DOUBLE POINTS AND THE PROPER TRANSFORM IN SYMPLECTIC GEOMETRY

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0. Introduction

In his book, *Partial Differential Relations*, Gromov introduced the symplectic analogue of the complex analytic operations of blowing up and blowing down. Of course, in complex geometry one of the primary uses of blowing up is to resolve singularities. Gromov proposed, in §3.4.4(D) of *Partial Differential Relations*, a program for resolving the singularities of symplectic immersions with symplectic crossings via blowing up, in exact analogue with the well known complex analytic technique. The purpose of this note is to show that this program cannot work. We show that there are symplectically immersed surfaces in symplectic 4-manifolds which do not have a symplectically embedded proper transform in any blow up of the four manifold. In particular, there are double points of a symplectically immersed surface which cannot be resolved symplectically using blowing up. We produce examples of such surfaces with only double point singularities. Interestingly, double points of immersed surfaces can be resolved using blowing up in the topological category and in the complex category. Symplectic immersions, however, are flexible enough to allow negative double points yet rigid enough to satisfy a symplectic version of the adjunction formula. This combination leads to our result. While all techniques used in this paper are elementary, the main result (Corollary 4.1) that singularities of symplectic immersions cannot be resolved by blowing up is not obvious. This result shows that the resolution of symplectic singularities will require further techniques.

1. Blowing Up in Symplectic 4-Manifolds

In this section we briefly review the essential ideas of blowing up (and down) in the case of symplectic 4-manifolds. These ideas have been worked out in detail by Guillemin and Sternberg [G-S] and McDuff [McD1].

In complex analysis blowing up replaces a point p, in a complex surface X^2, by the space of complex lines through p; that is, by $\mathbb{C}P^1$. More precisely, blowing up constructs a complex manifold $\tilde{X}$ and a holomorphic map

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$\pi : \tilde{X} \to X$ such that $\pi^{-1}(p) = \mathbb{C}P^1$ and $\pi^{-1} : M \setminus p \to \tilde{M} \setminus \mathbb{C}P^1$ is a biholomorphism. The complex curve $\pi^{-1}(p) \subset \tilde{X}$, called *the exceptional curve* and usually denoted E, is an imbedded $2-$sphere with self-intersection -1.

In the symplectic category it is not possible to carry out such an infinitesimal operation, rather the construction must be made locally. The idea is to replace, not a point p, but a symplectic ball around p, by a neighborhood of $\mathbb{C}P^1$. Precisely, consider $\mathbb{C}P^1$ with area form σ normalized so that $\int_{\mathbb{C}P^1} \sigma = \pi\lambda^2$. Symplectically imbed $(\mathbb{C}P^1, \sigma)$ into a symplectic 4-manifold so that its self-intersection is -1. Denote such an embedded surface by E. By the symplectic neighborhood theorem a sufficiently small tubular neighborhood of E is determined, up to symplectomorphism, by σ and the self-intersection number -1, alone. Consider the ball $B^4(\lambda + \epsilon) \subset \mathbb{R}^4$ equipped with the standard symplectic structure on $\mathbb{R}^4$.

Proposition 1.1. *For ϵ sufficiently small there is a symplectomorphism ϕ from $B^4(\lambda+\epsilon)\setminus B^4(\lambda)$ onto $\mathcal{N}_\epsilon(E)\setminus E$, where $\mathcal{N}_\epsilon(E)$ is a tubular neighborhood of E. The map ϕ extends continuously over $\partial B^4(\lambda)$ to be the Hopf map.*

Proof. The proposition is proved by building a model for the symplectic structure on the tubular neighborhood $\mathcal{N}_\epsilon(E)$ and then giving ϕ explicitly. For details see [G-S] or [McD1]. □

The blow up of the symplectic 4-manifold (M, ω) at p depends on the choice of a symplectic embedding $\rho : B^4(\lambda) \to M,\ \rho(0) = p$. Given such an embedding, extend ρ to a symplectic embedding $\rho' : B^4(\lambda+\epsilon) \to M$. Delete $B^4(\lambda)$ and glue in $\mathcal{N}_\epsilon(E)$ via the symplectomorphism ϕ. It then follows that the resulting symplectic manifold $(\tilde{M}, \tilde{\omega})$ is (up to isotopy) independent of the choice of extension ρ' of ρ and of ϵ. $(\tilde{M}, \tilde{\omega})$ is called the *blow-up of* (M, ω) *at p of weight λ*.

Suppose (M, ω) is a symplectic 4-manifold containing a symplectically embedded $2-$sphere E of self-intersection -1 with area $\int_E \omega = \pi\lambda^2$. Then E can be *blown down* by reversing the above procedure. The resulting symplectic 4-manifold has a symplectically imbedded ball $B^4(\lambda)$ replacing the symplectic surface E.

2. The Adjunction Formula

Let (M, ω) be a symplectic 4-manifold. ω determines, up to homotopy, an almost complex structure on M and hence the Chern classes $c_i(M, \omega)$ are well-defined. Consider a symplectic immersion $f : (\Sigma, \eta) \to (M, \omega)$. By a small perturbation of f we can assume that the immersion has only simple double points. Moreover since the condition that an immersion be symplectic is open it is possible to keep the map symplectic throughout the perturbation, if we also allow η to be perturbed. Hence the symplectic immersions with only double points are a family of maps of particular interest. We have:

Theorem 2.1. *Let $f : (\Sigma, \eta) \to (M, \omega)$ be a symplectic immersion of a compact symplectic surface (Σ, η) into a symplectic 4-manifold (M, ω). Suppose that f has only double points. Then:*

$$c_1(M, \omega)[\Sigma] = 2 - 2g + \Sigma \cdot \Sigma - 2D \tag{2.1}$$

where g is the genus of Σ, D is the number of double points of f counted with sign and $\Sigma \cdot \Sigma$ is the self-intersection of the class represented by $f(\Sigma)$.

Proof. Consider f^*TM as a symplectic vector bundle over Σ. Since f is a symplectic map $T\Sigma$ is a symplectic subbundle of f^*TM. A symplectic complement to $T\Sigma$ can be canonically defined using the symplectic orthogonal to $T_x\Sigma$ at each $x \in \Sigma$. Denote this bundle ν. Then:

$$f^*TM = T\Sigma \oplus \nu$$

as symplectic vector bundles and consequently as complex vector bundles. It follows that:

$$\begin{aligned} c_1(M, \omega)[\Sigma] &= f^*c_1(TM) \\ &= c_1(f^*TM) \\ &= c_1(T\Sigma) + c_1(\nu). \end{aligned}$$

Noting that ν is the normal bundle to $f(\Sigma)$ we have:

$$c_1(\nu) = \chi(\nu) = \Sigma \cdot \Sigma - 2D$$

and:

$$c_1(T\Sigma) = \chi(\Sigma) = 2 - 2g.$$

The result follows. □

We will often abbreviate $c_1(M, \omega)[\Sigma] = c_1(M)(\Sigma)$.

Formula 2.1 for holomorphically immersed curves in Kähler surfaces follows from the classical adjunction formula of algebraic geometry. Consequently we will call 2.1 the *symplectic adjunction formula.* Analogous formulas for J-holomorphic curves in symplectic 4-manifolds are given by McDuff [McD2]. These formulas are, it should be noted, deeper than 2.1. However, a symplectically immersed compact surface in a symplectic 4-manifold with only double points need not be J-holomorphic for any almost complex structure J. Hence, formula 2.1 does not follow from these deeper results.

3. Proper Transforms

Let (M, ω) be a compact symplectic 4-manifold with first Chern class $c_1(M) = c_1(M, \omega)$. Blowing up M at the points $p_1 \cdots p_k$ constructs a new symplectic 4-manifold that we denote $(\tilde{M}_k, \tilde{\omega}_k)$. Each point p_i determines a symplectically embedded $2-$sphere E_i of self intersection -1 in $\tilde{M}_k$. Let

$[E_i]$ denote the integral homology class determined by E_i and denote its Poincare dual by $[E_i]^\sharp$. Then it follows that:

$$c_1(\tilde{M}_k) = c_1(M) - \sum_{i=1}^{k} [E_i]^\sharp \tag{3.1}$$

Further, there are neighborhoods $\mathcal{N}_i(E_i) \subset \tilde{M}_k$ of the E_i and neighborhoods $B_i(p_i) \subset M$ of the p_i such that there is a symplectomorphism:

$$\pi : \tilde{M}_k \setminus \bigcup_i N_i(E_i) \to M \setminus \bigcup_i B_i(p_i).$$

Each B_i can be taken to be the symplectic ball around p_i that determines the blow-up at p_i, and each $N_i(E_i)$ is then the tubular neighborhood of E_i determined by the symplectomorphism of Proposition 1.1.

Let $f : (\Sigma, \eta) \to (M, \omega)$ be a symplectic immersion of the surface (Σ, η) into (M, ω). Let $\tilde{f} : (\tilde{\Sigma}, \tilde{\eta}) \to (\tilde{M}_k, \tilde{\omega}_k)$ be a symplectic immersion of the surface $(\tilde{\Sigma}, \tilde{\eta})$ into $(\tilde{M}_k, \tilde{\omega}_k)$. Denote the immersed surfaces by $f(\Sigma) \subset M$ and $\tilde{f}(\tilde{\Sigma}) \subset \tilde{M}_k$, respectively. We say $\tilde{f}$ is a *proper transform of* f if:

$$\pi(\tilde{f}(\tilde{\Sigma}) \setminus \bigcup_i (\tilde{f}(\tilde{\Sigma}) \cap \mathcal{N}_i(E_i))) = f(\Sigma) \setminus \bigcup_i (f(\Sigma) \cap B_i(p_i)).$$

A proper transform is not unique, unlike the analogous definition in complex geometry. A *resolution of the singularities* of the immersion f through blowing up is a symplectic embedding $\tilde{f} : (\tilde{\Sigma}, \tilde{\eta}) \to (\tilde{M}, \tilde{\omega})$ where $(\tilde{M}, \tilde{\omega})$ is some blow up of (M, ω), such that $\tilde{f}$ is a proper transform of f.

To resolve the singularities of a given symplectic immersion f we can, as remarked above, assume that f has only isolated double points, we henceforth make this assumption. Each double point p has a sign which we will call its index and denote $i(p)$. Thus, in formula 2.1, if $\{p_1 \cdots p_r\}$ denote the double points of the immersion then

$$D = \sum_{j=1}^{r} i(p_j).$$

Theorem 3.1. *Let $f : (\Sigma, \eta) \to (M, \omega)$ be a symplectic immersion of a compact surface into a symplectic 4-manifold such that f has only isolated double points, $\{p_1 \cdots p_r\}$. If $i(p_j) = -1$ for some j then there does not exist a resolution of f through blowing up.*

Proof. Since blow up is a local construction to prove the theorem it suffices to show that there does not exist a proper transform of f whose only double points are $\{p_1 \cdots \hat{p}_j \cdots p_r\}$. In other words it suffices to show that blowing up cannot resolve a double point with index -1.

Applying the symplectic adjunction formula to $f(\Sigma)$ we have

$$c_1(M)[\Sigma] = 2 - 2g + \Sigma \cdot \Sigma - 2D \tag{3.2}$$

where $g = g(\Sigma)$ is the genus of Σ and $D = \sum_{\ell=1}^{r} i(p_\ell)$. Suppose $\tilde{M}$ is obtained from M by blowing up the point p_j and that $\tilde{f} : (\tilde{\Sigma}, \tilde{\eta}) \to (\tilde{M}, \tilde{\omega})$

is a resolution of the singularity of f at p_j. In other words, suppose that $\tilde{f}$ is a proper transform of f with double points $\{p_1 \cdots \hat{p}_j \cdots p_r\}$. Applying the adjunction formula to $\tilde{f}$ we have

$$c_1(\tilde{X})(\tilde{\Sigma}) = 2 - 2\tilde{g} + \tilde{\Sigma} \cdot \tilde{\Sigma} - 2\tilde{D} \tag{3.3}$$

where $\tilde{g} = g(\tilde{\Sigma})$ is the genus of $\tilde{\Sigma}$ and

$$\tilde{D} = D + 1. \tag{3.4}$$

$\tilde{M}$ contains a new class represented by the exceptional curve $E = E_{p_j}$. Since $\tilde{f}$ is a proper transform of f the homology classes represented by f and $\tilde{f}$ satisfy:

$$[\tilde{f}(\tilde{\Sigma})] = [f(\Sigma)] + k[E] \tag{3.5}$$

for some integer k. The adjunction formula 3.3 becomes using 3.1, 3.4 and 3.5

$$(c_1(X) - [E]^\sharp)([f(\Sigma)] + k[E]) = 2 - 2\tilde{g} + ([f(\Sigma)] + k[E])^2 - 2(D+1). \tag{3.6}$$

Hence,

$$c_1(X)[\Sigma] + k = 2 - 2\tilde{g} + \Sigma \cdot \Sigma - k^2 - 2D - 2. \tag{3.7}$$

Using 3.2 this becomes:

$$\tilde{g} - g = -\frac{1}{2}(k(k+1)) - 1. \tag{3.8}$$

Thus, regardless of the value of k, we have:

$$\tilde{g} < g. \tag{3.9}$$

Therefore the proper transform $\tilde{f}$ has strictly smaller genus than f. This means that by a local operation we have reduced the number of handles of Σ. This is clearly impossible. □

Remark 3.1. In the topological category, double points of immersed surfaces can be resolved using blowing up, regardless of whether they are positive or negative. In this category, blowing up corresponds to connect sum with $\overline{\mathbb{C}P^2}$.

4. Examples

In this section we produce examples of symplectic immersions of compact surfaces into symplectic four manifolds which have only isolated double points, some of which must have negative index.

Let (T^2, ω_1) be the 2−torus with area form ω_1 and (S^2, ω_2) the 2−sphere with area form ω_2. Set $(M, \omega_\lambda) = (T^2 \times S^2, \omega_1 \oplus \lambda\omega_2)$ where $\lambda > 0$. Let Σ be a 2−torus and define $F = (f_n, g_{-k})$:

$$\begin{aligned} F : \Sigma &\rightarrow M \\ x &\mapsto (f_n(x), g_{-k}(x)) \end{aligned}$$

where $f_n : T^2 \to T^2$ is an $n-$fold covering map and $g_{-k} : T^2 \to S^2$ is a smooth map of degree $-k$. Since f_n is an immersion, F is an immersion. Moreover:

$$F^*(\omega_1 \oplus \lambda\omega_2) = f_n^*(\omega_1) + \lambda g_{-k}^*(\omega_2). \tag{4.1}$$

Clearly, choosing λ sufficiently small F is a symplectic map. Perturb F so that it remains a symplectic immersion and has only isolated double points. The symplectic adjunction formula 2.1 gives:

$$c_1(T^2 \times S^2)[F(\Sigma)] = \chi(T^2) + \Sigma \cdot \Sigma - 2D \tag{4.2}$$

where $D =$ the total number of double points counted with sign. It follows that :

$$-2k = 0 + 2n(-k) - 2D. \tag{4.3}$$

Hence:

$$D = k(1-n). \tag{4.4}$$

For $n > 1$, D is negative, so there must be double points with negative index.

From Theorem 3.1, we see that these surfaces provide counterexamples to Gromov's program for resolving the singularities of symplectic immersions with symplectic crossings via blowing up as outlined in §3.4.4(D) of *Partial Differential Relations*. Hence, these examples establish the following corollary of Theorem 3.1.

Corollary 4.1. *There exist symplectic immersions with symplectic crossings whose singularities cannot be resolved by blowing up.*

5. Resolving Double Points with Positive Index

Double points of a symplectic immersion with index $= +1$ can, unlike double points of index $= -1$, be resolved using blowing up. In fact there are different ways of doing this reflecting the nonuniqueness of the symplectic proper transform. In this section we outline one method which has a close analogy to resolution in the complex category.

Let p be a double point of the symplectic immersion $f : (\Sigma, \eta) \to (M, \omega)$ with index $= +1$. Consider a neighborhood U of p which we symplectically identify with a neighborhood of the origin in $\mathbb{R}^4$ equipped with the standard linear symplectic form ω_0. Such a neighborhood U is called a *Darboux neighborhood of* p. By perturbing f, if necessary, we can suppose that the image of f in a neighborhood $V \subset U$ coincides with the two tangent planes of f passing through the origin.

We are thus led to consider the geometry of a pair of symplectic planes in $\mathbb{R}^4$ which intersect positively and transversely in the origin. This problem has been analyzed by McDuff [McD3]. She shows that there is an almost complex structure J tamed by ω_0 so that both planes are $J-$holomorphic lines. J in general will not be ω_0-compatible (i.e., $\omega_0(J-,-)$ will not be a hermitian metric). However, if J is ω_0- compatible, then we have:

Proposition 5.1. *Let p be a double point given locally by the intersection of two transverse symplectic planes which are $J-$holomorphic lines for a ω_0-compatible almost complex structure J. Then p can be symplectically resolved by blowing up.*

Proof. There is a $3-$sphere, $S^3_\lambda = \partial B^4(\lambda)$, centered on the origin so that the planes intersect S^3_λ in distinct fibers of the Hopf fibration of S^3_λ. Any proper transform for the blow up of p of weight λ then must intersect $\partial\mathcal{N}(E)$ in two distinct fibers of the normal circle bundle of E. Thus a symplectic proper transform which resolves p can be defined by extending across the two fibers of the normal bundle. $\square$

Remark 5.1. This construction determines a unique proper transform; closely analogous to the proper transform of complex geometry.

In the general case when J is not ω_0-compatible, it can be shown that:

Proposition 5.2. *Let $f : (\Sigma, \eta) \to (M, \omega)$ be a symplectic immersion with a positive double point at $p \in M$. Suppose in a Darboux neighborhood V of p that the image of f is two symplectic planes intersecting transversely in the origin. Then there is a smooth family $f_t : \Sigma \to (M, \omega)$ of symplectic immersions $t \in [0, 1]$, such that*

(i) $f_0 = f$
(ii) $f_t|_{M\setminus V} = f_0$
(iii) $f_t|_{V\setminus\{p\}}$ *are imbeddings*
(iv) *There is a neighborhood $W \subset V$ such that the image of $f_1|_W$ consists of two $J-$holomorphic lines intersecting transversely in the origin, where J is a ω_0-compatible almost complex structure.*

Combining Propositions 5.1 and 5.2 shows that positive double points of symplectic immersions can be resolved.

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