we strengthen the hypothesis on the maxima of ranks of abelian subgroups in Theorem 3, we can allow S to be of genus one also, with only few exceptions.

Theorem 4 . *Let S and S' be compact connected orientable surfaces. Suppose that S has positive genus, S is not a torus with at most one hole, S' is not a closed surface of genus 2 and (S, S') is not a pair of tori with two holes. If the maxima of ranks of abelian subgroups of Mod_S and $\mathrm{Mod}_{S'}$ are equal and $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ is an injective homomorphism, then ρ is induced by a diffeomorphism $S \to S'$.*

In a previous version of this paper, we had a longer, albeit finite, list of pairs (S, S') which were excluded in the hypotheses of Theorem 4. We wish to thank the referee of that version for showing us how to reduce this list to a pair of tori with two holes.

Similar results are obtained when S' is a closed surface of genus 2. The statements involve the exceptional outer automorphism τ :

$\mathrm{Mod}_{S'} \to \mathrm{Mod}_{S'}$ which maps a Dehn twist about a nonseparating circle on S' to its product with the unique nontrivial element of the center of $\mathrm{Mod}_{S'}$ ([I2], [M]). No restrictions on the maxima of the ranks of abelian subgroups are needed in the following theorem because, in fact, under its assumptions the maxima of ranks are automatically equal ([IM]).

Theorem 5 . *Let S be a compact connected orientable surface of genus at least 2. Let S' be a closed surface of genus 2. Let τ be the exceptional outer automorphism of $\mathrm{Mod}_{S'}$. If $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ is an injective homomorphism, then either ρ or $\tau \circ \rho$ is induced by a diffeomorphism $S \to S'$.*

Theorem 6 . *Let S be a compact connected orientable surface of positive genus. Let S' be a closed surface of genus 2. Let τ be the exceptional outer automorphism of $\mathrm{Mod}_{S'}$. If the maxima of ranks of abelian subgroups of Mod_S and $\mathrm{Mod}_{S'}$ are equal and $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ is an injective homomorphism, then either ρ or $\tau \circ \rho$ is induced by a diffeomorphism $S \to S'$.*

Theorems 3–6 generalize the following results of [I2] and [M].

Corollary 1 . ([I2]) *Let S be a compact connected orientable surface of positive genus. Suppose that S is not a torus with at most two holes or a closed surface of genus 2. Then every automorphism of Mod_S is given by the restriction of an inner automorphism of Mod^*_S. In particular, $\mathrm{Out}(\mathrm{Mod}_S)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.*

Corollary 2 . ([M]) *Let S be a closed surface of genus 2. Let τ be the exceptional outer automorphism of Mod_S. Then every automorphism of Mod_S is given by the restriction of an inner automorphism of Mod^*_S or by the composition of such an automorphism with τ. In particular, $\mathrm{Out}(\mathrm{Mod}_S)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

We shall prove Theorems 2 and 3 in the present paper. This will establish Theorem 1, provided $g \geq 2$ and S is not a closed surface of genus 2. The rest of the results will be proved in the sequel to this paper [IM].

The techniques employed in this paper and its sequel [IM] are geometric in nature. Like those employed in [BLM], [I2] and [M], they are based upon Thurston's theory of surface diffeomorphisms. More precisely, the arguments of this paper play upon restrictions upon commuting elements in Mod_S which follow from Thurston's theory. We say that an injective homomorphism is *twist-preserving* if it sends Dehn twists about nonseparating circles to Dehn twists. The crucial step in the proof of Theorems 3–6, as in the proof of Corollary 1 in [I2], is to show that an injective homomorphism $\mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ is twist-preserving. The last property forces an injective homomorphism to be induced by a diffeomorphism $S \to S'$, provided the genus of S is at least 2, without any additional assumptions on S'. (This crucial step fails when S' is a closed surface of genus 2. However, as in the proof of Corollary 2 in [M], the failure is exactly compensated for by the exceptional outer automorphism $\tau : \mathrm{Mod}_{S'} \to \mathrm{Mod}_{S'}$ [IM].) Since our homomorphisms are only injective, the reduction to twist-preserving homomorphisms does not follow immediately from the algebraic characterization of Dehn twists given in [I2]. We do not know of an algebraic characterization of Dehn twists which would yield an immediate reduction in the present context. Nevertheless, the assumption on the maxima of the ranks of abelian subgroups allows us to complete this crucial step of the argument. At the same time, under this assumption, we are able to deal with the twist-preserving homomorphisms in the case when S has genus 1 also, with few exceptions [IM].

Here is an outline of the paper. In Section 2, we give examples of injective homomorphisms between modular groups which are not induced by diffeomorphisms. This section provides a contrast to our results on injective homomorphisms. The results of this section are not used in the rest of the paper.

In Section 3, we review the basic notions and results related to Teichmüller modular groups. We assume that the reader is familiar with the fundamentals of Thurston's theory of surfaces (cf. [FLP]). Section 4 concerns the relationship between Dehn twists supported on neighborhoods of boundary components of S and Dehn twists supported on nontrivial circles on S. The results of this section allow us to conclude that an injective homomorphism respects the distinction between boundary components and nonseparating circles. Section 5 is devoted to a discussion of centers of modular groups and closely related subgroups. Roughly speaking, the results of this section allow us to control the images of powers of Dehn twists under an injective homomorphism. Section 6 concerns systems of circles on S whose components are topologically equivalent on S. The results of this section allow us to conclude that an injective homomorphism respects the distinction between nonseparating and separating circles. Section 7 is devoted to a technical tool used in Sections 8 and 11: a special configuration of circles on S and its basic properties.

In Section 8, we prove that any injective twist-preserving homomorphism $\mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ is, in fact, induced by some diffeomorphism $S \to S'$ (and, in particular, is an isomorphism). We say that an injective homomorphism is *almost twist-preserving* if it sends some power of a Dehn twist about any nonseparating circle to a power of a Dehn twist. Section 9 is devoted to extending the results of Section 8 to almost twist-preserving injective homomorphisms. It is in this section that we see the assumption on maxima of ranks of abelian subgroups entering into our arguments. (This assumption is especially important in Lemma 9.5.) The main task of the next two sections is to reduce the proof of Theorem 3 to the case of almost twist-preserving homomorphisms. Theorem 3 appears as Theorem 11.7. The last section is devoted to the proof of Theorem 2, which appears as Theorem 12.2.

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2. Nongeometric injective homomorphisms

Our purpose, in this section, is to give examples of injective homomorphisms $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ between modular groups of compact connected orientable surfaces S and S' which are not induced by a diffeomorphism $S \to S'$. The results of this section are not used in the rest of the paper.

A simple construction of examples of nongeometric injective homomorphisms is provided by the classical topological construction of doubling a surface. Let S be a compact connected orientable surface with nonempty boundary and dS be the double of S. There is a well-defined injective *doubling homomorphism* $\delta : \mathrm{Mod}_S \to \mathrm{Mod}_{dS}$ given by the rule $\delta([F]) = [dF]$, where $dF : dS \to dS$ denotes the double of a diffeomorphism $F : S \to S$ and $[H]$ denotes the isotopy class of a diffeomorphism H. Since dS is closed and S is not, S is not diffeomorphic to dS. Hence, the injective homomorphism $\delta : \mathrm{Mod}_S \to \mathrm{Mod}_{dS}$ is not induced by a diffeomorphism $S \to dS$.

A second construction is provided by lifting to characteristic covers. Recall that a cover $X^\sim \to X$ is called *characteristic* if the image of the fundamental group $\pi_1(X^\sim)$ in $\pi_1(X)$ is a characteristic subgroup, (i.e. a subgroup invariant under all automorphisms of $\pi_1(X)$). For this construction, we consider a compact connected orientable surface S of genus $\mathbf{g} \geq 1$ with one boundary component C. Let R be the closed surface of genus $\mathbf{g}$ obtained by attaching a disc D to C. Choose a point p in the interior of D. Let $\pi : R' \to R$ be a characteristic cover of index $n \geq 2$ and let $p' \in \pi^{-1}(p)$. Note that S is naturally embedded in R.

Let $S' = \pi^{-1}(S)$. The covering π restricts to a covering $\pi|S' : S' \to S$. Given any diffeomorphism $F : S \to S$, there exists a unique lift $F' : S' \to S'$ subject to the condition that F' extends to a diffeomorphism $G' : R' \to R'$ such that $G'(p') = p'$. It is easy to see that there is a well-defined injective *lifting homomorphism* $\lambda : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ given by the rule $\lambda([F]) = [F']$. An Euler characteristic argument shows that S is not diffeomorphic to S'. Hence, the injective homomorphism $\lambda : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ is not induced by a diffeomorphism $S \to S'$.

The doubling construction can be generalized by replacing dS with the double of S along a submanifold C of ∂S. Likewise, the lifting construction can be generalized to surfaces S with several boundary components, provided we replace C with a component of ∂S. Both of these generalized constructions yield injective homomorphisms from the stabilizer of C in Mod_S into the mapping class groups of the corresponding surfaces S'. Finally, by choosing appropriate components of the boundary of various surfaces, it is possible to iteratively compose generalized doubling and lifting homomorphisms to obtain other interesting examples of injective homomorphisms $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$. Again, an Euler characteristic argument shows that, for most of these "hybrid" homomorphisms, S is not diffeomorphic to S' and, hence, ρ is not induced by a diffeomorphism $S \to S'$.

3. Preliminaries

In this section, we shall establish notation and discuss background material used throughout the paper.

3.1. **Notations and basic notions.** Let S be a compact connected orientable surface of genus $\mathbf{g}$ with $\mathbf{b}$ boundary components. Let ∂S denote the boundary of S. By $V(S)$ we denote the set of isotopy classes of nontrivial circles on S. $V_0(S)$ denotes the subset of $V(S)$ corresponding to nonseparating circles on S.

A one-dimensional submanifold C of S is called a *system of circles* on S, if the components of S are nontrivial and pairwise nonisotopic. A *multiwist* about C is any composition of powers of Dehn twists about the circles in C.

For every system of circles C on S, we denote by S_C the result of cutting S along C. Any diffeomorphism $F : S \to S$ preserving C determines a diffeomorphism $F_C : S_C \to S_C$. If f denotes the isotopy class of F, f_C denotes the isotopy class of F_C. If Q is a component of S_C and $F_C(Q) = Q$, we denote the restriction of F_C to Q by F_Q. If f denotes the isotopy class of F, f_Q denotes the isotopy class of F_Q. If at least one component of ∂Q corresponds to a component of ∂S, we say that Q is *peripheral*. Otherwise, we say that Q is *interior* to S. If no two components of ∂Q correspond to the same component of C, we say that Q is *embedded* in S.

The *complex of curves* $C(S)$ of S is the simplicial complex on $V(S)$ each of whose simplices correspond to a set of isotopy classes of components of some system of circles on S. If σ is a simplex of $C(S)$ corresponding to a system of circles C on S, we say that C is a *realization* of σ. If t is a multitwist about a realization C of σ, we will also say that t is a *multitwist* about σ.

For every pair of simplices σ and τ of $C(S)$, the *geometric intersection number* $i(\sigma, \tau)$ of σ and τ is the minimum number of points of $C \cap D$ over all realizations C of σ and D of τ. We shall also denote $i(\sigma, \tau)$ by $i(C, D)$. We say that C and D are in *minimal position* if the number of points of intersection of C and D is equal to $i(C, D)$ and C is transverse to D. We say that a configuration (i.e. a set) of circles is in *minimal position* if each pair of circles of the configuration is in minimal position.

3.2. **Pure elements.** A diffeomorphism $F : S \to S$ is called *pure* if there is a system of circles C on S such that (F, C) satisfies the following condition:

Condition P . All points of C and ∂S are fixed by F, F_C preserves each component of S_C and, for each component Q of S_C, F_Q is isotopic either to a pseudo-Anosov diffeomorphism or to the identity.

An element f of Mod_S is called *pure* if there is a pure diffeomorphism $F \in f$. If f is a pure element of Mod_S and $f^n(\alpha) = \alpha$ for a vertex α of $C(S)$ and some $n \neq 0$, then $f(\alpha) = \alpha$ (cf. [I3], Corollary 3.7). By Corollary 1.8 of [I3], there exists a subgroup Γ of finite index in Mod_S consisting entirely of pure elements.

3.3. **Reduction systems.** For each simplex σ of $C(S)$, we denote the stabilizer of σ in Mod_S by $M(\sigma)$. Note that σ is a *reduction system* for f if and only if $f(\sigma) = \sigma$, (i.e. $f \in M(\sigma)$). There is a natural *reduction homomorphism* $r_C : M(\sigma) \to \mathrm{Mod}_{S_C}$ given by the rule $r_C(f) = f_C$. The kernel of r_C is equal to T_C, the group generated by the Dehn twists about the components of C.

Let σ be a simplex of $C(S)$, let C be a realization of σ and $R = S_C$. For a component Q of R, we denote its stabilizer in Mod_R by $\mathrm{Mod}_R(Q)$. There is a canonical *restriction homomorphism* $\pi_Q : \mathrm{Mod}_R(Q) \to \mathrm{Mod}_Q$ given by the rule $\pi_Q(f) = f_Q$. If G is a subgroup of $M(\sigma)$ and Q is a component of S_C, we put $G_Q = \pi_Q(r_C(G) \cap \mathrm{Mod}_R(Q))$.

If $F : S \to S$ is a pure diffeomorphism, f is its isotopy class and C is a system of circles on S such that (F, C) satisfies Condition P, we say that C is a *pure reduction system* for F or f. Let Q be a component of S_C. If F_Q is isotopic to a pseudo-Anosov diffeomorphism of Q, we say that Q is a *pseudo-Anosov component of S_C* with respect to F or f. Otherwise, we say that Q is a *trivial component of S_C* with respect to F or f.

A reduction system σ for $f \in \mathrm{Mod}_S$ is called *pure* if there exists a diffeomorphism $F \in f$ and a realization C of σ such that C is a pure reduction system for F.

A vertex α of $C(S)$ is called an *essential reduction class* for a pure element f if (i) $f(\alpha) = \alpha$; (ii) if $i(\alpha, \beta) \neq 0$, then $f(\beta) \neq \beta$. We call the set of all essential reduction classes for f the *canonical reduction system* for f and denote it by $\sigma(f)$. By definition, the *canonical reduction system* of an arbitrary element f of Mod_S is the canonical reduction system for some pure power of f. This definition depends only upon f, and not upon the power involved (cf. [I3], Section 7.4).

Lemma 3.4. *If f is a pure element, then the canonical reduction system $\sigma(f)$ for f is pure and is contained in any other pure reduction system.*

Proof. Let F be a pure diffeomorphism representing f and C be a pure reduction system for F. After deleting some components from C, we will get a minimal pure reduction system C' for F, (i.e. such a reduction system that we cannot discard any component from C' without violating the Condition P). In the terminology of [I3], this is expressed by saying that C' does not have superfluous components. And, according to [I3], Section 7.19, $\sigma(f)$ is exactly the set of isotopy classes of components of C'. Clearly, this implies both statements of the lemma. $\square$

Corollary 3.5. *If f is a pure element, then $\sigma(f)$ is empty precisely when f is either trivial or pseudo-Anosov.*

Lemma 3.6. *If τ is a reduction system for a pure element f, then $\sigma(f) \cup \tau$ is a pure reduction system for f.*

Proof. It follows from the definition of essential reduction classes that $i(\sigma(f), \tau) = 0$. Hence, $\sigma(f) \cup \tau$ is a simplex of $C(S)$ and is a reduction system for f. In order to see that it is a pure reduction system, let us choose a realization C of $\sigma(f)$ and a diffeomorphism $F \in f$ as in Condition P. We can choose a realization D of $\sigma(f) \cup \tau$ containing C. Clearly, any component of D is contained either in C or in a component

Q of S_C such that F_Q is isotopic to the identity. This implies our assertion. $\square$

Theorem 3.7. *Let f be a pure element of Mod_S. Let σ be a reduction system for f, C be a realization of σ, and $F \in f$ such that $F(C) = C$. Then F leaves each component of $C \cup \partial S$ invariant, preserves their orientations, preserves the orientation of S, and also leaves each component of $S \setminus C$ invariant. In particular, if $f(\sigma) = \sigma$ for some simplex σ, then f fixes all vertices of σ.*

Theorem 3.8. *Let f be a pure element of Mod_S. Then f is either trivial or of infinite order.*

Theorem 3.9. *Let f be a pure element of Mod_S, τ be a reduction system for f and C be a realization of τ. Suppose that Q is a component of S_C. Then $f_C \in \mathrm{Mod}_{S_C}(Q)$ and f_Q is a pure element of Mod_Q.*

Proof. Given Lemmas 3.4 and 3.6, these three theorems are immediate. $\square$

Theorem 3.10. ([I3]) *Let Γ be a subgroup of Mod_S consisting of pure elements. If f is a pseudo-Anosov element of Γ, then its centralizer in Γ is an infinite cyclic group generated by a pseudo-Anosov element.*

Proof. See [I3], Lemma 8.13. $\square$

Theorem 3.11. ([I3]) *Suppose that S has negative Euler characteristic. Let G be a subgroup of Mod_S consisting of pure elements. Then G either contains a free group with two generators or is a free abelian group of rank $\leq 3\mathbf{g} - 3 + \mathbf{b}$, where $\mathbf{g}$ is the genus of S and $\mathbf{b}$ is the number of components of the boundary of S.*

Proof. This is a minor variation on Theorem 8.9 of [I3]. The changes may be summarized as follows: (i) eliminate all appeals to Corollary 1.8 of [I3] by replacing the hypothesis that certain subgroups of Mod_S act trivially on $H_1(S, \mathbb{Z}/m_0\mathbb{Z})$ for some integer $m_0 \geq 3$ with the assumption that the relevant subgroups consist entirely of pure elements, (ii) replace all appeals to Theorem 1.2 of [I3] with appeals to Theorem 3.7, all appeals to Corollary 1.5 of [I3] with appeals to Theorem 3.8, and all appeals to Lemma 1.6 of [I3] with appeals to Theorem 3.9. $\square$

3.12. **Reduction of subgroups.** Let Γ be a subgroup of Mod_S consisting of pure elements. If C is a system of circles on S and σ is the corresponding simplex of $C(S)$, we put $\Gamma(C) = M(\sigma) \cap \Gamma$. If $f \in \Gamma(C)$, then $f_C \in \mathrm{Mod}_{S_C}(Q)$ in view of Theorem 3.9. Now, let G be a subgroup of $M(\sigma)$ consisting entirely of pure elements. Then $G(C) = G$

and, hence, $r_C(G) \subset \mathrm{Mod}_{S_C}(Q)$ for every component Q of S_C. It follows that $G_Q = \pi_Q(r_C(G))$. Furthermore, by Theorem 3.9, G_Q consists entirely of pure elements of Mod_Q, and, by Theorem 3.8, G_Q is torsion free. Obviously, $r_C(G)$ lies in the product of the groups G_Q over all components of S_C. (This product naturally lies in Mod_{S_C}. Indeed, the intersection of the stabilizers $\mathrm{Mod}_{S_C}(Q)$ over all components Q is naturally isomorphic to the product of the groups Mod_Q over all components Q.) In the above setting, the homomorphism $r_C|G : G \to \mathrm{Mod}_{S_C}$ will be the main tool for studying G. Note that its kernel is equal to $T_C \cap G$.

3.13. **Relations between Dehn twists.** For the rest of this section, we assume that S is *oriented*. Our purpose, in this section, is to discuss basic relations between Dehn twists along circles on S. For each circle a on S, we denote by $t_a \in \mathrm{Mod}_S$ the *right* Dehn twist *about* a. We shall also denote t_a by t_α, where α is the isotopy class of a. Thus, t_α^{-1} is the left Dehn twist about a. It is well known that $f t_\alpha f^{-1} = t_{f(\alpha)}$ for any $f \in \mathrm{Mod}_S$, $\alpha \in V(S)$. Also, if $f \in \mathrm{Mod}_S^* \setminus \mathrm{Mod}_S$ (i.e., if f is orientation-reversing), then $f t_\alpha f^{-1} = t_{f(\alpha)}^{-1}$ for any $\alpha \in V(S)$. For each $\alpha \in V(S)$, t_α is a pure, reducible element of infinite order in Mod_S. Moreover, α is the canonical reduction system for t_α and its powers. These facts imply the following well-known result.

Theorem 3.14. *Let t_α, t_β be two right twists. Let j, k be two nonzero integers. Then $t_\alpha^j = t_\beta^k$ if and only if $\alpha = \beta$ and $j = k$.*

Theorem 3.15. *Let t_α, t_β be distinct right twists. Let j, k be two nonzero integers. Then:*
(i) $t_\alpha^j t_\beta^k = t_\beta^k t_\alpha^j$ if and only if $i(\alpha, \beta) = 0$,
(ii) $t_\alpha^j t_\beta^k t_\alpha^j = t_\beta^k t_\alpha^j t_\beta^k$ if and only if $j = k = \pm 1$ and $i(\alpha, \beta) = 1$.

Proof. This theorem generalizes Theorem 3.1 of [I2] and Lemma 4.3 of [M] and is proved by arguments similar to those used in establishing these results. □

Theorem 3.16. *Let a and b be two circles on S intersecting transversely at one point, and let U be a neighborhood of $a \cup b$ diffeomorphic to a torus with one hole. Let c be the boundary circle of U. Then*

$$(t_a t_b)^6 = t_c.$$

Moreover, if T_a, T_b and T_c are twist maps representing t_a, t_b and t_c respectively and supported in U, then $(T_a \circ T_b)^6$ is isotopic to T_c by an isotopy supported in *U.*

Proof. This theorem is well known and is essentially due to Dehn [D]. □

4. Peripheral Twists

In this section, S denotes a compact *oriented* surface. For a circle a on S, we denote by T_a a twist map supported on a neighborhood of a in S. So, T_a represents $t_a \in \mathrm{Mod}_S$. If a is a trivial circle on S, then t_a is the trivial element of Mod_S and, so, T_a represents the trivial element. Nevertheless, twist maps supported on neighborhoods of boundary components of S play an important role in the arguments of this paper. In this section, we develop a relationship between these *peripheral* twists and twists along nontrivial circles on S.

Let $\mathcal{M}_S$ denote the group of orientation-preserving diffeomorphisms $S \to S$ which fix ∂S pointwise modulo isotopies which fix ∂S pointwise. Let a be a nontrivial circle on S or a component of ∂S. Let $\tilde{t}_a$ denote the (isotopy) class of T_a in $\mathcal{M}_S$. Naturally, we call $\tilde{t}_a$ the *Dehn twist about a in $\mathcal{M}_S$*. There is a natural homomorphism $\mathcal{M}_S \to \mathrm{PMod}_S$. This homomorphism is surjective and its kernel is equal to the group $\tilde{T}_{\partial S}$ generated by the Dehn twists $\tilde{t}_a$ about the components a of ∂S. These Dehn twists freely generate $\tilde{T}_{\partial S}$. Thus, $\tilde{T}_{\partial S}$ is a free abelian group of rank $\mathbf{b}$. The natural homomorphism $\mathcal{M}_S \to \mathrm{PMod}_S$ maps $\tilde{t}_a$ to t_a.

Theorem 4.1. *Let S be a compact connected orientable surface. Suppose that $\mathcal{C}$ is a collection of nonseparating circles on S such that PMod_S is generated by the Dehn twists t_c along the circles c of $\mathcal{C}$. Then $\mathcal{M}_S$ is generated by the Dehn twists $\tilde{t}_c$ along the circles c of $\mathcal{C}$ and $\tilde{T}_{\partial S}$. Moreover, $\tilde{T}_{\partial S}$ is a central subgroup of $\mathcal{M}_S$.*

Proof. Let $\tilde{G}$ be the subgroup of $\mathcal{M}_S$ generated by the Dehn twists $\tilde{t}_c$ along the circles c of $\mathcal{C}$. Since the natural homomorphism $\mathcal{M}_S \to \mathrm{PMod}_S$ sends $\tilde{t}_c$ to t_c and has kernel $\tilde{T}_{\partial S}$, $\mathcal{M}_S$ is generated by $\tilde{G}$ and $\tilde{T}_{\partial S}$. Clearly, $\tilde{T}_{\partial S}$ is central in $\mathcal{M}_S$. This completes the proof. □

4.2. **Lantern relation.** Let us recall the well-known "lantern" relation discovered by M. Dehn [D] (cf. [D] §7 g) 1)) and rediscovered and popularized by Johnson [J]. Let S_0 be a sphere with four holes. Label the boundary components of S_0 by $C_0, .., C_3$ and write T_i for a twist map supported on a neighborhood of C_i in S_0. For $1 \leq i < j \leq 3$, let C_{ij} denote a circle encircling C_i and C_j as in Figure 4.1. Let T_{ij} denote a twist map supported on a neighborhood of C_{ij} in S_0. Then $T_0 \circ T_1 \circ T_2 \circ T_3$ is isotopic to $T_{12} \circ T_{13} \circ T_{23}$ by an isotopy which is fixed on ∂S_0.

Suppose that S_0 is embedded in S. The diffeomorphisms T_i and T_{ij} may be extended by the identity to all of S. In this sense, we may regard T_i and T_{ij} as twist maps on S supported on neighborhoods of

C_2 C_{12} C_{23} C_0 C_1 C_3 C_{13}

Figure 4.1

circles C_i and C_{ij}. Let $\tilde{t}_i \in \mathcal{M}_{\mathcal{S}}$ denote the Dehn twist $\tilde{t}_{C_i}$ and $\tilde{t}_{ij} \in \mathcal{M}_{\mathcal{S}}$ denote the Dehn twist $\tilde{t}_{C_{ij}}$. Any isotopy on S_0 which is fixed on ∂S_0 extends by the identity to all of S. Hence, the above discussion provides a relation in $\mathcal{M}_{\mathcal{S}}$:

$$\tilde{t}_0\tilde{t}_1\tilde{t}_2\tilde{t}_3 = \tilde{t}_{12}\tilde{t}_{13}\tilde{t}_{23}. \tag{4.1}$$

Theorem 4.3. *Let S be a compact connected orientable surface of genus $\mathbf{g} \geq 2$. Let $\mathcal{C}$ be a collection of nonseparating circles on S such that* PMod_S *is generated by the Dehn twists t_c along the circles c of $\mathcal{C}$. Then $\mathcal{M}_{\mathcal{S}}$ is generated by the Dehn twists $\tilde{t}_c$ along the circles c of $\mathcal{C}$. Moreover, $\tilde{T}_{\partial S}$ is contained in the commutator subgroup of $\mathcal{M}_{\mathcal{S}}$.*

Proof. Let $\tilde{G}$ be the subgroup of $\mathcal{M}_{\mathcal{S}}$ generated by the Dehn twists $\tilde{t}_c$ along the circles c of $\mathcal{C}$. By Theorem 4.1, $\mathcal{M}_{\mathcal{S}}$ is generated by $\tilde{G}$ and $\tilde{T}_{\partial S}$. It suffices to show that $\tilde{t}_a \in \tilde{G}$ for each component a of ∂S. Note that, since $\tilde{T}_{\partial S}$ is central in $\mathcal{M}_{\mathcal{S}}$, $\tilde{G}$ is a normal subgroup of $\mathcal{M}_{\mathcal{S}}$.

We may assume that ∂S is nonempty. Let a be a component of ∂S. As is well known (cf., for example, [H]), since $\mathbf{g} \geq 2$, we may embed S_0 in S so that: (i) $C_0 = a$; (ii) C_i is nonseparating on S for $1 \leq i \leq 3$; (iii) C_{ij} is nonseparating on S for $1 \leq i < j \leq 3$. Let $\tilde{t}_i$ denote the Dehn twist along C_i in $\mathcal{M}_{\mathcal{S}}$ and $\tilde{t}_{ij}$ denote the Dehn twist along C_{ij} in $\mathcal{M}_{\mathcal{S}}$. Since C_i for $1 \leq i \leq 3$ and all C_{ij} are nonseparating, $\tilde{t}_i$ for $1 \leq i \leq 3$ and all $\tilde{t}_{ij}$ are conjugate in $\mathcal{M}_{\mathcal{S}}$ to $\tilde{t}_1$. Hence, equation (4.1) implies that $\tilde{t}_0$ is equal to 0 in $H_1(\mathcal{M}_S)$ and, hence, $\tilde{t}_0$ is contained in the commutator subgroup of $\mathcal{M}_{\mathcal{S}}$. Since $\tilde{t}_0 = \tilde{t}_a$, this implies that $\tilde{T}_{\partial S}$ is contained in the commutator subgroup of $\mathcal{M}_{\mathcal{S}}$.

Since $\tilde{G}$ and $\tilde{T}_{\partial S}$ generate $\mathcal{M}_{\mathcal{S}}$, we may choose an element $\tilde{g}_1 \in \tilde{G}$ and an element $\tilde{t} \in \tilde{T}_{\partial S}$ such that $\tilde{t}_1 = \tilde{g}_1\tilde{t}$. Since $\tilde{G}$ is a normal subgroup of $\mathcal{M}_{\mathcal{S}}$ and $\tilde{t}$ is a central element of $\mathcal{M}_{\mathcal{S}}$, we conclude that there exist elements $\tilde{g}_i$ and $\tilde{g}_{ij}$ of $\tilde{G}$ such that:

$$\tilde{t}_i = \tilde{g}_i \tilde{t} \text{ for } 1 \leq i \leq 3\,; \;\; \tilde{t}_{ij} = \tilde{g}_{ij} \tilde{t} \tag{4.2}$$

(recall that $\tilde{t}_i$ for $1 \leq i \leq 3$ and all $\tilde{t}_{ij}$ are conjugate to $\tilde{t}_1$). Since $\tilde{t}$ is a central element of $\mathcal{M}_{\mathcal{S}}$, equations (4.1) and (4.2) imply that:

$$\tilde{t}_0 \tilde{g}_1 \tilde{g}_2 \tilde{g}_3 = \tilde{g}_{12} \tilde{g}_{13} \tilde{g}_{23}. \tag{4.3}$$

Since $\tilde{g}_i$ and $\tilde{g}_{ij}$ are elements of $\tilde{G}$, equation (4.3) implies that $\tilde{t}_0 \in \tilde{G}$. Since $\tilde{t}_0 = \tilde{t}_a$, this completes the proof. □

5. Centers of Modular Groups and Centralizers of mapping classes

In this section, S denotes a compact connected orientable surface. Let Γ be a subgroup of finite index in Mod_S consisting entirely of pure elements. The goal of this section is to describe the centers of the centralizers of elements of Γ. The main results are Theorems 5.9 and 5.10. To a large extent, they are contained implicitly in [I2].

For any group G and subset A of G, we denote by $C_G(A)$ the centralizer $\{g \in G : ga = ag \text{ for all } a \in A\}$ of A in G, and by $C(G)$ the center $C_G(G)$ of G. For $f \in G$, we denote by $C_G(f)$ the centralizer $\{g \in G : gf = fg\}$ of f in G. We are interested mainly in subgroups $C(C_G(f))$ consisting of all elements of G which commute with every element of G commuting with f.

Lemma 5.1. *The centralizer $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$ of PMod_S in Mod_S is equal to the kernel of the action of Mod_S on $V(S)$. If S has positive genus, then $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$ is equal to the kernel of the action of Mod_S on $V_0(S)$.*

Lemma 5.2. *The centralizer $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$ of PMod_S in Mod_S is a finite subgroup of Mod_S. It contains the centers $C(\mathrm{Mod}_S)$ and $C(\mathrm{PMod}_S)$ of Mod_S and PMod_S, and is normal in Mod_S.*

Theorem 5.3.

(i) *If S is an annulus, then $C(\mathrm{PMod}_S) = \mathrm{PMod}_S = \{1\}$ and $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S) = C(\mathrm{Mod}_S) = \mathrm{Mod}_S = \mathbb{Z}/2\mathbb{Z}$.*

(ii) *If S is a disc with two holes, then $C(\mathrm{PMod}_S) = C(\mathrm{Mod}_S) = \mathrm{PMod}_S = \{1\}$ and $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S) = \mathrm{Mod}_S$.*

(iii) *If S is a torus with at most one hole or a closed surface of genus 2, then $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S) = C(\mathrm{PMod}_S) = C(\mathrm{Mod}_S) \cong \mathbb{Z}/2\mathbb{Z}$.*

(iv) *If S is a sphere with four holes, then $C(\mathrm{PMod}_S) = C(\mathrm{Mod}_S) = \{1\}$ and $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.*

(v) *If S is a torus with two holes, then $C(\mathrm{PMod}_S) = \{1\}$ and $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S) = C(\mathrm{Mod}_S) \cong \mathbb{Z}/2\mathbb{Z}$.*

(vi) *Otherwise, $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S) = C(\mathrm{PMod}_S) = C(\mathrm{Mod}_S) = \{1\}$.*

The preceding three results do not appear in complete form in the literature, but are well known. Hence, we shall omit their proofs.

Lemma 5.4. *Let $f \in \Gamma$, $\sigma = \sigma(f)$ be the canonical reduction system for f, and C be a realization of σ. Then $C_\Gamma(f) \subset \Gamma(C)$ (cf. 3.12 for the notations).*

Proof. If h commutes with f, then $h(\sigma) = h(\sigma(f)) = \sigma(hfh^{-1}) = \sigma(f) = \sigma$ and, hence, $h \in M(\sigma)$. Hence, $C_\Gamma(f) \subset \Gamma \cap M(\sigma) = \Gamma(C)$. □

Lemma 5.5. *Let C be a system of circles on S and let B be a subgroup of $\Gamma(C)$. If $r_C(B)$ is abelian, then B is also abelian (cf. 3.3, 3.12 for notations).*

Proof. Recall that the kernel T_C of $r_C : M(\sigma) \to \mathrm{Mod}_{S_C}$, where σ is the simplex corresponding to C, is abelian. Hence, the kernel of $r_C|B : B \to \mathrm{Mod}_{S_C}$ is abelian. Since $r_C(B)$ is abelian, this implies that B is solvable. Finally, Theorem 3.11 implies that B is abelian. □

Lemma 5.6. *Let $f, h \in \Gamma$, $\sigma = \sigma(f)$ be the canonical reduction system for f, and C be a realization of σ. Then $h \in C_\Gamma(f)$ if and only if $h \in \Gamma(C)$ and h_Q commutes with f_Q for every component Q of S_C (cf. 3.3, 3.12 for notations).*

Proof. If h commutes with f, then $h \in \Gamma(C)$ by Lemma 5.4. Since h commutes with f, h_Q commutes with f_Q for every component Q of S_C.

Suppose, on the other hand, that $h \in \Gamma(C)$ and h_Q commutes with f_Q for every component Q of S_C. Let B be the subgroup of $\Gamma(C)$ generated by f and h. By the assumption, B_Q is abelian for every component Q of S_C. This implies that $r_C(B)$ is abelian. Now, Lemma 5.5 implies that B is abelian. In particular, h commutes with f. This completes the proof. □

Lemma 5.7. *Let $f \in \Gamma$, $\sigma = \sigma(f)$, and C be a realization of σ. Let Q be a trivial component of S_C with respect to f (cf. 3.3). Then $(C(C_\Gamma(f)))_Q$ is trivial.*

Proof. Let $G = C(C_\Gamma(f))$. If $g \in G$, then $g \in \Gamma$ and, in particular, g is a pure element. Since $f \in C_\Gamma(f)$, g commutes with f and, hence, $g \in \Gamma(C)$ by Lemma 5.4. It follows that σ is a reduction system for g. In particular, g_Q is defined and is a pure element of Mod_Q by Theorem 3.9.

Let $\alpha \in V(Q)$ and let a be a circle on Q in the isotopy class α. We may consider a also as a nontrivial circle on S. Since a is, clearly, disjoint from C, the Dehn twist $t_a \in \mathrm{Mod}_S$ belongs to $M(\sigma)$. Since Γ has finite index in Mod_S, some power $h = t_a^n$, $n \neq 0$, belongs to $\Gamma(C)$. Lemma 5.6 implies that $h \in C_\Gamma(f)$. Since $g \in G = C(C_\Gamma(f))$, g commutes with h. Hence, g_Q commutes with h_Q. The element $h_Q \in \mathrm{Mod}_Q$ is a nontrivial power of the Dehn twist about a on Q. Hence, $\sigma(h_Q) = \{\alpha\}$. Since g_Q commutes with h_Q, it must fix α. This implies that g_Q is in the kernel of the action of Mod_Q on $V(Q)$. Now, Lemmas 5.1 and 5.2 imply that g_Q has finite order. It follows that $g_Q = 1$ (because g_Q is pure). This completes the proof. $\square$

Lemma 5.8. *Let $f \in \Gamma$, $\sigma = \sigma(f)$, and C be a realization of σ. Let Q be a pseudo-Anosov component of S_C with respect to f (cf. 3.3). Then $(C(C_\Gamma(f)))_Q$ is an infinite cyclic group.*

Proof. Let $G = C(C_\Gamma(f))$. By Lemma 5.4, $G \subset \Gamma(C) \subset M(\sigma)$. Hence, $G_Q \subset \mathrm{Mod}_Q$ is defined. By Theorem 3.9, G_Q consists entirely of pure elements. Since Q is a pseudo-Anosov component, f_Q is a pseudo-Anosov element. Since G is abelian, G_Q is abelian. The lemma follows now from Theorem 3.10. $\square$

Theorem 5.9. *Let $f \in \Gamma$, $\sigma = \sigma(f)$ be the canonical reduction system for f, and C be a realization of σ. Let c be the number of components of C and p be the number of pseudo-Anosov components of f_C. Then $C(C_\Gamma(f))$ is a free abelian group of rank $c + p$.*

Proof. The lemma is easily established if S is a sphere with at most two holes or a closed torus. Hence, we may assume that S has negative Euler characteristic.

Let $G = C(C_\Gamma(f))$. Since $G \subset \Gamma$, G consists entirely of pure elements of Mod_S. Since G is abelian, Theorem 3.11 implies that G is a free abelian group of rank bounded above by $3\mathbf{g} - 3 + \mathbf{b}$. It remains to determine the rank of G.

By Lemmas 5.7 and 5.8, the torsion free rank of $r_C(G)$ is bounded above by p. Since the kernel of r_C is a free abelian group of rank c, we conclude that the rank of G is bounded above by $c + p$.

Let $f^{\mathbb{Z}}$ be the cyclic group generated by f. For each component Q of S_C, let us consider the cyclic group $f_Q^{\mathbb{Z}}$ generated by $f_Q \in \mathrm{Mod}_Q$. Let Φ be the product of the groups $f_Q^{\mathbb{Z}}$ over all components Q of S_C. This product naturally lies in Mod_{S_C} (cf. 3.12). Clearly, Φ is a free abelian group of rank p.

Let $\Pi = r_C^{-1}(\Phi)$. Because all elements of Π are obviously pure (C is a pure reduction system for appropriate representatives of all of them), Lemma 5.5 implies that Π is abelian. As we will see in a moment, the restriction $r_C \mid \Pi : \Pi \to \Phi$ is surjective. Given this, the exact sequence $0 \to T_C \to \Pi \to \Phi \to 0$ implies that Π is a free abelian subgroup of rank $p + c$.

In order to show that $\Pi \to \phi$ is surjective, let us choose a diffeomorphism $F \in f$ such that (F, C) satisfies condition P. For each component Q of S_C, let us extend F_Q to a diffeomorphism $\overline{F_Q} : S \to S$ by the identity. If $\overline{f_Q} \in \mathrm{Mod}_S$ is the isotopy class of $\overline{F_Q}$, then $r_C(\overline{f_Q})$ has f_Q as the Q-th coordinate and 1 as all other coordinates (we consider $r_C(\overline{f_Q})$ as an element of the product of groups Mod_R over all components R of S_C (cf. 3.12)). The surjectivity follows.

Let $h \in C_\Gamma(f)$. By Lemma 5.6, $h \in \Gamma(C)$ and h_Q commutes with f_Q for every component Q of S_C. Let B be the subgroup of $\Gamma(C)$ generated by $\Pi \cap \Gamma$ and h. Clearly, B_Q is abelian for every component Q of S_C and, hence, $r_C(B)$ is abelian. Since the kernel of r_C is an abelian group, B is a solvable subgroup of Γ. Now, Theorem 3.11 implies that B is abelian and, in particular, h commutes with all elements of $\Pi \cap \Gamma$. In other words, $\Pi \cap \Gamma \subset G$.

Since Γ is of finite index in Mod_S, the intersection $\Pi \cap \Gamma$ is a free abelian group of the same rank $p + c$ as Π. It follows that the rank of G is bounded not only above, but also below, by $p + c$. This completes the proof. □

Theorem 5.10. *Suppose that S is not a sphere with at most two holes or a closed torus. Let $f \in \Gamma$, $\sigma = \sigma(f)$ be the canonical reduction system for f, and C be a realization of σ. For any component Q of S_C, let $\mathbf{g}_Q$ be its genus and $\mathbf{b}_Q$ be the number of boundary components. Let c be the number of components of C, p be the number of pseudo-Anosov components of S_C with respect to f, and t be the sum of the numbers $3\mathbf{g}_Q - 3 + \mathbf{b}_Q$ over the trivial components Q of S_C. Then any abelian subgroup of $C_\Gamma(f)$ is a free abelian group of rank at most $c + p + t$.*

Proof. Since S is not a sphere with at most two holes or a closed torus, S has negative Euler characteristic. Let A be an abelian subgroup of $C_\Gamma(f)$. Since $A \subset C_\Gamma(f) \subset \Gamma$, the subgroup A consists entirely of pure elements. Hence, by Theorem 3.11, A is a free abelian group of finite rank. By Lemma 5.6, $A \subset \Gamma(C)$. This allows us to consider subgroups $A_Q \subset \Gamma(C)_Q \subset \mathrm{Mod}_Q$ for each component Q of S_C. All subgroups A_Q are abelian and, in view of 3.12, $r_C(A)$ is naturally contained in the product of the groups A_Q over all components Q of S_C. Hence, the rank of $r_C(A)$ is bounded by the sum of the ranks of groups A_Q. Since the kernel of r_C is a free abelian group of rank c (cf. 3.1, 3.3), the rank of A is bounded by $c + a$, where a is the above sum.

Let a_Q be the rank of A_Q. By Theorem 3.11, $a_Q \leq 3\mathbf{g}_Q - 3 + \mathbf{b}_Q$. Clearly, it is sufficient to show that, moreover, $a_Q \leq 1$ if Q is a pseudo-Anosov component. Note that $\Gamma(C)_Q$ consists entirely of pure elements by Theorem 3.9. Since $A \subset C_\Gamma(f)$, the group A_Q is contained in the centralizer of f_Q in $\Gamma(C)_Q$. Hence, Theorem 3.10 implies that $a_Q \leq 1$. This completes the proof. $\square$

6. Systems of Separating Circles

In this section, S denotes a compact connected orientable surface. We call two circles a, b on S *topologically equivalent* if there is a diffeomorphism $F : S \to S$ such that $F(a) = b$. The goal of this section is to show that the maximal or "almost" maximal systems of *separating* circles on S with all components topologically equivalent are, in fact, very special. The main features of such systems of circles are described in Theorems 6.1 and 6.2.

Theorem 6.1. *Let S be a compact connected orientable surface of genus $\mathbf{g}$ with $\mathbf{b}$ boundary components. Let C be a system of topologically equivalent separating circles on S. Suppose that C has $3\mathbf{g} - 4 + \mathbf{b}$ components. Then, for each component a of C, there exists a disc with two holes P_a embedded in S such that ∂P_a consists of a and two components of ∂S. Moreover, S is either a sphere with five, six, seven or eight holes or a torus with two holes.*

Proof. Let $n = 3\mathbf{g} - 4 + \mathbf{b}$. Since $n \geq 1$, $3\mathbf{g} + \mathbf{b} \geq 5$. Thus, S is not a sphere with at most four holes or a torus with at most one hole.

Let R be the surface obtained by cutting S along C. Since each component of C is a separating circle on S, the surface R has exactly $n+1$ components. Moreover, the genus of S is the sum of the genera of the components of R. Hence, there must be a system D of $\mathbf{g}$ nonseparating circles on R. The union $C \cup D$ is a system of circles on S with $n + \mathbf{g}$ components. Hence, $n + \mathbf{g} \leq 3\mathbf{g} - 3 + \mathbf{b}$. Since $3\mathbf{g} - 4 + \mathbf{b} = n$, we conclude that one of the following conditions must hold:

(i) $\mathbf{g} = 1$ and $n = \mathbf{b} - 1$,

(ii) $\mathbf{g} = 0$ and $n = \mathbf{b} - 4$.

For each component Q of R, let C_Q be a maximal system of circles on Q and m_Q be the number of components of C_Q. Let m be the sum of m_Q over all components Q of R. The union of C and the systems of circles C_Q over the components Q of R is a maximal system of circles on S. Hence, $3\mathbf{g} - 3 + \mathbf{b} = n + m$. Since $n = 3\mathbf{g} - 4 + \mathbf{b}$, we conclude that $m = 1$. Thus, every component of R is either a disc with two holes, a sphere with four holes or a torus with one hole. Moreover, there is exactly one component of R which is not a disc with two holes.

Since R has $n+1$ components, at least one component of R is a disc with two holes.

Consider the case $\mathbf{g} = 1$ and $n = \mathbf{b} - 1$ first. Since the genus of S is equal to the sum of the genera of the components of R, exactly one component Q of R has genus one. By the preceding considerations, Q is a torus with one hole. Since S is not a torus with one hole, ∂Q must correspond to a component of C. It follows that there exists a component of C which bounds a torus with one hole embedded in S. Since all components of C are topologically equivalent, it follows that each component a of C bounds a torus Q_a with one hole embedded in S. Note that every nontrivial circle on Q_a is nonseparating. It follows that $\operatorname{int} Q_a$ does not contain any components of C and, hence, Q_a is a component of R. Since ∂Q_a corresponds to a, Q_a and Q_b are distinct components of R for each pair a and b of distinct components of C. It follows that R has at least n components which are tori with one hole. Since R has exactly one component which is not a disc with two holes, $n = 1$ and C consists of a single separating circle a on S. Thus, R has two components, P and Q, which meet along a. Since Q is a torus with one hole and R has exactly one component which is not a disc with two holes, P is a disc with two holes. Hence, S is a torus with two holes. Moreover, P is embedded in S in such a way that ∂P consists of a and the two components of ∂S.

Let us consider now the case $\mathbf{g} = 0$ and $n = \mathbf{b} - 4$. Let a be a component of C. The circle a separates S into two spheres with holes P_a and Q_a. Let $p+1$ and $q+1$ be the number of boundary components of P_a and Q_a respectively. Since a is nontrivial, $p, q \geq 2$. We may assume that $p \leq q$. Since the components of C are topologically equivalent circles on S, the pair of integers p, q does not depend upon the component a of C. Moreover, $p + q = b$.

Suppose that $n = 1$. Since $n = b - 4$, $\mathbf{b}$ is equal to 5. Hence, S is a sphere with five holes. The system C consists of a single separating circle a. Since $p, q \geq 2$ and $p \leq q$, we have $p = 2$ and $q = 3$. Hence, P_a is a disc with two holes embedded in S such that ∂P_a consists of a and two components of ∂S.

Suppose now that $n \geq 2$. Let a and b be distinct components of C. If b is contained in P_a, then either P_b or Q_b is contained in P_a. Since both P_b and Q_b contain at least as many boundary components of S as P_a (namely, p or $q \geq p$), this implies that b is isotopic to a. Contradiction with the fact that C is a system of circles implies that b is contained in Q_a. A similar argument implies that P_b is contained in Q_a and $q > p$ (if $q = p$, then b is isotopic to a again). It follows that P_a and P_b are disjoint for every pair of distinct components a, b of C.

Hence, R is the union of a component Q_0 and the components P_a over the components a of C. Each component of ∂Q_0 is either a component of C or a component of ∂S. Moreover, Q_0 and P_a meet along a for every component a of C. Thus, no two components of ∂Q_0 correspond to the same component a of C. Hence, Q_0 is embedded in S. Since exactly one component of R is not a disc with two holes and $n \geq 2$, P_a is a disc with two holes for each component a of C. Since the genus of S is zero, Q_0 is not a torus with one hole. Hence, Q_0 is either a disc with two holes or a sphere with four holes.

Thus, $2 \leq n \leq 4$. Since $n = b - 4$, S is a sphere with six, seven or eight holes. This completes the proof. $\square$

Theorem 6.2. *Let S be a compact connected orientable surface of genus $\mathbf{g}$ with $\mathbf{b}$ boundary components. Let C be a system of topologically equivalent separating circles on S. Suppose that C has $3\mathbf{g} - 3 + \mathbf{b}$ components. Then, for each component a of C, there exists a disc with two holes P_a embedded in S such that ∂P_a consists of a and two components of ∂S. Moreover, S is a sphere with four, five or six holes.*

Proof. Let $n = 3\mathbf{g} - 3 + \mathbf{b}$. Since $n \geq 1$, S is not a sphere with at most three holes or a torus with at most one hole.

Let R be the surface obtained by cutting S along C. Since each component of C is a separating circle on S, R has exactly $n+1$ components. Moreover, the genus of S is the sum of the genera of the components of R. Hence, there must be a system D of $\mathbf{g}$ nonseparating circles on R. The union $C \cup D$ is a system of circles on S with $n + \mathbf{g}$ components. Hence, $n + \mathbf{g} \leq 3\mathbf{g} - 3 + \mathbf{b}$. Since $n = 3\mathbf{g} - 3 + \mathbf{b}$, we conclude that $\mathbf{g} = 0$ and $n = \mathbf{b} - 3$.

Since $n = 3\mathbf{g} - 3 + \mathbf{b}$, C is a maximal system of circles on S. Hence, each component of R is a disc with two holes.

Let a be a component of C. The circle a separates S into two spheres with holes P_a and Q_a. Let $p+1$ and $q+1$ be the number of boundary components of P_a and Q_a respectively. Since a is nontrivial, $p, q \geq 2$. We may assume that $p \leq q$. Since the components of C are topologically equivalent circles on S, the pair of integers p, q does not depend upon the component a of C. Moreover, $p + q = b$.

Suppose that $n = 1$. Since $n = b - 3$, $\mathbf{b}$ is equal to 4. In this case, S is a sphere with four holes and C consists of a single separating circle a. Both P_a and Q_a are discs with two holes embedded in S such that their boundaries consist of a and two components of ∂S.

Suppose that $n \geq 2$. Exactly the same argument as in the proof of Theorem 6.1 now implies that P_a and P_b are disjoint for every pair of distinct components a, b of C. Hence, R is the union of a component Q_0 and the components P_a over the components a of C. Each component of ∂Q_0 is either a component of C or a component of ∂S. Moreover, Q_0 and P_a meet along a for every component a of C. Thus, no two components of ∂Q_0 correspond to the same component a of C. Hence, Q_0 is embedded in S. Since each component of R is a disc with two holes, P_a is a disc with two holes for each component a of C and Q_0 is a disc with two holes.

Hence, $2 \leq n \leq 3$. Since $n = b - 3$, S is a sphere with five or six holes. This completes the proof. $\square$

7. A configuration of circles

In this section, we introduce a special configuration of circles, which will play an important role in Sections 8 and 11. The most important property of this configuration is the fact that Dehn twists about the circles (of a subconfiguration) of this configuration generate the pure modular group PMod_S (cf. Theorem 7.3).

7.1. **The configuration $\mathcal{C}$.** Let S be a compact orientable surface of genus $g \geq 2$. Let $\mathbf{g}$ be the genus of S and $\mathbf{b}$ the number of boundary components. We are interested in the configuration of circles $\mathcal{C}$ presented in Figure 7.1. The configuration $\mathcal{C}$ is in minimal position, distinct circles in $\mathcal{C}$ are not isotopic on S, and the intersection number $i(a, b)$ is 0 or 1 for each pair of circles a, b in $\mathcal{C}$. There are $2g$ circles $a_1, a_2, \ldots, a_{2g}$ in $\mathcal{C}$ which form a chain in the sense that $i(a_i, a_{i+1}) = 1$ for $1 \leq i \leq 2\mathbf{g} - 1$ and $i(a_j, a_k) = 0$ if $|j - k| > 1$. For any circle a_{2i} with $i \geq 2$, there are two circles ¿b_{2i} and c_{2i} in $\mathcal{C}$ having intersection number 1 with a_{2i} and not belonging to the above chain. Moreover, if $b \geq 1$, for the last circle a_{2g} of the chain, there are an additional $\mathbf{b} - 1$ circles $d_1, d_2, \ldots, d_{b-1}$ in $\mathcal{C}$ having the intersection number 1 with a_{2g}. All unmentioned intersection numbers are 0.

Note that if $\mathbf{b} = 1$, then there are no circles d_i; and if $\mathbf{b} = 0$, we replace the circles b_{2g} and c_{2g} by a circle a_{2g+1} having intersection number 1 with a_{2g}.

The even-numbered circles a_{2i} of the above chain are called the *dual* circles of $\mathcal{C}$. If we remove these circles from $\mathcal{C}$, we obtain a maximal system of circles C.

Clearly, all components of S_C are discs with two holes. Moreover, all components of S_C are embedded in S. For each component P of S_C, the complement $S \setminus P$ is connected. Moreover, either ∂P consists of three components of C or ∂P consists of two components of C and one component of ∂S. In the first case, ∂P is a system of circles on

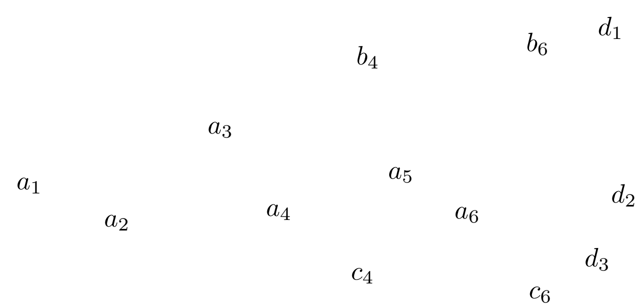

Figure 7.1

S. Recall that we call P an *interior* component in the first case and a *peripheral* component in the second (cf. 3.1).

Two distinct components of a system of circles C are said to be *adjacent* if they correspond to boundary components of a common component of S_C.

Lemma 7.2. *Let S be a compact orientable surface of genus $g \geq 2$. Let S' be some other compact orientable surface. If S is a closed surface of genus 2, let us assume that S' is also a closed surface of genus 2. Let $x \mapsto x'$ be an injective map from the set of circles of the configuration $\mathcal{C}$ to the set of nontrivial circles on S'. Suppose that the configuration of circles $\mathcal{C}'$ formed by these circles x' is in minimal position, distinct circles in $\mathcal{C}'$ are not isotopic on S', and:*

(i) *$i(x,y) = i(x',y')$ for all x, y in $\mathcal{C}$;*

(ii) *if three distinct circles x, y, z of $\mathcal{C}$ bound a disc with two holes in S, then the circles x', y', z' bound a disc with two holes in S'.*

Then there exists an embedding $H : S \to S'$ such that the image $H(S)$ contains all circles x'.

Suppose that, in addition, S' is diffeomorphic to S and:

(iii) *if x, y are adjacent circles of the system of circles C, then x', y' are adjacent circles of the corresponding system of circles C'.*

Then there exists a diffeomorphism $H : S \to S'$ such that $H(x) = x'$ for any circle x in $\mathcal{C}$.

Proof. We shall prove the result when S is not a closed surface of genus 2. The result may be established when S is a closed surface of genus 2 by arguments completely similar to those given here.

We begin by proving a slightly strengthened form of the first assertion. Namely, we prove that there exists a permutation $\sigma : \{1, 2, \ldots, b-1\} \to \{1, 2, \ldots, b-1\}$ and an embedding $H : S \to S'$ such that:

(a) $H(x) = x'$ if x is a circle of C and $x \neq d_i$, $1 \leq i \leq b-1$;

(b) $H(d_i) = d'_{\sigma(i)}$ for all $i = 1, 2, \ldots, b-1$;

(c) the image $H(S)$ contains the circles a'_{2i}, $1 \leq i \leq g$.

Let us orient S and S'. If P is an interior component of S_C and x, y, z are the circles of C corresponding to the components of ∂P, then, by (ii), x', y', z' bound a disc with two holes in S'. For each interior component P choose such a disc with two holes P'. Note that P' is an interior component of $S'_{C'}$. In this way, we establish a correspondence $P \mapsto P'$ between the interior components of S_C and certain interior components of $S'_{C'}$. The correspondence $P \mapsto P'$ is clearly 1–1, since different interior components of S_C have different sets of boundary circles.

For each interior component P of S_C, let us choose an orientation-preserving diffeomorphism $P \to P'$ respecting the correspondence $x \mapsto x'$ (i.e. such that if a component of ∂P corresponds to x, then its image corresponds to x'). Since these diffeomorphisms $P \to P'$ are all orientation-preserving, it is easy to see that, up to an isotopy of each such diffeomorphism, we can glue these diffeomorphisms $P \to P'$ into a diffeomorphism $H_0 : S_0 \to S'_0$, where S_0 (respectively S'_0) is the result of gluing of all interior components P (respectively of the corresponding components P'). If ∂S is nonempty, then S_0 is a surface of genus $\mathbf{g} - 1$ bounded by b_{2g} and c_{2g}, and S'_0 is a surface of genus $\mathbf{g} - 1$ bounded by b'_{2g} and c'_{2g}. If S is closed, then $S = S_0$ and $S' = S'_0$. In particular, if S is closed, we can take the diffeomorphism H_0 as the embedding H we are looking for now.

Our next step is to extend the diffeomorphism $H_0 : S_0 \to S'_0$ to the (image in S of the) peripheral components. We may assume that S is not closed. Note that a_{2g} meets the complement of the interior of S_0 in an arc I_{2g} beginning at b_{2g} and ending at c_{2g}. The circles $b_{2g}, d_1, d_2, \dots, d_{b-1}, c_{2g}$ meet a_{2g} in I_{2g}, in the indicated order, each exactly at one point. Note that (i) implies that a'_{2g} meets the complement of the interior of S'_0 in an arc I'_{2g} beginning at b'_{2g} and ending at c'_{2g}. Note that, by the construction, the components of $S'_0 \cap C'$ lying in the interior of S'_0 form a maximal system of circles on S'_0. Hence, (i) implies that the circles $b'_1, d'_1, d'_2, \dots, d'_{b-1}, c'_{2g}$ meet a'_{2g} in I'_{2g}, each exactly at one point, though not necessarily in the indicated order. Note that S is a regular neighborhood of the union $S_0 \cup I_{2g} \cup d_1 \cup d_2 \cup \ldots \cup d_{b-1}$. It follows that we may extend H_0 to an embedding $H : S \to S'$ so that $H(I_{2g}, d_1, ..., d_{b-1}) = (I'_{2g}, d'_{\sigma(1)}, d'_{\sigma(2)}, \dots, d'_{\sigma(b-1)})$, for some permutation $\sigma : \{1, 2, \dots, b-1\} \to \{1, 2, \dots, b-1\}$. Compare this situation with the discussion in [I2], Lemma 5.1, or [M], Lemma 4.9. Clearly, H has the properties (a) and (b). Also, it follows from (i) that the circles a'_{2i}, $1 \le i \le \mathbf{g} - 1$ are contained in $S'_0 = H(S_0) \subset H(S)$. Since $a'_{2g} \subset S'_0 \cup I'_{2g} = H(S_0 \cup I_{2g}) \subset H(S)$, property (c) follows. This completes the proof of our strengthened form of the first assertion.

Suppose now that S' is diffeomorphic to S. If S is closed, then the embedding $H : S \to S'$ is a diffeomorphism. Suppose, on the other hand, that S is not closed. The construction of H, in this case, allows us to assume that the image $H(S)$ is contained in the interior of S'. Under this assumption, each of the components of ∂S must correspond, via H, to a separating circle on S'. Otherwise, the genus of S' would be greater than that of S. It follows that $\partial H(S) = H(\partial S)$ divides S' into $b+1$ submanifolds, $H(S)$ and b surfaces $A'_1, A'_2, \dots, A'_b$, where A_i intersects $H(S)$ exactly at one common boundary component of $H(S)$ and A_i. Since the genus of S is equal to the genus of S', each A_i is a disc with holes. Note that none of the surfaces A_i is a disc. Otherwise, the image under H of two consecutive circles from the sequence of circles $b_{2g}, d_1, d_2, \dots, d_{b-1}, c_{2g}$ would be isotopic in S'. It follows that each A_i is an annulus. Otherwise, the Euler characteristic of S' would be less than that of S. It follows that $H(S)$ is obtained from S' by deleting all collars A_i. Since the configuration of circles C is contained in the interior of S, we may modify H, without changing it on C, to a diffeomorphism $H : S \to S'$. Therefore, we may assume that the embedding $H : S \to S'$ constructed above is a diffeomorphism.

We now prove that the diffeomorphism $H : S \to S'$ has the following property:

(a-b) $H(x) = x'$ for all circles x of the system of circles C.

Since H satisfies property (a), it suffices to establish property (a-b) for each of the circles d_i. Since $H(d_i) = d_{\sigma(i)}$ for some permutation

$\sigma : \{1, 2, \dots, b-1\} \to \{1, 2, \dots, b-1\}$, we may assume that $b > 2$. By construction, $H : S \to S'$ is a diffeomorphism such that $H(C) = C'$. It follows that x and y are adjacent circles of C if and only if $H(x)$ and $H(y)$ are adjacent circles of C'. Since $b > 1$, the circles adjacent to b_{2g} are exactly a_{2g-1} and d_1. Hence, the circles adjacent to $b'_{2g} = H(b_{2g})$ are exactly $a'_{2g-1} = H(a_{2g-1})$ and $d'_{\sigma(1)} = H(d_1)$. On the other hand, by (iii), a'_{2g-1} and d'_1 are adjacent to b'_{2g}. It follows that $H(d_1) = d'_1$. Likewise, by considering the circles adjacent to d_1, we may show that $H(d_2) = d'_2$. Continuing in this manner, we may prove, by induction on i, that $H(d_i) = d'_i$ for $1 \leq i \leq b-1$. This completes the proof of property (a-b).

It remains to consider the dual circles $a_2, a_4, \dots, a_{2g}$. If x is one of them, then $H(x)$ and x' intersect the same circles of C' and all these intersections are transverse and one-point. Since $S'_{C'}$ is a union of discs with two holes, the well-known Dehn–Thurston classification of multi-circles implies that, up to an isotopy preserving C', the collections of circles $\{H(x) : x \text{ is a dual circle }\}$ and $\{x' : x \text{ is a dual circle }\}$ differ by a composition of twist maps about components of C'. Hence, by composing H with such a composition and then applying some isotopy, we get a new diffeomorphism H such that $H(x) = x'$ for each circle x in $\mathcal{C}$. This proves the second assertion. □

Theorem 7.3. PMod_S *is generated by the Dehn twists along the circles of the configuration* $\mathcal{C}$.

Proof. This result may be proved by the techniques outlined in [I1], Section 5. As a short cut, one can use the results of Lickorish [L] directly (together with the well-known exact sequences relating the mapping class groups of a surface with the mapping class group of the same surface with a hole made in it). □

8. Twist-preserving homomorphisms

In this section, S and S' denote compact connected *oriented* surfaces and $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ or $\mathrm{PMod}_S \to \mathrm{Mod}_{S'}$ is an injective homomorphism. We assume that the genus g of S is at least 2.

We say that ρ is *twist-preserving* if $\rho(t_\alpha)$ is a right Dehn twist about a nontrivial circle on S' for each $\alpha \in V_0(S)$. In other words, ρ is twist-preserving if for each $\alpha \in V_0(S)$, there exists an isotopy class $\rho(\alpha) \in V(S')$ such that $\rho(t_\alpha) = t_{\rho(\alpha)}$. By Theorem 3.14, $\rho(\alpha)$ is uniquely determined by the identity $\rho(t_\alpha) = t_{\rho(\alpha)}$.

This section is devoted to injective twist-preserving homomorphisms $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ or $\mathrm{PMod}_S \to \mathrm{Mod}_{S'}$. We shall prove that such a

a d b e c

Figure 8.1

N

Figure 8.2

homomorphism is, in fact, induced by a diffeomorphism $S \to S'$. This is proved in Theorem 8.9, the main result of this section. The proof will require a lot of preliminary work, done in Lemmas and Corollaries 8.1–8.8.

For the remainder of this section, we assume that ρ is twist-preserving. Let a be a nonseparating circle on S. For a nonseparating circle a on S, we will denote by $\rho(a)$ some representative of the isotopy class $\rho(\alpha)$, where α is the isotopy class of a. Then $\rho(a)$ is well defined up to an isotopy on S' and $\rho(t_a) = t_{\rho(a)}$.

Lemma 8.1. *$\rho(\alpha) = \rho(\beta)$ if and only if $\alpha = \beta$.*

Proof. This follows from Theorem 3.14. □

Lemma 8.2. *Let a and b be distinct nonseparating circles on S. Then:*
(i) *$i(\rho(a), \rho(b)) = 0$ if and only if $i(a, b) = 0$;*
(ii) *$i(\rho(a), \rho(b)) = 1$ if and only if $i(a, b) = 1$.*

Proof. This follows from Theorem 3.15. □

Corollary 8.3. *$\rho(a)$ is nonseparating for every nonseparating circle a on S.*

Proof. A circle a is nonseparating on S if and only if there exists a circle b on S such that $i(a, b) = 1$. □

Corollary 8.4. *Let C be a system of nonseparating circles on S and σ be the corresponding simplex of $C(S)$. Let $\rho(\sigma) = \{\rho(\alpha) : \alpha \in \sigma\}$. Then $\rho(\sigma)$ is a simplex of $C(S')$.*

Lemma 8.5. *Let P be a disc with two holes embedded in S such that $S \setminus P$ is connected and ∂P is a system of circles on S. Let a, b and c be the three boundary components of P. In view of Corollary 8.4, $\{\rho(a), \rho(b), \rho(c)\}$ is a simplex and, hence, we may assume that the circles $\rho(a)$, $\rho(b)$ and $\rho(c)$ are disjoint. Then $\rho(a)$, $\rho(b)$ and $\rho(c)$ bound a disc with two holes P' embedded in S' such that $S' \setminus P'$ is connected and $\partial P'$ is a system of circles on S'.*

If, in addition, S is a closed surface of genus 2, then S' is also a closed surface of genus 2 and, hence, S' is a union of P' and another disc with two holes Q' meeting P' along their common boundary.

Proof. Since S and $S \setminus P$ are connected, the topological type of $S \setminus \operatorname{int} P$ is determined by the topological type of S (and the fact that P is a disc with two holes). It follows that, up to a diffeomorphism, the pair (S, P) is determined by S. By looking at one such pair in Figure 8.1,

$T_d(a)$

$T_b(T_d(a))$ $T_b(T_d(a))$

$F(a)$

Figure 8.3

$G(a)$

$F(a)$

Figure 8.4

$G(a)$

$F(a)$

Figure 8.5

we see that we may choose a pair of nontrivial circles d and e on S such that

$$i(a,d) = i(d,b) = i(b,e) = i(e,c) = 1$$

and

$$i(d,e) = i(d,c) = i(e,a) = 0.$$

We assume that the circles a, b, c, d and e are in minimal position. It is clear that all the circles a, b, c, d and e are nonseparating (since any one of them intersects some other transversely at one point).

By Lemma 8.2, the circles $\rho(a)$, $\rho(b)$, $\rho(c)$, $\rho(d)$ and $\rho(e)$ have the same pairwise geometric intersection numbers as a, b, c, d and e. Clearly, we may assume that $\rho(a)$, $\rho(b)$, $\rho(c)$, $\rho(d)$ and $\rho(e)$ are in minimal position.

Let N (respectively N') be a neighborhood of the union $a \cup b \cup c \cup d \cup e$ (respectively $\rho(a) \cup \rho(b) \cup \rho(c) \cup \rho(d) \cup \rho(e)$) diffeomorphic to a genus two surface with two holes and containing this union as a deformation retract (cf. Figure 8.2).

As is well known, our assumptions on the intersection numbers and the fact that our circles are in minimal position imply that there exists a diffeomorphism $H : N \to N'$ such that $H(x) = \rho(x)$ for each $x = a$, b, c, d or e. Compare [I2], Lemma 5.1, or [M], Lemma 4.9. (In both [I2] and [M], only *maximal chains* are considered, but the proofs work with trivial changes for all *chains*. The sequence of circles a, d, b, e, c is a particular example of a chain; the reader can figure out the general definition without any trouble.)

If $F : S \to S$ is a diffeomorphism with support in N, then we will denote by F^H the diffeomorphism $S' \to S'$ equal to $H \circ F|N \circ H^{-1}$ on N' and equal to the identity outside N'. For example, if T_x is a twist map about a circle in N (always assumed to have support in N), then T_x^H is a twist map about the circle $H(x)$. In particular, if $x = a$, b, c, d or e, then T_x^H is a twist map about $\rho(a)$, $\rho(b)$, $\rho(c)$, $\rho(d)$ or $\rho(e)$ respectively. We denote it also by $T_{H(x)}$.

Let $F = T_c \circ T_e \circ T_b \circ T_d$ and $G = T_c^{-1} \circ T_e^{-1} \circ T_b^{-1} \circ T_d^{-1}$, and let us consider the isotopy classes of the circles $F(a)$ and $G(a)$ in N and in S. A circle isotopic to $F(a)$ in N is found in Figure 8.3. This circle together with a circle isotopic to $G(a)$ is shown on Figure 8.4. It is clear that the circles in Figure 8.4 are in minimal position in N and, hence, have the intersection number 2 there. On the other hand, these circles are isotopic on S to the disjoint circles shown in Figure 8.5, and, hence, have the intersection number 0 on S. These properties are crucial for our proof.

Let f and g be the isotopy classes of F and G respectively. Clearly, $f = t_c t_e t_b t_d$ and $g = t_c^{-1} t_e^{-1} t_b^{-1} t_d^{-1}$. Consider now $\varphi = f t_a f^{-1}$ and $\psi = g t_a g^{-1}$. These two elements are Dehn twists about $F(a)$ and $G(a)$ respectively. On the other hand, φ and ψ are products of several Dehn twists about the circles a, b, c, d and e. This implies that $\rho(\varphi)$ and $\rho(\psi)$ are similar products of Dehn twists about the circles $\rho(a)$, $\rho(b)$, $\rho(c)$, $\rho(d)$ and $\rho(e)$. This implies, in turn, that we can take $(F \circ T_a \circ F^{-1})^H$ and $(G \circ T_a \circ G^{-1})^H$ as representatives of $\rho(\varphi)$ and $\rho(\psi)$ respectively. Since $F \circ T_a \circ F^{-1}$ (respectively $G \circ T_a \circ G^{-1}$) is a twist map about $F(a)$ (respectively $G(a)$), it follows that $\rho(\varphi)$ (respectively $\rho(\psi)$) is a Dehn twist about $H(F(a))$ (respectively $H(G(a))$).

Clearly, the circles $H(F(a))$ and $H(G(a))$ have the same intersection number in N' as the circles $F(a)$ and $G(a)$ have in N. If none of the two boundary components of N' bounds a disc in S', these circles have the same intersection number in S' also. Hence, in this case the intersection number of $H(F(a))$ and $H(G(a))$ is 2 and Dehn twists $\rho(\varphi)$ and $\rho(\psi)$ about these circles do not commute (cf. Theorem 3.15). On the other hand, the elements φ and ψ are Dehn twists about the circles $F(a)$ and $G(a)$, and, since these circles have the intersection number 0 in S, these two elements φ and ψ commute in Mod_S. Since ρ is a homomorphism, this implies that $\rho(\varphi)$ and $\rho(\psi)$ commute. The contradiction we reached means that at least one of the boundary components of N' bounds a disc in S'. Clearly, this implies the first assertion of the lemma.

Let us prove the second assertion. So, we assume now that S is a closed surface of genus 2. Let U (respectively V) be a neighborhood of $a \cup d$ (respectively $c \cup e$) in S diffeomorphic to a torus with one hole and containing this union as a deformation retract. We may assume that U and V are disjoint. Let u, v be the boundary circles of U, V respectively. By Theorem 3.16, $(t_a t_d)^6 = t_u$ and $(t_e t_c)^6 = t_v$. On the other hand, u is isotopic to v on S because S is a closed surface of genus 2 (cf. Figure 8.6). It follows that $t_u = t_v$ and $(t_a t_d)^6 = (t_e t_c)^6$. Hence, $(\rho(t_a)\rho(t_d))^6 = (\rho(t_e)\rho(t_c))^6$.

Note that $(\rho(t_a)\rho(t_d))^6$ is represented by $(T_a^H \circ T_d^H)^6 = (T_{H(a)} \circ T_{H(d)})^6$. Applying Theorem 3.16 to S', we see that $(T_{H(a)} \circ T_{H(d)})^6$ is isotopic to a twist map $T_{H(u)}$ about $H(u)$. Hence, $(\rho(t_a)\rho(t_d))^6 = t_{H(u)}$. Similarly, $(\rho(t_e)\rho(t_c))^6 = t_{H(v)}$. Since $(\rho(t_a)\rho(t_d))^6 = (\rho(t_e)\rho(t_c))^6$, we conclude that $t_{H(u)} = t_{H(v)}$. By Theorem 3.14, $H(u)$ is isotopic to $H(v)$. Since $H(u)$ and $H(v)$ are disjoint (because U and V are), they bound an annulus. The union of $H(U)$, $H(V)$ and this annulus is a closed surface of genus 2 contained in S'. Clearly, it has to be equal to the surface S' itself. Now, the fact that S' is a closed surface of genus 2 and P'

is a disc with two holes implies that $Q' = S' \setminus \operatorname{int} P'$ is also a disc with two holes. This completes the proof of the second assertion of the lemma. □

Lemma 8.6. *Let C be a system of nonseparating circles on S and σ be the corresponding simplex of $C(S)$. Let $\rho(\sigma)$ be the corresponding simplex of $C(S')$ as in Corollary 8.4 and let $\rho(C)$ be a realization of $\rho(\sigma)$. If a and b are adjacent components of C, then $\rho(a)$ and $\rho(b)$ are adjacent components of $\rho(C)$.*

Proof. Recall (cf. 3.1) that a and b are adjacent if there exists a component Q of S_C such that a and b both correspond to components of ∂Q. In this case, there exists a nontrivial circle d on S such that $i(d,a), i(d,b) \neq 0$ and $i(d,c) = 0$ for every component $c \neq a, b$ (cf. Figure 8.7).

Let us show that we can always replace d by a nonseparating circle with the same properties. Suppose that d is separating. Let d' be the image of a under a twist map about d. Clearly, $i(d', c) = 0$ for every component c of C different from a, b. Since d' is the image of a under a diffeomorphism of S and a is nonseparating, d' is nonseparating. By a special case of Proposition 1 from [FLP], Exposé [4], Appendice, $i(d', a) = i(d, a)^2$ and $i(d', b) = i(d, a)i(d, b)$. Hence, d' is the desired circle.

So, we may assume that d is nonseparating. Then $\rho(d)$ is defined. By Lemma 8.2, $i(\rho(d), \rho(a)), i(\rho(d), \rho(b)) \neq 0$ and $i(\rho(d), \rho(c)) = 0$ for every component c of C different from a, b. Hence, $\rho(d)$ is isotopic to a circle whose intersection with each of $\rho(a)$ and $\rho(b)$ is nonempty, but whose intersection with $\rho(c)$ is empty for each component c of C different from a, b. The existence of such a circle implies that $\rho(a)$ and $\rho(b)$ are adjacent components of C'. This completes the proof. □

Lemma 8.7. *Let $\mathcal{C}$ be the configuration of circles on S introduced in 7.1. Then there exists an embedding $H : S \to S'$ such that (i) $H(a) = \rho(a)$ for every circle a of the configuration $\mathcal{C}$ and (ii) for every component c of ∂S, $H(c)$ is either a component of $\partial S'$ or a nontrivial circle on S'.*

Proof. Without any loss of generality, we may restrict our attention to the case when ρ is defined on PMod_S.

We may assume that the configuration of circles $\rho(a)$, where a runs over the circles of $\mathcal{C}$, is in minimal position. We would like to apply Lemma 7.2 to the correspondence $a \mapsto \rho(a)$ in the role of $x \mapsto x'$. Note that this correspondence is injective by Lemma 8.1, satisfies the

U V

Figure 8.6

d a

a d

Figure 8.7

condition (i) from Lemma 7.2 by Lemma 8.2 and satisfies the condition (ii) from Lemma 7.2 by Lemma 8.5. Lemma 8.5 also ensures that S' is a closed surface of genus 2 if S is. Hence, the first part of Lemma 7.2 applies and there exists an embedding $H : S \to S'$ such that the image $H(S)$ contains all circles $\rho(a)$ for a in $\mathcal{C}$.

Since $g \geq 2$, the system of circles C introduced in 7.1 contains at least three circles. By Lemmas 8.1 and 8.2, $\rho(C)$ is a system of circles on S' with at least 3 circles. It follows that S' is not a sphere with at most five holes or a torus with at most two holes.

Let us prove that, for every component c of ∂S, its image $H(c)$ does not bound a disc in S'. Suppose that, to the contrary, some of these images $H(c)$ bound discs in S'. Let R' be the result of adding these discs to $H(S)$. Note that, because S' is not a closed torus, R' is not a closed torus also. Clearly, R' has the same genus as S and (strictly) less boundary components than S. Hence, the maximal number of circles in a system of circles on R' is less than the corresponding number for S (cf. 3.1; the fact that R' is not a closed torus is important here). It follows that some of the circles $\rho(a)$, where a runs over the components of C, are isotopic in R' and, hence, in S'. But this contradicts Lemma 8.1. The contradiction proves our assertion.

It may happen that images $H(a)$ and $H(b)$ of two boundary components are isotopic. Then they are both nontrivial in S' and bound an annulus in S'. Let us choose one circle from each such pair and then add to them all nontrivial circles of the form $H(c)$, where c is a boundary component, which are not isotopic to other circles of this form. The result is a system of circles on S', which we denote by D. Let δ be the corresponding simplex of $C(S')$. Let S'' be the component of S'_D containing $H(S)$. Obviously, S'' is diffeomorphic to S (because S'' can be obtained from $H(S)$ by gluing several annuli along some boundary components).

By Theorem 7.3, PMod_S is generated by Dehn twists t_a about circles a in $\mathcal{C}$. Hence, $\operatorname{Im}\rho$ is generated by Dehn twists $t_{\rho(a)} = \rho(t_a)$. All these Dehn twists and, hence, all elements of the image $\operatorname{Im}\rho$ can be represented by diffeomorphisms supported in $H(S)$ and, in particular, fixing D. It follows that $\operatorname{Im}\rho \subset M(\delta)$ and the composition $r_D \circ \rho : \mathrm{PMod}_S \to \mathrm{Mod}_{S'_D}$ is defined. Moreover, $\operatorname{Im} r_D \circ \rho \subset \mathrm{Mod}_{S'_D}(S'')$ and the composition $\pi_{S''} \circ r_D \circ \rho$ is defined (cf. 3.3 for notations). In addition, any element of $\operatorname{Im} r_D \circ \rho$ can be represented by a diffeomorphism equal to the identity on $S'_D \setminus S''$. It follows that $\operatorname{Ker} \pi_{S''} \circ r_D \circ \rho = \operatorname{Ker} r_D \circ \rho$.

We claim that this kernel is, in fact, trivial. Recall (cf. 3.3) that the kernel of r_D is the free abelian group generated by Dehn twists about components of D. It follows that $\operatorname{Ker} r_D \cap \operatorname{Im} \rho$ is a free abelian group contained in the center of $\operatorname{Im} \rho$. But this center is finite in view of Lemma 5.2. Hence, the intersection $\operatorname{Ker} r_D \cap \operatorname{Im} \rho$ is free abelian and finite at the same time, so it has to be trivial. Hence, $\operatorname{Ker} r_D \cap \operatorname{Im} \rho = \{1\}$ and $\operatorname{Ker} r_D \circ \rho = \{1\}$. It follows that $\operatorname{Ker} \pi_{S''} \circ r_D \circ \rho = \{1\}$ and, hence, the homomorphism $\rho'' = \pi_{S''} \circ r_D \circ \rho$ is injective.

Now, we can apply previous results to $\rho'' : \mathrm{PMod}_S \to \mathrm{Mod}_{S''}$ in the role of ρ (clearly, ρ'' is twist-preserving). Since S'' is diffeomorphic to S and the condition (iii) of Lemma 7.2 is fulfilled by Lemma 8.6, now we are in the position to apply the second part of Lemma 7.2. It gives us an embedding $H_0 : S \to S''$ such that $H_0(a) = \rho(a)$ for all a in $\mathcal{C}$. After deforming, if necessary, H_0 a little, we may assume that $H_0(S) \subset \operatorname{int} S''$. Then the composition H of H_0 with the canonical map $S'' \to S'$ is also an embedding and if the deformation is small enough, then $H(a) = \rho(a)$ for all a in $\mathcal{C}$. Applying to H the results already proved, we see that the image $H(c)$ of any boundary component c does not bound a disc in S'. This means that if $H(c)$ is a trivial circle for a boundary component c, then $H(c)$ is parallel to a component of $\partial S'$. By deforming H in a neighborhood of such boundary components c, we now can fulfill the second condition of the lemma. This completes the proof. □

Lemma 8.8. *If $H : S \to S'$ is a diffeomorphism such that $\rho(a) = H(a)$ for all circles a in $\mathcal{C}$, then ρ is induced by H.*

Proof. Recall that the isomorphism $H_* : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ induced by H is defined by the formula $H_*[G] = [HGH^{-1}]$, where G is an orientation-preserving diffeomorphism $S \to S$ and, as before, $[F]$ denotes the isotopy class of a diffeomorphism F. Obviously, $H_*(\mathrm{PMod}_S) = \mathrm{PMod}_{S'}$.

By Theorem 7.3, the Dehn twists t_a over the circles a of $\mathcal{C}$ form a set of generators of PMod_S. Since $\rho(t_a) = t_{\rho(a)} = t_{H(a)} = H_*(t_a)$ for all such a, this implies that ρ agrees with H_* on PMod_S. In particular, this proves the lemma if ρ is defined on PMod_S.

It remains to consider the case where ρ is defined on Mod_S. Let $\sigma = H_*^{-1} \circ \rho : \mathrm{Mod}_S \to \mathrm{Mod}_S$. Then σ is equal to the identity on PMod_S. Recall that if $f \in \mathrm{Mod}_S$ and $\alpha \in V(S)$, then $f t_\alpha f^{-1} = t_{f(\alpha)}$. By applying σ to this equality, we get $\sigma(f)\sigma(t_\alpha)\sigma(f)^{-1} = \sigma(t_{f(\alpha)})$. Since $t_\alpha, t_{f(\alpha)} \in \mathrm{PMod}_S$, this implies that $\sigma(f) t_\alpha \sigma(f)^{-1} = t_{f(\alpha)}$ and, consequently, $t_{\sigma(f)(\alpha)} = t_{f(\alpha)}$. In view of Theorem 3.14, this in turn implies that $\sigma(f)(\alpha) = f(\alpha)$ or $\sigma(f)^{-1} f(\alpha) = \alpha$ for all $\alpha \in V(S)$. Now, it follows from Lemma 5.1 that $\sigma(f)^{-1} f \in C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$. Since $g \geq 2$, $C_{\mathrm{Mod}_S}(\mathrm{PMod}_S)$ is trivial by Theorem 5.3. Hence, $\sigma(f)^{-1} f = 1$

for all f and $\sigma = \mathrm{id}$. It follows that $\rho = H_*$. This completes the proof. $\square$

Theorem 8.9. *Suppose that S is a surface of genus $\mathbf{g} \geq 2$. If $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ is an injective twist-preserving homomorphism, then ρ is induced by a diffeomorphism $S \to S'$.*

Proof. It is especially simple (after all previous work) for closed surfaces. If S is closed, then the embedding H provided by Lemma 8.7 is a diffeomorphism. Hence, Lemma 8.8 implies that ρ is induced by H. Note that we need only the trivial part of (the proof of) Lemma 8.8 here, because $\mathrm{PMod}_S = \mathrm{Mod}_S$ in this case.

Let us consider now the general case. We will use the notations of Section 4.2. Let $H : S \to S'$ be the embedding provided by Lemma 8.7. If $F : S \to S$ is a diffeomorphism fixed on ∂S, then we can define a diffeomorphism $S' \to S'$ by extending the diffeomorphism $H \circ F \circ H^{-1} : H(S) \to H(S)$ by the identity to the whole surface S'. By passing to the isotopy classes, we get the (well-known) homomorphism $\mathcal{M}_S \to \mathcal{M}_{S'}$ induced by H. We will denote it by H_*. Let us consider the following diagram.

$$\begin{array}{ccc} \mathcal{M}_S & \xrightarrow{H_*} & \mathcal{M}_{S'} \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p'} \\ \mathrm{PMod}_S & \xrightarrow{\rho} & \mathrm{Mod}_{S'} \end{array}$$

The vertical maps are the canonical homomorphisms $p : \mathcal{M}_S \to \mathrm{PMod}_S$, $p' : \mathcal{M}_{S'} \to \mathrm{Mod}_{S'}$. According to Lemma 8.7, $\rho \circ p(\tilde{t}_a) = \rho(t_a) = t_{H(a)}$ for all a in $\mathcal{C}$. Also, clearly, $p' \circ H_*(\tilde{t}_a) = p'(\tilde{t}_{H(a)}) = t_{H(a)}$ for a in $\mathcal{C}$. It follows that $\rho \circ p$ and $p' \circ H_*$ agree on the set $\{\tilde{t}_a : a \in \mathcal{C}\}$. But, in view of Theorems 7.3 and 4.3, this set generates $\mathcal{M}_S$. (This reference to Theorem 4.3 is the only place in the proof where the assumption $\mathbf{g} \geq 2$ is used.) Hence, our diagram is commutative.

Now, if c is a boundary component of S such that $H(c)$ is a nontrivial circle on S', then $p' \circ H_*(\tilde{t}_c) = p'(\tilde{t}_{H(c)}) = t_{H(c)} \neq 1$. On the other hand, $\rho \circ p(\tilde{t}_c) = \rho(1) = 1$. The contradiction shows that $H(c)$ cannot be a nontrivial circle for a boundary component c. In view of Lemma 8.7, this means that $H(\partial S) \subset \partial S'$. It follows that H is a diffeomorphism. An application of Lemma 8.8 completes the proof. $\square$

9. Almost twist-preserving homomorphisms

As in Section 8, S and S' denote compact connected *oriented* surfaces. We assume that the genus g of S is at least 2 and S' is not a closed surface of genus 2. Let $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ be an injective homomorphism. We say that ρ is *almost twist-preserving* if, for each isotopy class $\alpha \in V_0(S)$, there exists an isotopy class $\rho(\alpha) \in V(S')$ and nonzero integers M and N such that $\rho(t_\alpha^M) = t_{\rho(\alpha)}^N$. The goal of this section is to prove that, with few exceptions, injective almost twist-preserving homomorphisms are actually induced by diffeomorphisms $S \to S'$ if the maxima of ranks of abelian subgroups of Mod_S and $\mathrm{Mod}_{S'}$ differ by at most one. See Theorem 9.6 for an exact statement.

For the remainder of this section, we assume that ρ is almost twist-preserving. By Theorem 3.14, the isotopy class $\rho(\alpha) \in V(S')$ is uniquely determined by the equality $\rho(t_\alpha^M) = t_{\rho(\alpha)}^N$, independently of M and N. Since Dehn twists about nonseparating circles are conjugate in Mod_S, the integers M and N may be chosen independently of $\alpha \in V_0(S)$. For each nonseparating circle a on S, let $\rho(a)$ be a realization of $\rho(\alpha)$, where α is the isotopy class of a. Then $\rho(a)$ is well defined up to an isotopy on S' and $\rho(t_a^M) = t_{\rho(a)}^N$. Note that, clearly, $\rho(a)$ is the canonical reduction system of $t_{\rho(a)}^N$ and, since $t_{\rho(a)}^N = \rho(t_a^M) = \rho(t_a)^M$, it is also the canonical reduction system of $\rho(t_a)$.

For the remainder of this section, we will denote by $\mathbf{g}$, $\mathbf{b}$ (respectively $\mathbf{g}'$, $\mathbf{b}'$) the genus and the number of boundary components of S (respectively S').

Lemma 9.1. (i) *$\rho(\alpha) = \rho(\beta)$ if and only if $\alpha = \beta$.*

(ii) *Let C be a system of nonseparating circles on S and σ be the corresponding simplex of $C(S)$. Then $\rho(\sigma) = \{\rho(\alpha) : \alpha \in \sigma\}$ is a simplex of $C(S')$.*

Proof. (i) The "if" clause is trivial. Since ρ is almost twist-preserving, $\rho(\alpha) = \rho(\beta)$ implies $\rho(t_\alpha^M) = \rho(t_\beta^M)$. Since ρ is injective, this, in turn, implies that $t_\alpha^M = t_\beta^M$. Hence, by Theorem 3.14, $\alpha = \beta$.

(ii) Let $\alpha, \beta \in \sigma$. Then t_α and t_β commute. This implies that $\rho(t_\alpha^M)$ and $\rho(t_\beta^M)$ commute or, what is the same, $t_{\rho(\alpha)}^N$ and $t_{\rho(\beta)}^N$ commute. By Theorem 3.15, this implies that $i(\rho(\alpha), \rho(\beta)) = 0$. The assertion (ii) follows. □

Lemma 9.2. *Suppose that there exists an injective homomorphism $\mathrm{Mod}_S \to \mathrm{Mod}_{S'}$. Then S' is not a sphere with at most five holes or a torus with at most two holes.*

Proof. Since, by our assumptions, the genus g of S is at least two, the maxima of ranks of abelian subgroups of Mod_S is equal to $3\mathbf{g} - 3 + \mathbf{b}$ (cf. 3.1). Again, since $g \geq 2$, $3\mathbf{g} - 3 + \mathbf{b} \geq 3$. On the other hand, if S' is a sphere with at most five holes or a torus with at most two holes, then the maxima of ranks of abelian subgroups of $\mathrm{Mod}_{S'}$ is at most 2. $\square$

Lemma 9.3. *Suppose that the maxima of ranks of abelian subgroups of* Mod_S *and* $\mathrm{Mod}_{S'}$ *differ by at most one (and that* $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ *is an injective almost twist-preserving homomorphism). Then* S' *is not a sphere with six, seven or eight holes.*

Proof. Let C be a maximal system of nonseparating circles on S and σ be the corresponding simplex of $C(S)$. Let $\rho(C)$ be a realization of the simplex $\rho(\sigma) = \{\rho(\alpha) : \alpha \in \sigma\}$ (cf. Lemma 9.1 (ii)).

Since the genus g of S is at least 2, C consists of $3\mathbf{g}-3+\mathbf{b}$ components and $3\mathbf{g}-3+\mathbf{b}$ is also the maxima of ranks of abelian subgroups of Mod_S. By Lemma 9.1 (i), $\rho(C)$ also consists of $3\mathbf{g}-3+\mathbf{b}$ components. Since Dehn twists along nonseparating circles are all conjugate in Mod_S, all elements $t_\alpha^M, \alpha \in \sigma$ are conjugate in Mod_S and, hence, all elements $t_{\rho(\alpha)}^N = \rho(t_\alpha^M), \alpha \in \sigma$ are conjugate in $\mathrm{Mod}_{S'}$. Now, Theorem 3.14 and the fact that $f t_{\rho(\alpha)}^N f^{-1} = t_{f(\rho(\alpha))}^N$ for any $f \in \mathrm{Mod}_{S'}$ imply that all components of $\rho(C)$ are topologically equivalent on S' in the sense of Section 6. Since S' is a sphere with holes in all our cases (i)–(iii), all components of $\rho(C)$ are separating. This fact, together with the assumption on the maxima of ranks of abelian subgroups, will allow us to apply the results of Section 6. After these preliminary remarks, we now proceed with the proof.

Suppose that S' is a sphere with six holes. Then the maxima of ranks of abelian subgroups of $\mathrm{Mod}_{S'}$ is equal to 3. By our assumption, $3\mathbf{g}-3+\mathbf{b} \leq 3 \leq 3\mathbf{g}-3+\mathbf{b}+1$ and, hence, $3\mathbf{g}+\mathbf{b} \leq 6 \leq 3\mathbf{g}+\mathbf{b}+1$. Since $\mathbf{g} \geq 2$, this implies that $\mathbf{g} = 2$ and $\mathbf{b} = 0$, (i.e. that S is a closed surface of genus 2). In this case, C and $\rho(C)$ consist of 3 components. Note that $3 = 3\mathbf{g}' - 3 + \mathbf{b}'$ because S' is a sphere with 6 holes. As we had seen, all components of $\rho(C)$ are topologically equivalent. Hence, we may apply Theorem 6.2 to S' and $\rho(C)$ in the role of S and C respectively. We conclude that, for each component a' of $\rho(C)$, there exists a disc with two holes P' embedded in S' such that $\partial P'$ consists of a' and two components of $\partial S'$. Clearly, these discs with two holes are disjoint and the closure of their complement in S' is another disc with holes embedded in S', which we will denote Q'. Obviously, $\partial Q' = \rho(C)$.

Let us consider now the hyperelliptic involution $i \in \mathrm{Mod}_S$ and its image $i' = \rho(i) \in \mathrm{Mod}_{S'}$. Together with i, the image i' is a non-trivial element of finite order (actually of order two). By a well-known theorem of Nielsen (cf. for example, [FLP], Exp. 11, §4), it can be realized by a non-trivial periodic diffeomorphism $F' : S' \to S'$. Moreover, we may assume that F' is an isometry of a hyperbolic metric on S' with geodesic boundary. In addition, we may assume that the components of $\rho(C)$ are geodesics with respect to this metric. Now, it is well known that i is the (unique) non-trivial element of the center of Mod_S. In particular, i commutes with the Dehn twists along the components of C. It follows that i' commutes with the N-th powers of the Dehn twists along the components of $\rho(C)$. Now, Theorem 3.14 and the relation $i' t_{\alpha'}^N (i')^{-1} = t_{i'(\alpha')}^N$ (where $\alpha' \in V(S')$) imply that i' preserves the isotopy classes of the components of $\rho(C)$. Hence, F' preserves the components of $\rho(C)$ themselves (we chose them to be the unique geodesic representatives of their isotopy classes). This, clearly, implies that F' preserves the disc with two holes Q'. The diffeomorphism $Q' \to Q'$ induced by F' preserves orientation and preserves each component of the boundary. Hence, it is isotopic to the identity (cf., for example, [FLP], Exp. 2, §III). Being an isometry, it is actually the identity. Hence, F' is equal to the identity on Q'. Because F' is an isometry, this implies that F' is equal to the identity on the whole surface S' and, hence, $i' = 1$. This contradicts the injectivity of ρ. Hence, S' cannot be a sphere with six holes.

Suppose that S' is a sphere with seven holes. As before, we conclude that $\mathbf{g} = 2$ and $\mathbf{b} = 0, 1$.

Suppose first that $\mathbf{g} = 2$ and $\mathbf{b} = 1$. In this case, C and $\rho(C)$ consist of 4 components. Note that $4 = 3\mathbf{g'} - 3 + \mathbf{b'}$ because S' is a sphere with 7 holes. As we have seen, all components of $\rho(C)$ are topologically equivalent, and, hence, we may apply Theorem 6.2 exactly as in the proof of (i). We conclude that, for each component a' of $\rho(C)$, there exists a disc with two holes P' embedded in S' such that $\partial P'$ consists of a' and two components of $\partial S'$. Clearly, these discs with two holes are disjoint. Each of them contributes two components to the boundary $\partial S'$. This implies that $\partial S'$ has at least 8 components. Contradiction with the assumption that S' is a sphere with 7 holes completes our consideration of the $\mathbf{g} = 2, \mathbf{b} = 1$ case.

Assume now that $\mathbf{g} = 2, \mathbf{b} = 0$, (i.e. that S is a closed surface of genus 2). In this case, C and $\rho(C)$ consist of 3 components. Note that $3 = 3\mathbf{g'} - 4 + \mathbf{b'}$ because S' is a sphere with 7 holes. Since all components of $\rho(C)$ are topologically equivalent, this means that we may apply Theorem 6.1 in this case. Again, we conclude that, for each component a' of $\rho(C)$, there exists a disc with two holes P' embedded in S' such that $\partial P'$ consists of a' and two components of $\partial S'$. Clearly, these discs with two holes are disjoint and the closure of their complement in S' is a sphere with four holes embedded in S', which we denote Q'. One component of the boundary $\partial Q'$ is a part of

$\partial S'$ and the other components of $\partial Q'$ are components of $\rho(C)$. Arguing exactly as in the proof of (i), we can realize the image $i' = \rho(i)$ of the hyperelliptic involution i by an isometry $F' : S' \to S'$ of a hyperbolic metric on S' with geodesic boundary such that F' preserves all the components of $\rho(C)$ (we assume that they are geodesic). Such an F' obviously preserves Q', and, preserving three of the four components of the boundary $\partial Q'$, it preserves them all. Since F' is orientation-preserving, it follows that F' acts trivially on the first homology group of Q' (with any coefficients). This implies that F' is equal to the identity on Q'. (Note that F' is periodic and use, for example, [I3], Theorem 1.3.) It follows that F' is equal to the identity on the whole surface S' and, hence, $i' = 1$. As in the proof of (i), this contradicts the injectivity of ρ. Hence, S' cannot be sphere with seven holes.

Finally, suppose that S' is a sphere with 8 holes. In this case, we conclude that $\mathbf{g} = 2$ and $\mathbf{b} = 1, 2$.

Suppose first that $\mathbf{g} = 2$ and $\mathbf{b} = 2$. In this case, C and $\rho(C)$ consist of 5 components. Note that $5 = 3\mathbf{g}' - 3 + \mathbf{b}'$ because S' is a sphere with 8 holes. Using Theorem 6.2 exactly as in the case $\mathbf{g} = 2, \mathbf{b} = 1$ of the proof of (ii), we conclude that $\partial S'$ has at least 10 components. The obvious contradiction completes our consideration of the $\mathbf{g} = 2, \mathbf{b} = 2$ case.

Suppose now that $\mathbf{g} = 2$ and $\mathbf{b} = 1$, (i.e. S is a surface of genus two with one boundary component). In this case, C and $\rho(C)$ consist of 4 components and $4 = 3\mathbf{g}' - 4 + \mathbf{b}'$. So, Theorem 6.1 applies. Hence, for each component a of C, there exists a disc with two holes P'_a embedded in S' such that $\partial P'_a$ consists of $\rho(a)$ and two components of $\partial S'$. These discs with two holes are disjoint and the closure of their complement in S' is a sphere with four holes, which we denote by Q'.

We may assume that C consists of the circles a_1, a_3, b_4, c_4 presented in Figure 7.1. Clearly, there exists a circle e such that $i(a_1, e) = i(b_4, e) = i(c_4, e) = 1$ and $i(a_3, e) = 0$. Let a be any of the circles a_1, b_4, c_4. By our assumptions, $\rho(t_a)^M = \rho(t_a^M) = t_{\rho(a)}^N$ is a power of a Dehn twist along $\rho(a)$. We will prove now that the element $\rho(t_a)$ itself is a power of a Dehn twist along $\rho(a)$.

If b is a component of C, then t_a commutes with t_b and, hence, $\rho(t_a)$ commutes with $\rho(t_b^M) = t_{\rho(b)}^N$. By the usual argument (compare the proof of the fact that i' preserves the isotopy classes of components of $\rho(C)$ in the proof of (i)), this implies that $\rho(t_a)$ preserves the isotopy classes of all components of $\rho(C)$. Hence, we can represent $\rho(t_a)$ by a diffeomorphism $H' : S' \to S'$ preserving all components of $\rho(C)$. Let R' be the result of cutting S' along $\rho(a)$. The surface R' consists of two components. One of them is a disc with two holes P'_a and the other is a sphere with seven holes Q''. Clearly, Q'' contains Q'. Since the diffeomorphism H' preserves $\rho(a)$, it induces a diffeomorphism $G' : R' \to R'$. Since $(H')^M$ represents $t^N_{\rho(a)} = \rho(t^M_a)$, $(G')^M$ is isotopic to the identity. Since the two components of R' are not diffeomorphic, G' preserves them both. Let G'' be the diffeomorphism $Q'' \to Q''$ induced by G'. By using the same theorem of Nielsen as in the proof of (i), we can find a hyperbolic metric with geodesic boundary on Q'' and an isometry F' of this metric isotopic to G''. In addition, we may assume that the components of $\rho(C) \setminus \rho(a)$ are geodesic with respect to this metric. Since F', together with G'' and H', preserves the isotopy classes of these components, it has to preserve the components themselves (because F' is an isometry). This implies that F' preserves Q' and all its boundary components. By the same token as in the proof of (ii), this implies that F' is equal to the identity on Q' and, hence, on the whole surface Q''. Hence, the restriction G'' of G' to Q'' is isotopic to the identity. If the restriction of G' to P'_a is also isotopic to the identity, then the diffeomorphism H' representing $\rho(t_a)$ is isotopic to a power of the Dehn twist along $\rho(a)$ as claimed.

The restriction of G' to P'_a preserves the boundary component of P'_a corresponding to $\rho(a)$. If this restriction of G' is not isotopic to the identity, then it transposes the other two boundary components of P'_a (cf. for example, [FLP], Exp. 2, §III). In other words, $\rho(t_a)$ transposes the components of $\partial S'$ contained in $\partial P'_a$ (and fixes the other components). Since all t_a for $a = a_1,\ b_4,\ c_4$ are conjugate, if this is true for one of them, then it is true for the remaining two. In this case we can label the components of $\partial S'$ by the numbers $1,\ 2,\ \dots,\ 8$ in such a way that, say, $\rho(t_{a_1})$ induces the transposition (12), $\rho(t_{b_4})$ induces the transposition (34) and $\rho(t_{c_4})$ induces the transposition (56). Since t_e is also conjugate to $t_{a_1},\ t_{b_4},\ t_{c_4}$ (because $i(a_1, e) = 1$, e is a nonseparating circle), its image $\rho(t_e)$ also induces some transposition (ij). Now, $t_a t_e t_a = t_e t_a t_e$ because $i(a, e) = 1$ for all $a = a_1,\ b_4,\ c_4$. It follows that

$$(12)(ij)(12) = (ij)(12)(ij)$$
$$(34)(ij)(34) = (ij)(34)(ij)$$
$$(56)(ij)(56) = (ij)(56)(ij).$$

Suppose that $\{i, j\}$ and $\{1, 2\}$ are disjoint. Then (ij) and (12) commute. Since $(12)(ij)(12) = (ij)(12)(ij)$, this implies that $\{i, j\} = \{1, 2\}$. Hence, $\{1, 2\}$ and $\{i, j\}$ cannot be disjoint. Likewise $\{i, j\}$ and $\{3, 4\}$ are not disjoint and $\{i, j\}$ and $\{5, 6\}$ are not disjoint. But, clearly, $\{i, j\}$ cannot intersect three disjoint sets $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$

simultaneously. The contradiction shows that $\rho(t_a)$ fixes all boundary components of $\partial S'$ for $a = a_1,\ b_4,\ c_4$. As we had seen, this means that $\rho(t_a)$ is a power of the Dehn twist along $\rho(a)$ for $a = a_1,\ b_4,\ c_4$.

Now, we are going to use the relation $t_a t_e t_a = t_e t_a t_e$ for, say, $a = a_1$ once more. Since t_e is conjugate to t_a, $\rho(t_e)$ is a power of the Dehn twist along $\rho(e)$. The above relation implies $\rho(t_a)\rho(t_e)\rho(t_a) = \rho(t_e)\rho(t_a)\rho(t_e)$. Since ρ is injective, $\rho(t_a)$ and $\rho(t_e)$ do not commute. Hence, $\rho(a) \neq \rho(e)$. Thus, Theorem 3.15 implies that $i(\rho(a),\ \rho(e)) = 1$. But, since S' is a sphere with holes, this is impossible (all circles on S' are separating!). The contradiction completes the proof. □

Lemma 9.4. *Suppose that the maxima of ranks of abelian subgroups of* Mod_S *and* $\mathrm{Mod}_{S'}$ *differ by at most one. If a is a nonseparating circle on S, then $\rho(a)$ is a nonseparating circle on S.*

Proof. Let C be a maximal system of nonseparating circles on S containing a and σ be the corresponding simplex of $C(S)$. Let $\rho(C)$ be a realization of $\rho(\sigma)$. By Lemma 9.1, $\rho(C)$ is a system of circles on S' with $3\mathbf{g} - 3 + \mathbf{b}$ components. Since Dehn twists about nonseparating circles on S are conjugate in Mod_S, the components of $\rho(C)$ are topologically equivalent circles on S'. Hence, the result follows from Theorems 6.1 and 6.2 and Lemmas 9.2 and 9.3. □

Lemma 9.5. *Suppose that the maxima of ranks of abelian subgroups of* Mod_S *and* $\mathrm{Mod}_{S'}$ *differ by at most one. Suppose, in addition, that S is not a closed surface of genus 2. Then $\rho(t_a)$ is equal to $t_{\rho(a)}$ or $t^{-1}_{\rho(a)}$.*

Proof. By Lemma 9.2, S' is not a sphere with at most five holes or a torus with at most two holes. By assumption, S' is not a closed surface of genus 2.

Let C be a maximal system of nonseparating circles containing a and let σ be the corresponding simplex of $C(S)$. Let $\rho(C)$ be a realization of $\rho(\sigma)$. We may assume that the circles $\rho(a)$, where a runs over components of C, are components of $\rho(C)$ (the circles $\rho(a)$ are well defined only up to an isotopy). By Lemma 9.4, these circles $\rho(a)$ are nonseparating.

By the usual argument (compare the proof of Lemma 9.3), $\rho(t_a)$ preserves all vertices of $\rho(\sigma)$. In particular, $\rho(t_a)$ preserves the isotopy class of $\rho(a)$. Hence, we can represent $\rho(t_a)$ by a diffeomorphism $H' : S' \to S'$ such that $H'(\rho(a)) = \rho(a)$. Let S'' be the surface obtained by cutting S' along $\rho(a)$ and let $G' : S'' \to S''$ be the diffeomorphism induced by H'. Note that S'' is connected (because $\rho(a)$ is nonseparating). Since $\rho(t_a)^M = \rho(t_a^M) = t^N_{\rho(a)}$ is a power of the Dehn twist along $\rho(a)$, the isotopy class of G' has finite order. Using the Nielsen theorem, as in the proof of Lemma 9.3, we choose a hyperbolic metric with geodesic boundary on S'' and an isometry $F' : S'' \to S''$ isotopic to G'.

In addition, we may assume that $\rho(b)$ is a geodesic on S'' for each component b of $C \setminus a$. Together with H', the diffeomorphism G' preserves the isotopy classes of all components $\rho(b)$ of $\rho(C) \setminus \rho(a)$. Since F' is an isometry isotopic to G', F' preserves the components $\rho(b)$ themselves.

Let R' be the surface obtained by cutting S'' along all circles $\rho(b)$, where b runs over components of $C \setminus a$. Note that, at the same time, R' is the result of cutting of S' along $\rho(C)$. The number of components of $\rho(C)$ is equal to the maxima of ranks of abelian subgroups of Mod_S, and it differs by at most one from the corresponding maxima for $\mathrm{Mod}_{S'}$. It follows that $\rho(C)$ is either a maximal system of circles on S' or has one circle less than such a maximal system. Hence, all components of R' are discs with two holes or spheres with four holes, and there is at most one sphere with four holes among them. If there is only one component of R' and it is a sphere with four holes, then S' is either a sphere with four holes, a torus with two holes or a closed surface of genus 2. We have already seen that these cases are impossible. Hence, at least one component Q' of R' is a disc with two holes. Since each component of C' is a nonseparating circle and S' is not a torus with one hole (as we saw above), Q' is embedded in S'.

If only one component of $\partial Q'$ corresponds to a component of $\rho(C)$, then this component of $\rho(C)$ is separating. As we had seen, all components of $\rho(C)$ are nonseparating. Hence, at least two components of $\partial Q'$ correspond to components of $\rho(C)$. Recall that F' preserves all the components of $\rho(C) \setminus \rho(a)$. In particular, F' preserves at least two components of $\partial Q'$, namely, the components corresponding to components of $\rho(C)$ (here we consider Q' as a subsurface of S''). If $F'(Q') \neq Q'$, then $F'(Q') \cup Q'$ is a subsurface of S'' with boundary contained in the boundary of S''. Clearly, $F'(Q') \cup Q' = S''$ in this case and, hence, S'' is either a sphere with four holes, a torus with two holes or a closed surface of genus two. Since S'' is the result of cutting S', it cannot be closed. So, the last case is impossible. In the first two cases, S is either a torus with two holes or a closed surface of genus 2. As we have already seen, this is impossible. Hence, $F'(Q') = Q'$. Because F' preserves each component of $\partial Q'$, the diffeomorphism $Q' \to Q'$ induced by F' is isotopic to the identity. Since F' is an isometry, this diffeomorphism is, in fact, the identity. So, the restriction of F' on Q' is the identity and, hence, F' is the identity itself.

Hence, $G' : S'' \to S''$ is isotopic to the identity and $H' : S' \to S'$ is isotopic to a power of the Dehn twist along $\rho(a)$. In other words, $\rho(t_a) = t^K_{\rho(a)}$ for some integer K, which has to be nonzero, because ρ is injective. Since a is a nonseparating circle on S, we may choose a nonseparating circle e on S such that $i(a, e) = 1$. Since t_a and t_e are conjugate, $\rho(t_e) = t^K_{\rho(e)}$. Now, $t_a t_e t_a = t_e t_a t_e$ and, hence, $\rho(t_a)\rho(t_e)\rho(t_a) = \rho(t_e)\rho(t_a)\rho(t_e)$, (i.e. $t^K_{\rho(a)} t^K_{\rho(e)} t^K_{\rho(a)} = t^K_{\rho(e)} t^K_{\rho(a)} t^K_{\rho(e)}$). In addition, $t^K_{\rho(a)}$ and $t^K_{\rho(e)}$ do not commute (because ρ is injective and t_a and t_e do not commute) and, thus, $\rho(a) \neq \rho(e)$. Hence, Theorem 3.15 implies that $K = \pm 1$. (Compare this argument with the end of the proof of Lemma 9.3 (iii)). This completes the proof. □

Theorem 9.6. *Let S and S' be compact connected orientable surfaces. Suppose that S has genus at least 2 and S' is not a closed surface of genus 2. If the maxima of ranks of abelian subgroups of* Mod_S *and* $\mathrm{Mod}_{S'}$ *differ by at most one and $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ is an injective almost twist-preserving homomorphism, then ρ is induced by a diffeomorphism $S \to S'$.*

Proof. Let a be a nonseparating circle on S. By Lemma 9.5, $\rho(t_a)$ is equal to $t_{\rho(a)}$ or $t^{-1}_{\rho(a)}$.

Suppose that $\rho(t_a) = t_{\rho(a)}$. Since Dehn twists along nonseparating circles are conjugate in S, it follows that $\rho(t_b) = t_{\rho(b)}$ for every nonseparating circle b on S. In other words, ρ is twist-preserving. Hence, if $\rho(t_a) = t_{\rho(a)}$, the result follows from Theorem 8.9.

Suppose that $\rho(t_a) = t^{-1}_{\rho(a)}$. Let $F' : S' \to S'$ be an orientation-reversing diffeomorphism. F' induces an automorphism $F'_* : \mathrm{Mod}_{S'} \to \mathrm{Mod}_{S'}$. Since F' is orientation-reversing, $F'(t_{a'}) = t^{-1}_{F'(a')}$ for every circle a' on S'. Let $\rho' = F'_* \circ \rho$. Then $\rho' : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ is an injective homomorphism and $\rho'(t_a) = F'_*(t^{-1}_{\rho(a)}) = t_{F'(\rho(a))}$. Hence, by the previous paragraph, ρ' is twist-preserving. By Theorem 8.9, $\rho' = H'_*$ for some diffeomorphism $H' : S \to S'$. Thus, $\rho = (F'_*)^{-1} \circ H'_*$. This implies that $\rho = ((F')^{-1} \circ H')_*$. This completes the proof.

Note that if S is a closed surface of genus at least 2, only the easy part of Theorem 8.9 is actually needed (cf. the first paragraph of the proof of Theorem 8.9). In particular, we don't need the results of Section 4 in this case. □

10. Injective homomorphisms I

As in Section 9, S and S' denote compact connected *oriented* surfaces. We assume that the genus of S is at least 2, that S' is not a closed surface of genus 2, and that the maxima of ranks of abelian subgroups of Mod_S and $\mathrm{Mod}_{S'}$ differ by at most one. In this section, ρ will be an injective homomorphism $\mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ or $\mathrm{Mod}^*_S \to \mathrm{Mod}^*_{S'}$. The second case is needed only for the proof of Theorem 2 (and will be used only in the proof of Lemma 12.1). The notion of an *almost twist-preserving* homomorphism is defined for the extended modular groups (i.e., in the case $\mathrm{Mod}^*_S \to \mathrm{Mod}^*_{S'}$) exactly as for the usual modular groups (cf. Section 9).

Our first goal in this section is to show that the image under ρ of a (sufficiently high) power of a Dehn twist about a nonseparating circle is a multitwist about at most two circles (cf. Lemma 10.6). After this, we study the basic properties of ρ when ρ is not almost twist-preserving (cf. 10.7 and Lemmas 10.8, 10.9, 10.12 and 10.13). Eventually, these properties will lead to a contradiction to the assumptions of Theorem 3 of the introduction (cf. the proof of Theorem 11.7).

For the remainder of this section, we will denote by $\mathbf{g}, \mathbf{b}$ (respectively $\mathbf{g}', \mathbf{b}'$) the genus and the number of boundary components of S (respectively S').

Since the genus g of S is at least two, the maxima of the ranks of abelian subgroups of Mod_S is equal to $3\mathbf{g} - 3 + \mathbf{b}$ (cf. 3.1). By Lemma 9.2, S' is not a sphere with at most 5 holes or a torus with at most two holes. In particular, the maxima of the ranks of abelian subgroups of $\mathrm{Mod}_{S'}$ is equal to $3\mathbf{g}' - 3 + \mathbf{b}'$. Our asumptions, together with the injectivity of ρ, imply

$$3\mathbf{g} + \mathbf{b} \leq 3\mathbf{g}' + \mathbf{b}' \leq 3\mathbf{g} + \mathbf{b} + 1.$$

Let us fix subgroups of finite index Γ, Γ' in Mod_S, $\mathrm{Mod}_{S'}$ respectively such that both Γ and Γ' consist entirely of pure elements and $\rho(\Gamma) \subset \Gamma'$. (It is sufficient to take a subgroup of finite index Γ' in $\mathrm{Mod}_{S'}$ consisting entirely of pure elements and let $\Gamma = \rho^{-1}(\Gamma') \cap \Gamma_0$, where Γ_0 is a subgroup of finite index in Mod_S consisting entirely of pure elements (cf. 3.2 for the existence of Γ', Γ_0).)

Lemma 10.1. *Let H be a subgroup of a group G and let $A \subset H$. Then*

$$C(C_G(A)) \cap H \subset C(C_H(A)).$$

Proof. We leave the (easy) proof to the reader. □

Lemma 10.2. *Let $G \subset \Gamma$ be a free abelian group of rank $3\mathbf{g} - 3 + \mathbf{b}$. If $f \in G$, then*

$$\operatorname{rank} C(C_{\Gamma'}(\rho(f))) \leq \operatorname{rank} C(C_\Gamma(f)) + 1.$$

Proof. Let $f' = \rho(f)$. Let B be the subgroup of Γ' generated by $\rho(G)$ and $C(C_{\Gamma'}(f'))$ and let $A = \rho(G) \cap C(C_{\Gamma'}(f'))$. Since $f \in G$ and G is abelian, $\rho(G) \subset C_{\Gamma'}(f')$. This implies that B is abelian. We have

$$\operatorname{rank} \rho(G) + \operatorname{rank} C(C_{\Gamma'}(f')) = \operatorname{rank} A + \operatorname{rank} B.$$

Since ρ is injective, $\operatorname{rank} \rho(G) = 3\mathbf{g} - 3 + \mathbf{b}$. Thus,

$$3\mathbf{g} - 3 + \mathbf{b} + \operatorname{rank} C(C_{\Gamma'}(f')) = \operatorname{rank} A + \operatorname{rank} B.$$

Since $B \subset \mathrm{Mod}_{S'}$, $\operatorname{rank} B \leq 3\mathbf{g}' - 3 + \mathbf{b}'$. Hence,

$$3\mathbf{g} + \mathbf{b} + \operatorname{rank} C(C_{\Gamma'}(f')) \leq \operatorname{rank} A + 3\mathbf{g}' + \mathbf{b}'.$$

Since $3\mathbf{g}' + \mathbf{b}' \leq 3\mathbf{g} + \mathbf{b} + 1$, this implies

$$\operatorname{rank} C(C_{\Gamma'}(f')) \leq \operatorname{rank} A + 1.$$

By Lemma 10.1, $C(C_{\Gamma'}(f')) \cap \rho(\Gamma) \subset C(C_{\rho(\Gamma)}(f'))$. It follows that $A \subset C(C_{\rho(\Gamma)}(f'))$. Since ρ is injective, the last group is isomorphic to $C(C_{\Gamma}(f))$. Hence,

$$\operatorname{rank} A \leq \operatorname{rank} C(C_{\Gamma}(f)).$$

The lemma follows. □

Corollary 10.3. *Let $f \in \Gamma$ be a power of a Dehn twist. Then*

$$\operatorname{rank} C(C_{\Gamma'}(\rho(f))) \leq 2.$$

Proof. If f is a power of a Dehn twist about a nontrivial circle a, then a is a realization of the canonical reduction system and $r_a(f) = 1$. Hence, by Theorem 5.9, $\operatorname{rank} C(C_{\Gamma}(f)) = 1$. It remains to apply Lemma 10.2. □

Lemma 10.4. *Let $f \in \Gamma$ be a power of a Dehn twist. Then $\rho(f)$ is reducible of infinite order.*

Proof. Since f is of infinite order and ρ is injective, $\rho(f)$ is of infinite order. Hence, $\rho(f)$ is either reducible or pseudo-Anosov.

If $\rho(f)$ is pseudo-Anosov, then $C_{\Gamma'}(\rho(f))$ is an infinite cyclic group by Theorem 3.10. Let $f = t_a^n$ for some $n \in \mathbb{Z}$ and some circle a. Let C be a maximal system of circles containing a. Recall that T_C is the subgroup of Mod_S generated by the Dehn twists about components of C. Thus, T_C is a free abelian group of rank $3\mathbf{g} - 3 + \mathbf{b}$ containing f. It follows that $\rho(T_C \cap \Gamma)$ is also free abelian of rank $3\mathbf{g} - 3 + \mathbf{b}$ and $\rho(T_C \cap \Gamma) \subset C_{\Gamma'}(\rho(f))$. Since $C_{\Gamma'}(\rho(f))$ is infinite cyclic, this implies that $3\mathbf{g} - 3 + \mathbf{b} \leq 1$ and, hence, $3\mathbf{g} + \mathbf{b} \leq 4$. Since $\mathbf{g} \geq 2$, by the assumptions of this section, this is impossible. Hence, $\rho(f)$ is reducible. □

Lemma 10.5. *If a, b are disjoint, nonseparating, nonisotopic circles on S, then there exists a nonseparating circle d on S such that $i(d, a) = 0$ and $i(d, b) \neq 0$. Similarly, if a, b, c are disjoint, nonseparating, nonisotopic circles on S, then there exists a nonseparating circle d on S such that $i(d, a) = i(d, b) = 0$ and $i(d, c) \neq 0$.*

Proof. We will prove only the first assertion, the proof of the second one being completely similar. Clearly, there exists a possibly separating circle e on S such that $i(e, a) = 0$ and $i(e, b) \neq 0$. By a special case of Proposition 1 from Exposé 4, Appendice of [FLP], we have $i(t_e(b), b) = i(e, b)^2$. (Compare this argument with the proof of Theorem 3.15.) It follows that $c = t_e(b)$ is the required nonseparating circle. $\square$

Lemma 10.6. *Let $f \in \Gamma$ be a power of a Dehn twist about a nonseparating circle a. Let C' be a realization of the canonical reduction system for $\rho(f)$. Then C' has at most two components and $\rho(f)$ is a multitwist about C' (i.e., an element of $T_{C'}$).*

Proof. Let $f' = \rho(f)$. By Theorem 5.9, $C(C_{\Gamma'}(f'))$ is a free abelian group of rank $c' + p'$, where c' is the number of components of C' and p' is the number of pseudo-Anosov components of $\rho_{C'}(f')$. By Corollary 10.3, $c' + p' \leq 2$. Hence, $c' \leq 2$. This proves the first assertion.

Lemma 10.4 implies that C' is nonempty. Suppose that $p' \neq 0$. Then $c' = 1$ and $p' = 1$. Hence, C' is a nontrivial circle on S' and there is exactly one component P of $S'_{C'}$ such that f'_P is pseudo-Anosov.

Suppose that P is the only component of $S'_{C'}$. Then $\mathbf{g}' \geq 1$. As in the proof of Lemma 10.4, let us consider a maximal system of circles C containing a. Then T_C is a free abelian group of rank $3\mathbf{g} - 3 + \mathbf{b}$ containing f. It follows that $\rho(T_C \cap \Gamma)$ is also free abelian of rank $3\mathbf{g} - 3 + \mathbf{b}$ and $\rho(T_C \cap \Gamma) \subset C_{\Gamma'}(\rho(f))$. By Theorem 5.10, rank $\rho(T_C \cap \Gamma) \leq 2$ and, hence, $3\mathbf{g} - 3 + \mathbf{b} \leq 2$. Since $\mathbf{g} \geq 2$, by the assumptions of this section, this is impossible. The contradiction with our assumptions shows that P cannot be the only component of $S'_{C'}$.

Thus, C' is a nontrivial separating circle on S'. Hence, $S'_{C'}$ has exactly two components, P and the other component which we denote Q. Moreover, f'_P is pseudo-Anosov and f'_Q is trivial.

Now, let C be a maximal system of nonseparating circles on S containing a. For each component b of C, choose a power $f_b \in \Gamma$ of the Dehn twist about b and let $f'_b = \rho(f_b)$. We may assume that f_a is a power of f. Since Dehn twists about nonseparating circles on S are all conjugate in Mod_S, we may assume that all elements f_b are conjugate in Mod_S. It follows that the images f'_b are all conjugate in $\mathrm{Mod}^*_{S'}$ and, moreover, are all conjugate to a power of f'. Hence, for each component b of C, the canonical reduction system $\sigma(f')$ can be realized by a nontrivial separating circle $\rho(b)$ on S' dividing S' into two parts P_b and Q_b such that $(f'_b)_{P_b}$ is pseudo-Anosov and $(f'_b)_{Q_b}$ is trivial. (Clearly, $(\rho(a), P_a, Q_a) = (C', P, Q)$ and the triples $(\rho(b), P_b, Q_b)$ are all topologically equivalent.) Note that $\rho(b)$ is the unique component of $\rho(C)$ lying on the boundary ∂P_b of P_b.

The elements f'_b generate a free abelian group F'_C of rank $3\mathbf{g} - 3 + \mathbf{b}$. It follows that the circles $\rho(b)$ can be chosen to be pairwise disjoint or equal and then the union $\rho(C)$ of these circles $\rho(b)$ is a realization of a reduction system for F'_C.

Suppose that $\rho(b) \neq \rho(d)$ for each pair of distinct components b and d of C. Then $\rho(C)$ is a system of $3g - 3 + b$ circles on S'. Since $3g' - 3 + b' \leq (3g - 3 + b) + 1$, there exists at most one component of $S'_{\rho(C)}$ which contains a nontrivial circle. Note that, for any component b of C, the component P_b of $S'_{\rho(b)}$ must be simultaneously a component of $S'_{\rho(C)}$, because $\rho(C)$ is a realization of a reduction system for $f'_b \in F'_C$ and $(f'_b)_{P_b}$ is pseudo-Anosov. Moreover, P_b contains a nontrivial circle (because it carries a pseudo-Anosov element). We conclude that $P_b = P_d$ for any pair of distinct components b and d of C. Clearly, $\rho(b) = \rho(d)$ for each pair of distinct components b and d of C. Since $g \geq 2$, by the assumptions of this section, there is at least one such pair. The obvious contradiction implies that there exists a pair of distinct components b and d of C such that $\rho(b) = \rho(d)$. Clearly, for any such pair, $\{P_b, Q_b\} = \{P_d, Q_d\}$. Hence, $P_b = P_d$ or Q_d.

Suppose that $P_d = P_b$. Then $Q_d = Q_b$. By Lemma 10.5, we may choose a third nonseparating circle e on S such that $i(e, b) = 0 \neq i(e, d)$. Choose a power $f_e \in \Gamma$ of the Dehn twist about e and let $f'_e = \rho(f_e)$. Then f_d and f_e commute with f_b, but f_e does not commute with f_d. This implies that f'_d and f'_e commute with f'_b, but f'_e does not commute with f'_d (the last is because ρ is injective). Let B be the subgroup of Γ' generated by f'_b, f'_d and f'_e. Since the generators of B all commute with f'_b, they all preserve the isotopy class of $\rho(b)$. Hence, we have a reduction homomorphism $r_{\rho(b)} : B \to \mathrm{Mod}_{R'}$, where $R' = S'_{\rho(b)}$. Since B_{P_b} contains the pseudo-Anosov element $(f'_b)_{P_b}$ and every element of B_{P_b} commutes with this element, Theorem 3.10 implies that B_{P_b} is infinite cyclic. On the other hand, since $Q_b = Q_d$ is a trivial component of both $r_{\rho(b)}(f'_b)$ and $r_{\rho(d)}(f'_d)$, the group B_{Q_b} is generated by $(f'_e)_{Q_b}$. Hence, $B_{Q'}$ is abelian for both components Q' of R'. This implies that $r_{\rho(b)}(B)$ is abelian. Now, Lemma 5.5 implies that B is abelian. In particular, f'_e commutes with f'_d, in contradiction with the above.

Hence, $P_d = Q_b$. Thus, the two components of R' are P_b and P_d. Let $A = \{f_b, f_d\}$ and $A' = \rho(A) = \{f'_b, f'_d\}$. Each element of the centralizer $G = C_{\Gamma'}(A')$ preserves the isotopy class of the circle $\rho(b)$. Hence, we have a reduction homomorphism $r_{\rho(b)} : G \to \mathrm{Mod}_{R'}$. Since G_{P_b} contains the pseudo-Anosov element $(f'_b)_{P_b}$ and every element of G_{P_b} commutes with this element, Theorem 3.10 implies that G_{P_b} is infinite cyclic. Likewise, G_{P_d} is infinite cyclic. Hence, $G_{Q'}$ is abelian for every component Q' of R' and, hence, $r_{\rho(b)}(G)$ is abelian. Hence, by Lemma 5.5, G is abelian. Because ρ maps $C_\Gamma(A)$ injectively into G, this implies that $C_\Gamma(A)$ is abelian.

Since $g \geq 2$, by the assumptions of this section, there exists a third component e of C. Since b, d and e are disjoint nontrivial circles on S, we may choose a nontrivial circle h on S such that $i(h,b) = i(h,d) = 0 \neq i(h,e)$. Then f_e and f_h are noncommuting elements of $C_\Gamma(A)$. This contradicts the previous paragraph. The contradiction shows that our assumption $p' \neq 0$ is not true and, hence, proves the lemma. □

10.7. **Action of ρ on (the isotopy classes of) circles.** Let $\alpha \in V_0(S)$ be the isotopy class of some nonseparating circle a. Let us choose some $n \neq 0$ such that $t_\alpha^n \in \Gamma$ and put $\rho(\alpha) = \sigma(\rho(t_\alpha^n))$. In fact, $\rho(\alpha) = \sigma(\rho(t_\alpha^n)) = \sigma(\rho(t_\alpha)^n) = \sigma(\rho(t_\alpha))$ by the definition of the canonical reduction systems (cf. 3.3). In particular, $\rho(\alpha)$ does not depend on the choice of $n \neq 0$. Let $\rho(a)$ be a realization of $\rho(\alpha)$. By Lemma 10.6, $\rho(a)$ consists of one or two components and $\rho(t_\alpha^n)$ is a multitwist about $\rho(a)$ (or, what is the same, about $\rho(\alpha)$) if $t_\alpha^n \in \Gamma, n \neq 0$.

For $\sigma \subset V_0(S)$, we define $\rho(\sigma)$ as the union of simplices $\rho(\alpha)$ over $\alpha \in \sigma$, disagreeing slightly with the usual set-theoretic notation. As we will see in a moment (cf. Lemma 10.8), if $\sigma \subset V_0(S)$ is a simplex of $C(S)$, then $\rho(\sigma)$ is a simplex of $C(S')$. If C is a system of nonseparating circles on S, then we will denote by $\rho(C)$ a realization of the simplex $\rho(\sigma)$, where σ is the simplex of $C(S)$ corresponding to C. The system of circles $\rho(C)$ is well defined up to an isotopy on S'.

Lemma 10.8. *Let $\alpha, \beta \in V_0(S)$. Then $i(\alpha,\beta) = 0$ if and only if $i(\rho(\alpha),\rho(\beta)) = 0$. If $\sigma \subset V_0(S)$ is a simplex, then $\rho(\sigma)$ is a simplex.*

Proof. Clearly, the second assertion follows from the first one. In order to prove the first assertion, let us choose $m, n \neq 0$ such that $t_\alpha^n, t_\beta^m \in \Gamma$. Let $f_\alpha = t_\alpha^n, f_\beta = t_\beta^m$.

If $i(\alpha,\beta) = 0$, then f_α commutes with f_β by Theorem 3.15 and, hence,

$$\rho(f_\beta)(\rho(\alpha)) = \rho(f_\beta)(\sigma(\rho(f_\alpha))) = \sigma(\rho(f_\beta)\rho(f_\alpha)\rho(f_\beta)^{-1}) =$$

$$= \sigma(\rho(f_\beta f_\alpha f_\beta^{-1})) = \sigma(\rho(f_\alpha)) = \rho(\alpha).$$

By combining this with Theorem 3.7, we see that $\rho(f_\beta)$ fixes all vertices of $\rho(\alpha)$. For such a vertex $\gamma \in \rho(\alpha)$, let us choose $k \neq 0$ such that $t_\gamma^k \in \Gamma'$. Since $\rho(f_\beta)(\gamma) = \gamma$, we have

$$t^k_{\rho(f_\beta)(\gamma)} = t^k_\gamma$$

and $\rho(f_\beta) t_\gamma^k \rho(f_\beta)^{-1} = t_\gamma^k$. In other words, t_γ^k commutes with $\rho(f_\beta)$. Arguing as above, we can deduce from this that t_γ^k commutes with some nontrivial power $t_\delta^l \in \Gamma', l \neq 0$, of t_δ for any $\delta \in \rho(\beta)$. Now, Theorem 3.15 implies that $i(\gamma, \delta) = 0$ for any $\gamma \in \rho(\alpha), \delta \in \rho(\beta)$. Hence, $i(\rho(\alpha), \rho(\beta)) = 0$. Conversely, if $i(\rho(\alpha), \rho(\beta)) = 0$, then $\rho(f_\alpha)$ and $\rho(f_\beta)$ commute because they are multitwists about $\rho(\alpha)$ and $\rho(\beta)$ respectively, in view of Lemma 10.6. Since ρ is injective, in this case f_α and f_β commute. Hence, $i(\alpha, \beta) = 0$ by Theorem 3.15. □

Lemma 10.9. *If ρ is not almost twist-preserving, then $\rho(\alpha)$ is an edge of $C(S')$ for any $\alpha \in V_0(S)$.*

Proof. If $\alpha, \beta \in V_0(S)$, then t_α, t_β are conjugate in Mod_S. Hence, for appropriate $n \neq 0$, the powers t_α^n, t_β^n are both in Γ and are conjugate in Mod_S. Then $\rho(t_\alpha^n), \rho(t_\beta^n)$ are conjugate in $\mathrm{Mod}^*_{S'}$ and, hence, $\rho(\alpha)$ and $\rho(\beta)$ are equivalent under the action of $\mathrm{Mod}^*_{S'}$. It follows that either all $\rho(\alpha)$, $\alpha \in V_0(S)$, consist of one vertex, and, in this case, ρ is almost twist-preserving, or all $\rho(\alpha)$, $\alpha \in V_0(S)$, consist of two vertices, (i.e. all $\rho(\alpha)$ are edges of $C(S')$). □

Lemma 10.10. *Suppose that ρ is not almost twist-preserving. Let $\alpha, \beta \in V_0(S)$. If $i(\alpha, \beta) = 0$ and $\alpha \neq \beta$, then $\rho(\alpha) \cup \rho(\beta)$ is a triangle of $C(S')$. In particular, $\rho(\alpha)$ and $\rho(\beta)$ have a unique common vertex.*

Proof. Clearly, $\{\alpha, \beta\}$ is a simplex of $C(S)$. In view of Lemma 10.8, this implies that $\rho(\alpha) \cup \rho(\beta) = \rho(\{\alpha, \beta\})$ is a simplex of $C(S')$.

Suppose that $\rho(\alpha) = \rho(\beta)$. Let C' be a realization of $\rho(\alpha)$ and let $R' = S'_{C'}$. In view of Lemma 10.5, we may choose a vertex $\delta \in V_0(S)$ such that $i(\delta, \alpha) = 0$ and $i(\delta, \beta) \neq 0$. Let f_α, f_β, and $f_\delta \in \Gamma$ be some nontrivial powers of Dehn twists about α, β, and δ respectively. Then f_β and f_δ commute with f_α, but f_δ does not commute with f_β. Let $f'_\alpha = \rho(f_\alpha)$, $f'_\beta = \rho(f_\beta)$, and $f'_\delta = \rho(f_\delta)$. Clearly, f'_β and f'_δ commute with f'_α, but f'_δ does not commute with f'_β (the last is because ρ is injective). Let G be the subgroup of Γ' generated by f'_α, f'_β, and f'_δ. Since the generators of G all commute with f'_α, they all preserve $\rho(\alpha) = \sigma(\rho(f_\alpha))$.

Hence, $G \subset M(\rho(\alpha)) \cap \Gamma' = \Gamma'(C')$ (cf. 3.12 for the notations) and we can consider the reduction homomorphism $r_{C'}|G : G \to \mathrm{Mod}_{R'}$. Since

f'_α and f'_β are multitwists about $\rho(\alpha) = \rho(\beta)$, the reductions $r_{C'}(f'_\alpha)$ and $r_{C'}(f'_\beta)$ are both trivial. Thus, $r_{C'}(G)$ is generated by $r_{C'}(f'_\delta)$. Hence, $r_{C'}(G)$ is cyclic and, in particular, abelian. By Lemma 5.5, G is abelian. In particular, f'_δ commutes with f'_β in contradiction with the above. Hence, $\rho(\alpha) \neq \rho(\beta)$.

Suppose now that $\rho(\alpha) \cap \rho(\beta) = \emptyset$. Again, let $f_\alpha, f_\beta \in \Gamma$ be some nontrivial powers of Dehn twists about α, β respectively. Let $h = f_\alpha f_\beta$ and $h' = \rho(h)$. Note that $h \in T_C \cap \Gamma$, where C is some maximal system of circles such that the corresponding simplex contains α and β. The group $T_C \cap \Gamma$ is a free abelian group of rank $3\mathbf{g} - 3 + \mathbf{b}$, because T_C is such a group and Γ is of finite index in Mod_S. Hence, Lemma 10.2 implies that

$$\operatorname{rank} C(C_{\Gamma'}(h')) \leq \operatorname{rank} C(C_\Gamma(h)) + 1. \tag{10.1}$$

Clearly, $\{\alpha, \beta\}$ is the canonical reduction system for h and h is a multi-twist about $\{\alpha, \beta\}$. Hence, Theorem 5.9 implies that $\operatorname{rank} C(C_\Gamma(h)) = 2$.

Another application of Theorem 5.9 shows that rank $C(C_{\Gamma'}(h')) = 4$ (note that $\sigma(h') = \rho(\alpha) \cup \rho(\beta)$). Contradiction with (10.1) shows that the intersection $\rho(\alpha) \cap \rho(\beta)$ cannot be empty. Thus, the edges $\rho(\alpha)$ and $\rho(\beta)$ are not disjoint and not equal. This means that the simplex $\rho(\alpha) \cup \rho(\beta)$ has exactly three vertices and the edges $\rho(\alpha)$ and $\rho(\beta)$ have exactly one common vertex. $\square$

Lemma 10.11. *Suppose that ρ is not almost twist-preserving. Let $\{\alpha, \beta, \gamma\} \subset V_0(S)$ be a simplex of $C(S)$. Then $\rho(\alpha) \cap \rho(\beta) = \rho(\beta) \cap \rho(\gamma) = \rho(\gamma) \cap \rho(\alpha)$.*

Proof. Of course, it is sufficient to prove that $\rho(\alpha) \cap \rho(\beta) = \rho(\alpha) \cap \rho(\gamma)$. Note that, by Lemma 10.9, $\rho(\alpha)$, $\rho(\beta)$, and $\rho(\gamma)$ are edges of $C(S')$. By Lemma 10.10, each pair of these edges has exactly one common vertex. Let $\{\alpha'\} = \rho(\beta) \cap \rho(\gamma)$, $\{\beta'\} = \rho(\alpha) \cap \rho(\gamma)$, and $\{\gamma'\} = \rho(\alpha) \cap \rho(\beta)$.

Suppose that $\beta' \neq \gamma'$. Then $\rho(\alpha) = \{\beta', \gamma'\}$, because $\beta', \gamma' \in \rho(\alpha)$. Moreover, $\alpha' \neq \beta'$ in this case, because, otherwise, $\beta' = \alpha' \in \rho(\beta)$ and $\gamma' \in \rho(\beta)$ and, hence, $\rho(\alpha)$ and $\rho(\beta)$ have two common vertices β', γ' in contradiction with Lemma 10.10. Similarly, $\alpha' \neq \gamma'$ in this case. It follows that $\rho(\alpha) = \{\beta', \gamma'\}$, $\rho(\beta) = \{\alpha', \gamma'\}$, and $\rho(\gamma) = \{\alpha', \beta'\}$.

Lemma 10.5 implies that there exists $\delta \in V_0(S)$ such that $i(\delta, \alpha) = i(\delta, \beta) = 0$ and $i(\delta, \gamma) \neq 0$. As usual, let f_α, f_β, f_γ, and $f_\delta \in \Gamma$ be some nontrivial powers of Dehn twists about α, β, γ, and δ respectively. Since $i(\delta, \alpha) = 0$, the elements f_δ and f_α commute. It follows that $\rho(f_\delta)$ and $\rho(f_\alpha)$ commute and, since $\rho(\alpha) = \sigma(\rho(f_\alpha))$, that $\rho(f_\delta)(\rho(\alpha)) = \rho(\alpha)$. Because $\rho(\alpha)$ consists of only two vertices, this implies that

$\rho(f_\delta^2) = \rho(f_\delta)^2$ fixes both vertices of $\rho(\alpha)$. In particular, $\rho(f_\delta^2)(\beta') = \beta'$. Similarly, $i(\delta, \beta) = 0$ implies that $\rho(f_\delta^2)(\alpha') = \alpha'$. Since $\rho(f_\gamma)$ is a multitwist about $\rho(\gamma) = \{\alpha', \beta'\}$ in view of Lemma 10.6, it follows that $\rho(f_\delta^2)$ commutes with $\rho(f_\gamma)$. Since ρ is injective, this implies that f_δ^2 commutes with f_γ. By Theorem 3.15, this implies that $i(\delta, \gamma) = 0$ in contradiction with the above.

Hence, our assumption that $\beta' \neq \gamma'$ is not true. In other words, $\beta' = \gamma'$ and $\rho(\alpha) \cap \rho(\gamma) = \rho(\alpha) \cap \rho(\beta)$. This completes the proof. $\square$

Lemma 10.12. *Suppose that ρ is not almost twist-preserving. Let σ be a simplex of $C(S)$ contained in $V_0(S)$ and having at least two vertices. Then there exists a unique isotopy class $\rho_\sigma \in V(S')$ such that $\rho_\sigma \in \rho(\alpha)$ for each $\alpha \in \sigma$. If $\alpha, \beta \in \sigma$ and $\alpha \neq \beta$, then $\{\rho_\sigma\} = \rho(\alpha) \cap \rho(\beta)$.*

Proof. Let $\alpha, \beta, \gamma, \delta \in \sigma$ such that $\alpha \neq \beta$, $\gamma \neq \delta$. If $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ have a common element, then $\rho(\alpha) \cap \rho(\beta) = \rho(\gamma) \cap \rho(\delta)$ by Lemma 10.11. Otherwise, $\rho(\alpha) \cap \rho(\beta) = \rho(\alpha) \cap \rho(\gamma) = \rho(\gamma) \cap \rho(\delta)$, again by Lemma 10.11. In addition, for any $\alpha, \beta \in \sigma$, $\alpha \neq \beta$, the intersection $\rho(\alpha) \cap \rho(\beta)$ consists of exactly one vertex, by Lemma 10.10. So, we can take ρ_σ to be this vertex. $\square$

Lemma 10.13. *If ρ is not almost twist-preserving, then $3\mathbf{g}' + \mathbf{b}' = 3\mathbf{g} + \mathbf{b} + 1$. Hence, the maxima of ranks of abelian subgroups of $\mathrm{Mod}_{S'}$ is bigger by one than the maxima of ranks of abelian groups of Mod_S.*

Proof. Let C be a maximal system of nonseparating circles on S and σ be the corresponding simplex. Let $\rho(C)$ be a realization of $\rho(\sigma)$. By Lemma 10.9, all $\rho(\alpha), \alpha \in \sigma$ are edges. By Lemma 10.12, there is one vertex common to all these edges, and the remaining vertices of these edges are all distinct. This implies that the union $\rho(\sigma)$ of these edges has one more vertex than σ. Hence, $\rho(C)$ has $3\mathbf{g}-3+\mathbf{b}+1 = 3\mathbf{g}-2+\mathbf{b}$ components. Since $\rho(C)$ is a system of circles on S', this implies that $3\mathbf{g}-2+\mathbf{b} \leq 3\mathbf{g}'-3+\mathbf{b}'$ and, hence, $3\mathbf{g}+\mathbf{b}+1 \leq 3\mathbf{g}'+\mathbf{b}'$. On the other hand, as we noticed in the beginning of this section, $3\mathbf{g}'+\mathbf{b}' \leq 3\mathbf{g}+\mathbf{b}+1$. The lemma follows. $\square$

11. Injective homomorphisms II

As in Section 10, S and S' denote compact connected *oriented* surfaces. As in section 10, we assume that the genus of S is at least two, that S' is not a closed surface of genus two, and that the maxima of ranks of abelian subgroups of Mod_S and $\mathrm{Mod}_{S'}$ differ by at most one. As in Section 10, we will denote by $\mathbf{g}, \mathbf{b}$ (respectively $\mathbf{g}', \mathbf{b}'$) the genus and the number of boundary components of S (respectively S'). As usual, let $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ be an injective homomorphism. As in

Section 10, we conclude that the maxima of the ranks of abelian subgroups of Mod_S is equal to $3\mathbf{g} - 3 + \mathbf{b}$, S' is not a sphere with at most five holes or a torus with at most two holes, the maxima of the ranks of abelian subgroups of $\mathrm{Mod}_{S'}$ is equal to $3\mathbf{g}' - 3 + \mathbf{b}'$, and

$$3\mathbf{g} + \mathbf{b} \leq 3\mathbf{g}' + \mathbf{b}' \leq 3\mathbf{g} + \mathbf{b} + 1.$$

The goal of this section is to prove Theorem 3 of the Introduction. This theorem appears below as Theorem 11.7. We will continue the line of arguments started in Section 10. Note that, since S is of genus at least two, all results of Section 10 are valid under our current assumptions. In particular, we may use the notations introduced in 10.7. As we saw in Section 10, if ρ is not almost twist-preserving, then $\rho(\alpha)$ is an edge of $C(S')$ for any $\alpha \in V_0(S)$ (cf. Lemma 10.9). Moreover, if σ is a simplex of $C(S)$ contained in $V_0(S)$ and having at least two vertices, then there is one vertex common to all edges $\rho(\alpha)$, $\alpha \in \sigma$, and the other vertices of these edges are all distinct (cf. Lemma 10.12). As in Lemma 10.12, we will denote this unique common vertex by ρ_σ.

Lemma 11.1. *Suppose that ρ is not almost twist-preserving. Let C be the maximal system of circles introduced in 7.1, and let σ be the simplex of $C(S)$ corresponding to C. Then $\rho(\mathrm{PMod}_S)$ is contained in the stabilizer of ρ_σ in $\mathrm{Mod}_{S'}$.*

Proof. All components of C are obviously nonseparating. Hence, $\sigma \subset V_0(S)$. Also, C has $3\mathbf{g} - 3 + \mathbf{b}$ components and $\mathbf{g} \geq 2$. Hence, σ has at least three vertices. In particular, ρ_σ is indeed well defined.

Recall that $\rho(\alpha) = \sigma(\rho(t_\alpha))$ for any $\alpha \in V_0(S)$ (cf. 10.7). It follows that $\rho(\alpha)$ is invariant under $\rho(t_\alpha)$. In addition, if $i(\alpha, \beta) = 0$, then $t_\alpha t_\beta t_\alpha^{-1} = t_\beta$ and $\rho(t_\alpha)(\rho(\beta)) = \rho(t_\alpha)(\sigma(\rho(t_\beta))) = \sigma(\rho(t_\alpha)\rho(t_\beta)\rho(t_\alpha)^{-1}) = \sigma(\rho(t_\alpha t_\beta t_\alpha^{-1})) = \sigma(\rho(t_\beta)) = \rho(\beta)$. So, if $i(\alpha, \beta) = 0$, then $\rho(\beta)$ is also invariant under $\rho(t_\alpha)$.

If we apply these remarks to two vertices $\alpha, \beta \in \sigma$, we conclude $\rho(t_\alpha)$ preserves both $\rho(\alpha)$ and $\rho(\beta)$. Since ρ_σ is the unique common vertex of $\rho(\alpha)$ and $\rho(\beta)$ by Lemma 10.12, it follows that ρ_σ is preserved by $\rho(t_\alpha)$. Thus, ρ_σ is preserved by all $\rho(t_\alpha)$, $\alpha \in \sigma$.

Now, let β be the isotopy class of one of the *dual* circles of the configuration $\mathcal{C}$ (cf. 7.1). We would like to prove that ρ_σ is preserved also by $\rho(t_\beta)$.

Since $\mathbf{g} \geq 2$, there exist two distinct vertices $\alpha, \gamma \in \sigma$ such that $i(\alpha, \beta) = 1$ and $i(\gamma, \beta) = 0$ (cf. Figure 7.1). In view of the above remarks, $\rho(\alpha)$ and $\rho(\gamma)$ are invariant under $\rho(t_\alpha)$. Also, $\rho(t_\alpha)$ fixes ρ_σ and ρ_σ is the unique common vertex of the edges $\rho(\alpha)$, $\rho(\gamma)$ (by Lemma 10.12). It follows that $\rho(t_\alpha)$ fixes each vertex of $\rho(\alpha)$ and

$\rho(\gamma)$. In addition, $\rho(\gamma)$ is invariant under $\rho(t_\beta)$ because $i(\gamma, \beta) = 0$. Now, $t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta$ by Theorem 3.15 and, hence, $\rho(t_\alpha)\rho(t_\beta)\rho(t_\alpha) = \rho(t_\beta)\rho(t_\alpha)\rho(t_\beta)$. Since $\rho(t_\alpha)$ is equal to the identity on $\rho(\gamma)$ and $\rho(\gamma)$ is invariant under $\rho(t_\beta)$, the last equality implies that $\rho(t_\beta)$ is also equal to the identity on $\rho(\gamma)$. In particular, $\rho(t_\beta)$ fixes $\rho_\sigma \in \rho(\gamma)$.

Since the configuration $\mathcal{C}$ consists of the components of C and the dual circles (cf. 7.1), we have shown that ρ_σ is fixed by the images under ρ of the Dehn twists along all circles of the configuration $\mathcal{C}$. Hence, the result follows from Theorem 7.3. □

Lemma 11.2. *Suppose that ρ is not almost twist-preserving. Let C be the maximal system of circles introduced in 7.1, and let σ be the simplex of $C(S)$ corresponding to C. Then $\rho_\sigma \in \rho(\gamma)$ for all $\gamma \in V_0(S)$.*

Proof. Let $\gamma \in V_0(S)$ and $\alpha \in \sigma$. Since both α and γ are isotopy classes of nonseparating circles, $f(\alpha) = \gamma$ for some $f \in \mathrm{PMod}_S$. Thus, $ft_\alpha f^{-1} = t_\gamma$ and $\rho(f)(\rho(\alpha)) = \rho(f)(\sigma(\rho(t_\alpha))) = \sigma(\rho(f)\rho(t_\alpha)\rho(f)^{-1}) = \sigma(\rho(ft_\alpha f^{-1})) = \sigma(\rho(t_\gamma)) = \rho(\gamma)$. Because $\rho_\sigma \in \rho(\alpha)$ by the definition of ρ_σ (cf. Lemma 10.12), this implies that $\rho(f)(\rho_\sigma) \in \rho(\gamma)$. On the other hand, $\rho(f)(\rho_\sigma) = \rho_\sigma$ by Lemma 11.1. The lemma follows. □

Lemma 11.3. *Suppose that ρ is not almost twist-preserving. Let C be the maximal system of circles introduced in 7.1, and let σ be the simplex of $C(S)$ corresponding to C. Then $\rho(\mathrm{Mod}_S)$ is contained in the stabilizer of ρ_σ in $\mathrm{Mod}_{S'}$.*

Proof. If $\alpha, \beta \in V_0(S)$, $\alpha \neq \beta$, and $i(\alpha, \beta) = 0$, then $\rho(\alpha)$ and $\rho(\beta)$ have a unique common vertex by Lemma 10.12 (namely, ρ_τ, where $\tau = \{\alpha, \beta\}$). Lemma 11.2 implies that this common vertex is equal to ρ_σ for all such α, β.

Now, let us choose such a pair α, β. Let $f \in \mathrm{Mod}_S$. Then γ, δ, where $\gamma = f(\alpha), \delta = f(\beta)$, is another such pair. Thus, ρ_σ is the unique common vertex of $\rho(\gamma), \rho(\delta)$. On the other hand, $\rho(f)(\rho(\alpha)) = \rho(f)(\sigma(\rho(t_\alpha))) = \sigma(\rho(f)\rho(t_\alpha)\rho(f)^{-1}) = \sigma(\rho(ft_\alpha f^{-1})) = \sigma(\rho(t_\gamma)) = \rho(\gamma)$ and, similarly, $\rho(f)(\rho(\beta)) = \rho(\delta)$. It follows that $\rho(f)$ maps the unique common vertex of $\rho(\alpha)$ and $\rho(\beta)$ into the unique common vertex of $\rho(\gamma)$ and $\rho(\delta)$. Since both of them are equal to ρ_σ, we have $\rho(f)(\rho_\sigma) = \rho_\sigma$. This completes the proof. □

Lemma 11.4. *Suppose that ρ is not almost twist-preserving. Let σ be the simplex of $C(S)$ considered in Lemmas 11.1—11.3. Let z be some circle on S' in the isotopy class ρ_σ and let $R' = S'_z$ be the result of cutting S' along z.*

(i) *If z is nonseparating, then R' is a connected surface of genus $\mathbf{g}' - 1$ with $\mathbf{b}' + 2$ boundary components.*

(ii) *If z is separating, then R' consists of two components. One of them is a disc with two holes, and the other is a connected surface of genus $\mathbf{g}'$ with $\mathbf{b}' - 1$ boundary components.*

Proof. (i) This part of the lemma is obvious.

(ii) For any $\alpha \in V_0(S)$, we can realize $\rho(\alpha)$ by a system of circles having z as one of the two components, because $\rho_\sigma \in \rho(\alpha)$ by Lemma 11.2 (there are exactly two components by Lemma 10.9). Let us denote by $C(\alpha)$ the other component of this system of circles.

It follows from Lemma 10.8 that $i(\alpha, \beta) = 0$ if and only if $i(C(\alpha), C(\beta)) = 0$. In particular, if $i(\alpha, \beta) \neq 0$, then $i(C(\alpha), C(\beta)) \neq 0$ and, hence, $C(\alpha)$ and $C(\beta)$ are contained in the same component of R'.

Now, note that, for any two vertices $\alpha, \beta \in V_0(S)$, there exists a vertex $\gamma \in V_0(S)$ such that $i(\alpha, \gamma) \neq 0$ and $i(\gamma, \beta) \neq 0$. For example, it is sufficient to take $\gamma = f^N(\gamma')$, where f is a pseudo-Anosov element, $\gamma' \in V_0(S)$ and N is sufficiently big. It follows that $C(\alpha)$ and $C(\beta)$ are contained in the same component of R' (namely, in the component containing $C(\gamma)$). Thus, all circles $C(\alpha), \alpha \in V_0(S)$, are contained in the same component of R'.

Let us consider now some maximal system of nonseparating circles on S and the corresponding simplex τ in $C(S)$. For any two vertices $\alpha, \beta \in \tau$, we have $i(C(\alpha), C(\beta)) = 0$ (because $i(\alpha, \beta) = 0$). Hence, we may assume that the circles $C(\alpha), \alpha \in \tau$, are pairwise disjoint. By Lemma 10.12, these circles are pairwise non-isotopic and none of them are isotopic to z. Hence, the circles $C(\alpha), \alpha \in \tau$, together with z form a system of circles on S'. It has $3\mathbf{g} - 3 + \mathbf{b} + 1 = 3\mathbf{g} - 2 + \mathbf{b}$ components, because τ has $3\mathbf{g} - 3 + \mathbf{b}$ vertices. On the other hand, $3\mathbf{g} - 2 + \mathbf{b} = 3\mathbf{g}' - 3 + \mathbf{b}'$ by Lemma 10.13. Hence, this system of circles is a maximal system of circles on S'. Since all components of this maximal system of circles other than z are contained in the same component of S'_z, the other component of S'_z is a disc with two holes.

So, we proved that one of the components of $R' = S'_z$ is a disc with two holes. Obviously, this implies that the other component has genus $\mathbf{g}'$ (the same as S') and $\mathbf{b}' - 1$ boundary components. This completes the proof. □

Lemma 11.5. *Let Q be a compact connected orientable surface, c be a nontrivial circle on Q, and $R = Q_c$. Let $M(c)$ be the stabilizer in* Mod_Q *of the isotopy class of c.*

(i) *If c is nonseparating, then the kernel of the reduction homomorphism $r_c : M(c) \to \mathrm{Mod}_R$ is an infinite cyclic subgroup contained in the center of $M(c)$.*

(ii) *If c divides Q into two parts P and P_0 such that P_0 is a disc with two holes and P is not a disc with two holes, then* Mod_R *fixes the component P of R and the kernel of the composition $\pi_P \circ r_c : M(c) \to$* Mod_P *is an infinite cyclic subgroup contained in the center of $M(c)$.*

Proof. (i) The kernel of r_c is generated by the Dehn twist t_c about c, which is clearly central in $M(c)$ (cf. 3.3). This proves (i).

(ii) Since P and P_0 are not diffeomorphic, Mod_R fixes both components P and P_0 of R. This implies that $\pi_P \circ r_c$ is well defined (formally, $r_c(M(c)) \subset \mathrm{Mod}_R = \mathrm{Mod}_R(P)$).

Every element of the kernel of $\pi_P \circ r_c$ obviously can be represented by a diffeomorphism $F : Q \to Q$ equal to the identity on P. Such a diffeomorphism F is uniquely defined by the induced diffeomorphism $F_0 : P_0 \to P_0$. Clearly, F_0 is equal to the identity on the component c of the boundary ∂P_0, but may interchange two other boundary components. Moreover, any diffeomorphism $P_0 \to P_0$ equal to the identity on c can arise in this way. The group G of such diffeomorphisms $P_0 \to P_0$, considered up to isotopies fixed on c, is known to be infinite cyclic: it is generated by the so-called *half-twist* about c; the square of this generator is a Dehn twist about c. The obvious map from this group G to the kernel of $\pi_P \circ r_c$ is surjective by the above remarks; it is injective because its restriction to the infinite cyclic subgroup of powers of the Dehn twist about c is obviously injective. It follows that the kernel of $\pi_P \circ r_c$ is infinite cyclic.

Finally, any element of $M(c)$ can be represented by a diffeomorphism $F : Q \to Q$ preserving c. Such a diffeomorphism preserves the sides of c, because P and P_0 are not diffeomorphic, and, hence, preserves the orientation of c (because F is orientation-preserving). Therefore, replacing F by an isotopic diffeomorphism if necessary, we may assume that F is equal to the identity on c. Now, the description of the kernel of $\pi_P \circ r_c$, given in the previous paragraph (and, in particular, the fact that G is abelian) implies that the isotopy class of F commutes with all elements of this kernel. Hence, the kernel of $\pi_P \circ r_c$ is contained in the center of $M(c)$. □

Lemma 11.6. *Suppose that ρ is not almost twist-preserving. Let σ be the simplex of $C(S)$ considered in Lemmas 11.1—11.3. Let z be some circle on S' in the isotopy class ρ_σ and let $R' = S'_z$ be the result of cutting S' along z.*

If z is nonseparating, let $P' = R'$.

If z is separating, then there is a unique component of R' which is not a disc with two holes. Let us denote by P' this component.

In both cases, $\mathrm{Mod}_{R'}$ *fixes the component* P' *of* R' *and, thus, the induced homomorphism* $\rho' = \pi_{P'} \circ r_z \circ \rho : \mathrm{Mod}_S \to \mathrm{Mod}_{P'}$ *is well defined. Moreover,* ρ' *is an injective almost twist-preserving homomorphism.*

Proof. First, note that $r_z \circ \rho$ is well defined because $\rho(\mathrm{Mod}_S)$ is contained in the stabilizer of ρ_σ by Lemma 11.3.

If z is nonseparating, then $P' = R'$ and $\pi_{P'} = \mathrm{id}$. It follows that $\rho' = r_z \circ \rho$ and, so, ρ' is well defined.

If z is separating, then one of the components of R' is a disc with two holes by Lemma 11.4. The other component cannot be a disc with two holes, because it has genus $\mathbf{g}'$ and $\mathbf{b}' - 1$ boundary components by Lemma 11.4 and $3\mathbf{g}' + \mathbf{b}' - 1 = 3\mathbf{g} + \mathbf{b} \geq 6$ by Lemma 10.13 and the assumption $\mathbf{g} \geq 2$. So, one component of R' is a disc with two holes and the other is not. It follows that $\mathrm{Mod}_{R'}$ fixes both components of R'. Hence, $\rho' = \pi_{P'} \circ r_z \circ \rho$ is, indeed, well defined.

It remains to prove that ρ' is injective and almost twist-preserving.

The kernel of ρ' is isomorphic via ρ to the intersection of $\rho(\mathrm{Mod}_S)$ with the kernel of $\pi_{P'} \circ r_z$. If z is nonseparating, then $\pi_{P'} \circ r_z = r_z$ and, hence, the kernel of $\pi_{P'} \circ r_z$ is infinite cyclic by Lemma 11.5 (i). If z is separating, then the kernel of $\pi_{P'} \circ r_z$ is infinite cyclic by Lemma 11.5 (ii). It follows that the kernel of ρ' is a subgroup of an infinite cyclic group and, thus, is either trivial or infinite cyclic. Since no infinite cyclic subgroup of Mod_S can be normal (this follows easily from Thurston's classification of elements of Mod_S; alternatively, one may use [I3], Exercises 5.a, 5.b and Lemma 9.12), the kernel of ρ' is trivial.

So, ρ' is injective. If $\alpha \in V_0(S)$, then $\rho(t_\alpha^n)$ is a multitwist about $\rho(\alpha)$ for some $n \neq 0$ (cf. 10.7). Since $\rho(\alpha)$ consists of two vertices and one of them is the isotopy class ρ_σ of z (by Lemma 11.2), it follows that $r_z(\rho(t_\alpha^n))$ is a power of a Dehn twist (about a circle representing the other vertex). Hence, $\pi_{P'}(r_z(\rho(t_\alpha^n)))$ is a power of a Dehn twist. This proves that $\rho' = \pi_{P'} \circ r_z \circ \rho$ is almost twist-preserving and, hence, completes the proof. $\square$

Theorem 11.7. *Let* S *and* S' *be compact connected orientable surfaces. Suppose that the genus of* S *is at least* 2 *and* S' *is not a closed surface of genus* 2*. Suppose that the maxima of ranks of abelian subgroups of* Mod_S *and* $\mathrm{Mod}_{S'}$ *differ by at most one. If* $\rho : \mathrm{Mod}_S \to \mathrm{Mod}_{S'}$ *is an injective homomorphism, then* ρ *is induced by a diffeomorphism* $S \to S'$.

Proof. In view of Theorem 9.6, it is sufficient to consider the case when ρ is not almost twist-preserving.

Let σ be the simplex of $C(S)$ considered in Lemmas 11.1—11.4 and 11.6. As in Lemma 11.6, let z be some circle on S' in the isotopy class ρ_σ and let $R' = S'_z$. Let P' be the component of R' introduced in Lemma 11.6. Clearly, P' is not a closed surface of genus 2. By Lemma 11.6, the homomorphism $\rho' = \pi_{P'} \circ r_z \circ \rho : \mathrm{Mod}_S \to \mathrm{Mod}_{P'}$ is well defined and is an injective almost twist-preserving homomorphism. In addition, Lemma 11.4 implies that the maxima of ranks of abelian subgroups of $\mathrm{Mod}_{P'}$ is one less than the maxima of ranks of abelian subgroups of $\mathrm{Mod}_{S'}$ and, hence, is equal to the maxima of ranks of abelian subgroups of Mod_S (it cannot be less than this maxima for Mod_S, because ρ' is injective). This means, in particular, that Theorem 9.6 applies to ρ' and implies that ρ' is induced by some diffeomorphism $H : S \to P'$.

In the remaining part of the proof, we will use the notations (and the results) of Section 4.

If $F : S \to S$ is a diffeomorphism fixed on ∂S, then the diffeomorphism $H \circ F \circ H^{-1} : P' \to P'$ gives rise to a diffeomorphism $S' \to S'$ by gluing in the case when z is nonseparating (note that $H \circ F \circ H^{-1}$ is fixed on $\partial P'$) and by extending by the identity in the case when z is separating. By passing to the isotopy classes, we get a homomorphism $\mathcal{M}_\mathcal{S} \to \mathcal{M}_{\mathcal{S}'}$ induced by H. We will denote this homomorphism by H_*. (Compare this argument with the proof of Theorem 8.9.) By choosing the orientations of S and S' appropriately, we may assume that H is orientation-preserving. Then $H_*(\tilde{t}_c) = \tilde{t}_{H(c)}$ for any circle c on S. As in the proof of Theorem 8.9, let us consider the following diagram.

$$\begin{array}{ccc}
\mathcal{M}_S & \xrightarrow{H_*} & \mathcal{M}_{S'} \\
\Big\downarrow p & & \Big\downarrow p' \\
\mathrm{PMod}_S & \xrightarrow{\rho} & \mathrm{Mod}_{S'}
\end{array}$$

The vertical maps are the canonical homomorphisms $p : \mathcal{M}_S \to \mathrm{PMod}_S$, $p' : \mathcal{M}_{S'} \to \mathrm{Mod}_{S'}$. Note that we cannot claim that this diagram is commutative. But, in fact, it is quite close to being commutative. Namely, the compositions $\pi_{P'} \circ r_z \circ p' \circ H_*$ and $\pi_{P'} \circ r_z \circ \rho \circ p : \mathcal{M}_S \to \mathrm{Mod}_{P'}$ are equal, because $\rho' = \pi_{P'} \circ r_z \circ \rho$ is induced by H. Therefore, $p' \circ H_*(f)$ and $\rho \circ p(f)$ differ by an element of the kernel of $\pi_{P'} \circ r_z$ for any $f \in \mathcal{M}_S$. By Lemma 11.5, this kernel is an infinite cyclic group, contained in the center of $M(z)$, where $M(z)$ is the stabilizer in $\mathrm{Mod}_{S'}$ of the isotopy class of z (i.e., of ρ_σ). We will denote this kernel by K.

Let us consider the Dehn twist $\tilde{t}_c \in \mathcal{M}_S$, where c is a boundary component of S corresponding to z under H. By Theorem 4.3, the element $\tilde{t}_c \in \mathcal{M}_S$ belongs to the commutator subgroup of $\mathcal{M}_S$. (Here, we use the assumption that the genus of S is at least 2 in a crucial way.) In other words,

$$\tilde{t}_c = \prod_{i=1}^{n} [f_i, g_i],$$

for some $f_i, g_i \in \mathcal{M}_S$, $1 \le i \le n$, where $[f, g]$ denotes the commutator $fgf^{-1}g^{-1}$. Now,

$$\begin{aligned} t_z = p'(\tilde{t}_z) = p'(\tilde{t}_{H(c)}) = p'(H_*(\tilde{t}_c)) = \\ = p' \circ H_*(\tilde{t}_c) = p' \circ H_*(\prod_{i=1}^{n} [f_i, g_i]) = \\ = \prod_{i=1}^{n} [p' \circ H_*(f_i), p' \circ H_*(g_i)] = \\ = \prod_{i=1}^{n} [\rho \circ p(f_i) k_i, \rho \circ p(g_i) l_i] \end{aligned}$$

for some $k_i, l_i \in K$, $1 \le i \le n$, by the discussion in the previous paragraph. Since $\rho(\mathrm{PMod}_S)$ is contained in $M(z)$ by Lemma 11.1 and K is contained in the center of $M(z)$, as we noticed above, the last expression is equal to

$$\begin{aligned} \prod_{i=1}^{n} [\rho \circ p(f_i), \rho \circ p(g_i)] = \rho \circ p(\prod_{i=1}^{n} [f_i, g_i]) = \\ = \rho \circ p(\tilde{t}_c) = \rho(1) = 1 \end{aligned}$$

(since c is a boundary component, $p(\tilde{t}_c) = 1$). We conclude that $t_z = 1$, contradicting the fact that z is a nontrivial circle.

Thus, the assumption that ρ is not almost twist-preserving leads to a contradiction. In view of the above, this completes the proof. □

12. Proof of Theorem 2

In this section, we deduce Theorem 2 of the Introduction from Theorem 11.7 (i.e., from Theorem 3 of the Introduction) (cf. Theorem 12.2).

Lemma 12.1. *Let S, S' be a pair of surfaces satisfying the assumptions of the first paragraph of Section 10. If $\rho : \mathrm{Mod}^*_S \to \mathrm{Mod}^*_{S'}$ is an injective homomorphism, then $\rho(\mathrm{Mod}_S) \subset \mathrm{Mod}_{S'}$.*

Proof. Suppose that, to the contrary, $\rho(\mathrm{Mod}_S)$ is not contained in $\mathrm{Mod}_{S'}$. Since Mod_S is generated by the Dehn twists about nonseparating circles, in this case $\rho(t_\alpha) \in \mathrm{Mod}^*_{S'} \setminus \mathrm{Mod}_{S'}$ for some $\alpha \in V_0(S)$. In other words, $\rho(t_\alpha)$ is an orientation-reversing element for some $\alpha \in V_0(S)$. Let us fix such an α.

Suppose that ρ is almost twist-preserving. Then $\rho(t_\alpha^N) = t_{\rho(\alpha)}^M$ for some $M, N \neq 0$. Clearly, $\rho(\alpha) = \sigma(\rho(t_\alpha^N)) = \sigma(\rho(t_\alpha t_\alpha^N t_\alpha^{-1})) = \sigma(\rho(t_\alpha)\rho(t_\alpha^N)\rho(t_\alpha)^{-1}) = \rho(t_\alpha)(\sigma(\rho(t_\alpha^N))) = \rho(t_\alpha)(\rho(\alpha))$. Thus, $\rho(t_\alpha)$ preserves the isotopy class $\rho(\alpha)$. Since $\rho(t_\alpha)$ is orientation-reversing, we have $\rho(t_\alpha)t_{\rho(\alpha)}^M\rho(t_\alpha)^{-1} = t_{\rho(\alpha)}^{-M}$. On the other hand, $\rho(t_\alpha)$ obviously commutes with $\rho(t_\alpha^N) = t_{\rho(\alpha)}^M$. Hence, we have a contradiction in this case and, so, ρ cannot be almost twist-preserving.

Since ρ is not almost twist-preserving, $\rho(\gamma)$ is an edge of $C(S')$, for any $\gamma \in V_0(S)$, in view of Lemma 10.9. Let us choose $\beta \in V_0(S)$ such that $i(\beta, \alpha) = 0$ and $\beta \neq \alpha$. Then $\rho(\alpha)$ and $\rho(\beta)$ have a unique common vertex by Lemma 10.10. Since $\rho(t_\alpha)$ commutes with both $\rho(t_\alpha^N)$ and $\rho(t_\beta^N)$ for any N, the image $\rho(t_\alpha)$ preserves both $\rho(\alpha)$ and $\rho(\beta)$ and, hence, fixes the unique common vertex of $\rho(\alpha)$ and $\rho(\beta)$. It follows that $\rho(t_\alpha)$ fixes all the vertices of $\rho(\alpha)$ and $\rho(\beta)$ (because $\rho(\alpha)$ and $\rho(\beta)$ are edges). Now, $\rho(t_\alpha^n)$ is a multitwist about $\rho(\alpha)$ for some $n \neq 0$ (cf. 10.7). Thus, $\rho(t_\alpha^n) = t_\gamma^l t_\delta^m$ for some $l, m \neq 0$, where γ, δ are the vertices of the edge $\rho(\alpha)$. Because $\rho(t_\alpha)$ fixes both γ and δ and $\rho(t_\alpha)$ is orientation-reversing, we have $\rho(t_\alpha)t_\gamma^l t_\delta^m \rho(t_\alpha)^{-1} = t_\gamma^{-l} t_\delta^{-m}$. On the other hand, $\rho(t_\alpha)$ obviously commutes with $\rho(t_\alpha^n) = t_\gamma^l t_\delta^m$. We again reached a contradiction. This completes the proof. $\square$

Theorem 12.2. *Let S be a closed orientable surface of genus at least 2. Then there is no injective homomorphism $\mathrm{Out}(\pi_1(S)) \to \mathrm{Aut}(\pi_1(S))$. In particular, the natural epimorphism $\mathrm{Aut}(\pi_1(S)) \to \mathrm{Out}(\pi_1(S))$ is nonsplit.*

Proof. Let x be a basepoint on S. Since the genus of S is at least two, the center of $\pi_1(S, x)$ is trivial. Hence, we have a short exact sequence:

$$1 \to \pi_1(S, x) \xrightarrow{\partial} \mathrm{Aut}(\pi_1(S, x)) \to \mathrm{Out}(\pi_1(S, x)) \to 1.$$

According to the the Baer-Dehn-Nielsen Theorem (cf. for example [Z], Corollary 11.7), the natural homomorphism $\pi_0(\mathrm{Diff}(S)) \to \mathrm{Out}(\pi_1(S, x))$ is an isomorphism. Hence, $\mathrm{Mod}_S^* = \pi_0(\mathrm{Diff}(S))$ is naturally isomorphic to $\mathrm{Out}(\pi_1(S, x))$.

According to Theorem 4.3 of [B], we have a short exact sequence:

$$1 \to \pi_1(S, x) \xrightarrow{\partial} \pi_0(\mathrm{Diff}(S, x)) \to \pi_0(\mathrm{Diff}(S)) \to 1.$$

By the Baer-Dehn-Nielsen Theorem and the Five Lemma, the natural homomorphisms $\pi_0(\mathrm{Diff}(S, x)) \to \mathrm{Aut}(\pi_1(S, x))$ and $\pi_0(\mathrm{Diff}(S)) \to \mathrm{Out}(\pi_1(S, x))$ yield an identification of this sequence with the previous sequence. Hence, $\pi_0(\mathrm{Diff}(S, x))$ is naturally isomorphic to $\mathrm{Aut}(\pi_1(S, x))$.

Let S' denote a surface of genus g with one hole. Clearly, there exists a map $(S', \partial S') \to (S, x)$ identifying S with the surface obtained from S' by collapsing the boundary $\partial S'$ of S' to a point. This identification induces an isomorphism $\mathrm{Mod}^*_{S'} \to \pi_0(\mathrm{Diff}(S, x))$. We conclude that $\mathrm{Mod}^*_{S'}$ is naturally isomorphic to $\mathrm{Aut}(\pi_1(S, x))$.

Suppose that there exists an injective homomorphism $\mathrm{Out}(\pi_1(S)) \to \mathrm{Aut}(\pi_1(S))$. By the previous discussion, this means that there is an injective homomorphism $\mathrm{Mod}^*_S \to \mathrm{Mod}^*_{S'}$. By Lemma 12.1, it maps Mod_S to $\mathrm{Mod}_{S'}$. Hence, there exists an injective homomorphism $\mathrm{Mod}_S \to \mathrm{Mod}_{S'}$.

Now, S is a closed surface of genus at least two and S' is a surface of the same genus with one boundary component. In particular, S' is not a closed surface of genus two and the maxima of ranks of abelian subgroups of Mod_S and $\mathrm{Mod}_{S'}$ differ by one. Hence, Theorem 11.7 implies that there exists a diffeomorphism $S \to S'$. The obvious contradiction proves that there are no injective homomorphisms $\mathrm{Out}(\pi_1(S)) \to \mathrm{Aut}(\pi_1(S))$. □

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Department of Mathematics, Michigan State University, East Lansing, MI 48824

E-mail address: ivanov@math.msu.edu

Department of Mathematics, Michigan State University, East Lansing, MI 48824

E-mail address: mccarthy@math.msu.edu