DISCRETENESS AND HOMOGENEITY OF THE TOPOLOGICAL FUNDAMENTAL GROUP

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ABSTRACT. For a locally path connected topological space, the topological fundamental group is discrete if and only if the space is semilocally simply-connected. While functoriality of the topological fundamental group, with target the category of topological groups, remains an open question in general, the topological fundamental group is always a homogeneous space.

1. INTRODUCTION

The concept of a natural topology for the fundamental group appears to have originated with Witold Hurewicz [8] in 1935. It received further attention in 1950 by James Dugundji [2] and more recently by Daniel K. Biss [1], Paul Fabel [3], [4], [5], [6], and others. The purpose of this note is to prove the following folklore theorem.

Theorem 1.1. Let $X$ be a locally path connected topological space. The topological fundamental group $\pi_1^{\text{top}}(X)$ is discrete if and only if $X$ is semilocally simply-connected.

Theorem 5.1 of [1] is Theorem 1.1 without the hypothesis of local path connectedness. However, a counterexample of Fabel [6] shows that this stronger result is false. Fabel [6] also proves a weaker
version of Theorem 1.1, assuming that $X$ is locally path connected and a metric space. In this note we remove the metric hypothesis.

Our proof proceeds from first topological principles, making no use of rigid covering fibrations [1] nor even of classical covering spaces. We make no use of the functoriality of the topological fundamental group, a property which was also a main result in [1, Corollary 3.4] but, in fact, is unproven [5, pp. 188–189]. Beware that the misstep in the proof of Proposition 3.1 in [1], namely the assumption that the product of quotient maps is a quotient map, is repeated in Theorem 2.1 of [7].

In general, the homeomorphism type of the topological fundamental group depends on a choice of basepoint. We say that $\pi_1^{\text{top}}(X)$ is *discrete*, without reference to a basepoint, provided $\pi_1^{\text{top}}(X, x)$ is discrete for each $x \in X$. If $x$ and $y$ are connected by a path in $X$, then $\pi_1^{\text{top}}(X, x)$ and $\pi_1^{\text{top}}(X, y)$ are homeomorphic. This fact was proved in Proposition 3.2 of [1], and a detailed proof is provided for completeness in section 4 of this paper. Theorem 1.1 now immediately implies the following.

**Corollary.** Let $X$ be a path connected and locally path connected topological space. The topological fundamental group $\pi_1^{\text{top}}(X, x)$ is discrete for some $x \in X$ if and only if $X$ is semilocally simply-connected.

As mentioned above, it is open whether $\pi_1^{\text{top}}$ is a functor from the category of pointed topological spaces to the category of topological groups. The unsettled question is whether multiplication

$$
\pi_1^{\text{top}}(X, x) \times \pi_1^{\text{top}}(X, x) \xrightarrow{\mu} \pi_1^{\text{top}}(X, x)
$$

$$
([f], [g]) \mapsto [f \cdot g]
$$

is continuous. By Theorem 1.1, if $X$ is locally path connected and semilocally simply-connected, then $\pi_1^{\text{top}}(X, x)$, and, hence, the product $\pi_1^{\text{top}}(X, x) \times \pi_1^{\text{top}}(X, x)$ are discrete and so $\mu$ is trivially continuous. Continuity of $\mu$, in general, remains an interesting question.
Lemma 5.1 below shows that if \((X, x)\) is an arbitrary pointed topological space, then left and right multiplication by any fixed element in \(\pi_1^{\text{top}}(X, x)\) are continuous self maps of \(\pi_1^{\text{top}}(X, x)\). Therefore, \(\pi_1^{\text{top}}(X, x)\) acts on itself by left and right translation as a group of self homeomorphisms. Clearly, these actions are transitive. Thus, we obtain the following result.

**Theorem 1.2.** Let \((X, x)\) be a pointed topological space. Then \(\pi_1^{\text{top}}(X, x)\) is a homogeneous space.

This note is organized as follows. Section 2 contains definitions and conventions, section 3 proves two lemmas and Theorem 1.1, section 4 addresses change of basepoint, and section 5 shows left and right translation are homeomorphisms.

## 2. Definitions and Conventions

By convention, neighborhoods are open. Unless stated otherwise, homomorphisms are inclusion induced.

Let \(X\) be a topological space and \(x \in X\). A neighborhood \(U\) of \(x\) is *relatively inessential* (in \(X\)) provided \(\pi_1(U, x) \to \pi_1(X, x)\) is trivial. \(X\) is *semilocally simply-connected* at \(x\) provided there exists a relatively inessential neighborhood \(U\) of \(x\). \(X\) is *semilocally simply-connected* provided it is so at each \(x \in X\). A neighborhood \(U\) of \(x\) is *strongly relatively inessential* (in \(X\)) provided \(\pi_1(U, y) \to \pi_1(X, y)\) is trivial for every \(y \in U\).

The fundamental group is a functor from the category of pointed topological spaces to the category of groups. Consequently, if \(A\) and \(B\) are any subsets of \(X\) such that \(x \in A \subset B \subset X\) and \(\pi_1(B, x) \to \pi_1(X, x)\) is trivial, then \(\pi_1(A, x) \to \pi_1(X, x)\) is trivial as well. This observation justifies the convention that neighborhoods are open.

If \(X\) is locally path connected and semilocally simply-connected, then each \(x \in X\) has a path connected relatively inessential neighborhood \(U\). Such a \(U\) is necessarily a strongly relatively inessential neighborhood of \(x\), as the reader may verify (see for instance, [9, Exercise 5, p. 330]).

Let \((X, x)\) be a pointed topological space and let \(I = [0, 1] \subset \mathbb{R}\). The space

\[ C_x(X) = \{ f : (I, \partial I) \to (X, x) \mid f \text{ is continuous} \} \]
is endowed with the compact-open topology. The function

\[
C_x(X) \xrightarrow{q} \pi_1(X, x) \\
f \quad \mapsto [f]
\]

is surjective, so \(\pi_1(X, x)\) inherits the quotient topology, and one writes \(\pi_1^{\text{top}}(X, x)\) for the resulting topological fundamental group. Let \(e_x \in C_x(X)\) denote the constant map. If \(f \in C_x(X)\), then \(f^{-1}\) denotes the path defined by \(f^{-1}(t) = f(1-t)\).

3. Proof of Theorem 1.1

We prove two lemmas and then Theorem 1.1.

**Lemma 3.1.** Let \((X, x)\) be a pointed topological space. If \([e_x]\) is open in \(\pi_1^{\text{top}}(X, x)\), then \(x\) has a relatively inessential neighborhood in \(X\).

**Proof:** The quotient map \(q\) is continuous and \([e_x]\) \(\subset \pi_1^{\text{top}}(X, x)\) is open, so \(q^{-1}([e_x]) = [e_x]\) is open in \(C_x(X)\). Therefore, \(e_x\) has a basic open neighborhood

\[(3.1) \quad e_x \in V = \bigcap_{n=1}^{N} V(K_n, U_n) \subset [e_x] \subset C_x(X),\]

where each \(K_n \subset I\) is compact, each \(U_n \subset X\) is open, and each \(V(K_n, U_n)\) is a subbasic open set for the compact-open topology on \(C_x(X)\). We will show that

\[U = \bigcap_{n=1}^{N} U_n\]

is a relatively inessential neighborhood of \(x\) in \(X\). Clearly, \(U\) is open in \(X\) and, by (3.1), \(x \in U\). Finally, let \(f : (I, \partial I) \to (U, x)\). For each \(1 \leq n \leq N\), we have

\[f(K_n) \subset U \subset U_n.\]

Thus, \(f \in [e_x]\) by (3.1), so \([f] = [e_x]\) is trivial in \(\pi_1(X, x)\). \(\square\)

**Lemma 3.2.** Let \((X, x)\) be a pointed topological space and let \(f \in C_x(X)\). If \(X\) is locally path connected and semilocally simply-connected, then \([f]\) is open in \(\pi_1^{\text{top}}(X, x)\).
Proof: As $q$ is a quotient map, we must show that $q^{-1}([f]) = [f]$ is open in $C_x(X)$. So let $g \in [f]$. For each $t \in I$, let $U_t$ be a path connected relatively inessential neighborhood of $g(t)$ in $X$. The sets $g^{-1}(U_t)$, where $t \in I$, form an open cover of $I$. Let $\lambda > 0$ be a Lebesgue number for this cover. Choose $N \in \mathbb{N}$ so that $1/N < \lambda$.

For each $1 \leq n \leq N$, let

$$I_n = \left[ \frac{n-1}{N}, \frac{n}{N} \right] \subset I.$$

Reindex the $U_t$'s so that

$$g(I_n) \subset U_n \text{ for each } 1 \leq n \leq N.$$

The $U_n$'s are not necessarily distinct, nor does the proof require this condition. For each $1 \leq n \leq N$, let $W_n$ denote the path component of $U_n \cap U_{n+1}$ containing $g(n/N)$, so

$$g \left( \frac{n}{N} \right) \in W_n \subset (U_n \cap U_{n+1}) \subset X. \quad (3.2)$$

Consider the basic open set

$$V = \left( \bigcap_{n=1}^{N} V(I_n, U_n) \right) \cap \left( \bigcap_{n=1}^{N-1} V \left( \left\{ \frac{n}{N} \right\}, W_n \right) \right) \subset C_x(X). \quad (3.3)$$

By construction, $g \in V$. It remains to show that $V \subset [f]$. So, let $h \in V$. As $[g] = [f]$, it suffices to show that $[h] = [g]$.

By (3.3) we have

$$h(I_n) \subset U_n \quad \text{for each } 1 \leq n \leq N \text{ and}$$

$$h \left( \frac{n}{N} \right) \in W_n \quad \text{for each } 1 \leq n \leq N - 1. \quad (3.4)$$

For each $1 \leq n \leq N - 1$, let $\gamma_n : I \rightarrow W_n$ be a continuous path such that

$$\gamma_n(0) = h \left( \frac{n}{N} \right) \quad \text{and}$$

$$\gamma_n(1) = g \left( \frac{n}{N} \right).$$
which exists by (3.2) and (3.4). Let $\gamma_0 = e_x$ and $\gamma_N = e_x$. For each $1 \leq n \leq N$, define

$$I \xrightarrow{s_n} I_n$$

$$t \xrightarrow{} \frac{1}{N} t + \frac{n-1}{N}$$

and let

$$g_n = g \circ s_n \quad \text{and} \quad h_n = h \circ s_n.$$ 

So, $g_n$ and $h_n$ are affine reparameterizations of $g|_{I_n}$ and $h|_{I_n}$, respectively. For each $1 \leq n \leq N$,

$$\delta_n = g_n \ast \gamma_n^{-1} \ast h_n^{-1} \ast \gamma_{n-1}^{-1}$$

is a loop in $U_n$ based at $g_n(0)$ (see Figure 1). As $U_n$ is a strongly rel-

![Figure 1. Loop $\delta_n = g_n \ast \gamma_n^{-1} \ast h_n^{-1} \ast \gamma_{n-1}^{-1}$ in $U_n$ based at $g_n(0)$.](image)

atively inessential neighborhood, $[\delta_n] = 1 \in \pi_1(X, g_n(0))$. Therefore, $g_n$ and $\gamma_{n-1}^{-1} \ast h_n \ast \gamma_n$ are path homotopic. In $\pi_1(X, x)$, we have

$$[h] = [h_1 \ast h_2 \ast \cdots \ast h_N]$$

$$= [\gamma_0^{-1} \ast h_1 \ast \gamma_1^{-1} \ast h_2 \ast \gamma_2^{-1} \ast \cdots \ast \gamma_{N-1}^{-1} \ast h_N \ast \gamma_N]$$

$$= [g_1 \ast g_2 \ast \cdots \ast g_N]$$

$$= [g],$$

proving the lemma. \square

In the previous proof, the second collection of subbasic open sets in (3.3) is essential. Figure 2 shows two loops $g$ and $h$ based
at $x$ in the annulus $X = S^1 \times I$. All conditions in the proof are satisfied, except $g(1/N)$ and $h(1/N)$ fail to lie in the same connected component of $U_1 \cap U_2$. Clearly, $g$ and $h$ are not homotopic loops.

![Figure 2. Loops $g$ and $h$ based at $x$ in the annulus $X$.](image)

**Proof of Theorem 1.1:** First, assume $\pi_1^{\text{top}}(X)$ is discrete and let $x \in X$. By definition, $\pi_1^{\text{top}}(X, x)$ is discrete, so $\{[e_x]\}$ is open in $\pi_1^{\text{top}}(X, x)$. By Lemma 3.1, $x$ has a relatively inessential neighborhood in $X$. The choice of $x \in X$ was arbitrary, so $X$ is semilocally simply-connected.

Next, assume $X$ is semilocally simply-connected and let $x \in X$. Points in $\pi_1^{\text{top}}(X, x)$ are open by Lemma 3.2, so $\pi_1^{\text{top}}(X, x)$ is discrete. The choice of $x \in X$ was arbitrary, so $\pi_1^{\text{top}}(X)$ is discrete. □

### 4. Basepoint change

**Lemma 4.1.** Let $X$ be a topological space and $x, y \in X$. If $x$ and $y$ lie in the same path component of $X$, then $\pi_1^{\text{top}}(X, x)$ and $\pi_1^{\text{top}}(X, y)$ are homeomorphic.

**Proof:** Let $\gamma : I \to X$ be a continuous path with $\gamma(0) = y$ and $\gamma(1) = x$. Define the function

$$
\begin{align*}
C_y(X) & \overset{r}{\longrightarrow} C_x(X) \\
\quad f & \quad \mapsto (\gamma^{-1} \ast f) \ast \gamma.
\end{align*}
$$


First, we show that $\Gamma$ is continuous. Let $I_1 = [0, 1/4]$, $I_2 = [1/4, 1/2]$, and $I_3 = [1/2, 1]$. Define the affine homeomorphisms

$$
\begin{align*}
I_1 & \xrightarrow{s_1} I \\
0 & \to 4t \\
I_2 & \xrightarrow{s_2} I \\
1/4 & \to 4t - 1 \\
I_3 & \xrightarrow{s_3} I \\
1/2 & \to 2t - 1
\end{align*}
$$

and note that

$$
\begin{align*}
\Gamma(t) & \in X \\
t & \mapsto \gamma^{-1} \circ s_1(t) & 0 \leq t \leq 1/4 \\
t & \mapsto f \circ s_2(t) & 1/4 \leq t \leq 1/2 \\
t & \mapsto \gamma \circ s_3(t) & 1/2 \leq t \leq 1.
\end{align*}
$$

Consider an arbitrary subbasic open set

$$
V = V(K, U) \subset C_x(X).
$$

Observe that $\Gamma(f) \in V$ if and only if

(4.1) $\gamma^{-1} \circ s_1(K \cap I_1) \subset U,$

(4.2) $f \circ s_2(K \cap I_2) \subset U,$ and

(4.3) $\gamma \circ s_3(K \cap I_3) \subset U.$

Define the subbasic open set

$$
V' = V(s_2(K \cap I_2), U) \subset C_y(X).
$$

Observe that $f \in V'$ if and only if (4.2) holds. As conditions (4.1) and (4.3) are independent of $f$, either $\Gamma^{-1}(V) = \emptyset$ or $\Gamma^{-1}(V) = V'$. Thus, $\Gamma$ is continuous. Next, consider the diagram

$$
\begin{array}{c}
C_y(X) \xrightarrow{\Gamma} C_x(X) \\
\downarrow \pi_1^{\text{top}} (X, y) & \pi_1^{\text{top}} (X, x) \\
q_y & \downarrow q_x \pi_1^{\text{top}} (X, y) \xrightarrow{\pi_1^{\text{top}}} \pi_1^{\text{top}} (X, x).
\end{array}
$$

The composition $q_x \circ \Gamma$ is constant on each fiber of $q_y$, so there is a unique set function making the diagram commute, namely $\pi_1^{\text{top}} (\Gamma)(f)$. As $q_y$ is a quotient map, the universal property of quotient maps [9, Theorem 11.1, p. 139] implies that $\pi_1^{\text{top}} (\Gamma)$ is continuous. It is well known that $\pi_1^{\text{top}} (\Gamma)$ is a bijection [9, Theorem 2.1, p. 327]. Repeating the above argument with the roles of
$x$ and $y$ interchanged and the roles of $\gamma$ and $\gamma^{-1}$ interchanged, we see that $\pi(\Gamma)^{-1}$ is continuous. Thus, $\pi(\Gamma)$ is a homeomorphism as desired.

\[\square\]

5. Translation

**Lemma 5.1.** Let $(X, x)$ be a pointed topological space. If $[f] \in \pi_1^{\top}(X, x)$, then left and right translation by $[f]$ are self homeomorphisms of $\pi_1^{\top}(X, x)$.

**Proof:** Fix $[f] \in \pi_1^{\top}(X, x)$ and consider left translation by $[f]$ on $\pi_1^{\top}(X, x)$

\[
\pi_1^{\top}(X, x) \xrightarrow{L[f]} \pi_1^{\top}(X, x)
\]

Plainly, $L[f]$ is a bijection of sets. Consider the commutative diagram

\[
\begin{array}{ccc}
C_x(X) & \xrightarrow{L_f} & C_x(X) \\
q \downarrow & & q \downarrow \\
\pi_1^{\top}(X, x) & \xrightarrow{L[f]} & \pi_1^{\top}(X, x),
\end{array}
\]

where $L_f$ is defined by

\[
C_x(X) \xrightarrow{L_f} C_x(X)
\]

\[
g \longmapsto f \ast g.
\]

First, we show $L_f$ is continuous. Let $I_1 = [0, 1/2]$ and $I_2 = [1/2, 1]$. Define the affine homeomorphisms

\[
I_1 \xrightarrow{s_1} I, \quad I_2 \xrightarrow{s_2} I
\]

\[
t \longmapsto 2t, \quad t \longmapsto 2t - 1
\]

and note that

\[
I \xrightarrow{f \ast g} X
\]

\[
t \longmapsto f \circ s_1(t) \quad 0 \leq t \leq \frac{1}{2}
\]

\[
t \longmapsto g \circ s_2(t) \quad \frac{1}{2} \leq t \leq 1.
\]
Consider an arbitrary subbasic open set
\[ V = V(K, U) \subset C_x(X). \]
Observe that \( f \ast g \in V \) if and only if
\[ f \circ s_1(K \cap I_1) \subset U \quad \text{and} \]
\[ g \circ s_2(K \cap I_2) \subset U. \]
(5.2)
(5.3)
Define the subbasic open set
\[ V' = V(s_2(K \cap I_2), U) \subset C_x(X). \]
Observe that \( g \in V' \) if and only if (5.3) holds. As condition (5.2) is independent of \( g \), either \( L_f^{-1}(V) = \emptyset \) or \( L_f^{-1}(V) = V' \). Thus, \( L_f \) is continuous. The composition \( q \circ L_f \) is constant on each fiber of the quotient map \( q \) and (5.1) commutes, so the universal property of quotient maps \([9, \text{Theorem } 11.1, \text{p. } 139]\) implies that \( L[f] \) is continuous.

Applying the previous argument to \( f^{-1} \), we get \( L_f^{-1} = L[f^{-1}] \) is continuous and \( L[f] \) is a homeomorphism. The proof for right translation is almost identical. \( \square \)

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