ON THE ASPHERICITY OF A SYMPLECTIC $M^3 \times S^1$

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ABSTRACT. C. H. Taubes asked whether a closed (i.e. compact and without boundary) connected oriented three-dimensional manifold whose product with a circle admits a symplectic structure must fiber over a circle. An affirmative answer to Taubes' question would imply that any such manifold either is diffeomorphic to the product of a two-sphere with a circle or is irreducible and aspherical. In this paper, we prove that this implication holds up to connect sum with a manifold which admits no proper covering spaces with finite index. It is pointed out that Thurston's geometrization conjecture and known results in the theory of three-dimensional manifolds imply that such a manifold is a three-dimensional sphere. Hence, modulo the present conjectural picture of three-dimensional manifolds, we have shown that the stated consequence of an affirmative answer to Taubes' question holds.

1. INTRODUCTION

An interesting question in symplectic topology, which was posed by C. H. Taubes, concerns the topology of closed (i.e. compact and without boundary) connected oriented three-dimensional manifolds whose product with a circle admits a symplectic structure. The only known examples of such manifolds are those which fiber over a circle. Taubes asked whether these examples are the only examples of such manifolds.

Let M be a closed oriented 3-manifold which fibers over the circle S^1. By definition, M is the total space of a locally trivial fiber bundle whose base space is the circle S^1 and whose fiber F is a closed oriented 2-manifold. Let $\pi : M \to S^1$ be the projection of this bundle. By replacing π by a lift $\tilde{\pi} : M \to S^1$ of π to an appropriate covering space $S^1 \to S^1$ of S^1 with finite index, we may assume that F is connected.

We may pull back the universal covering space $\mathbb{R} \to S^1$ of S^1 through π to obtain an infinite cyclic covering space Z of M which fibers over the line $\mathbb{R}$ with fiber F. It follows that Z is diffeomorphic to $F \times \mathbb{R}$. Thus, the universal covering space $\tilde{M}$ of M is diffeomorphic to $\tilde{F} \times \mathbb{R}$, where $\tilde{F}$ is the universal covering space of F. As is well known, $\tilde{F}$ is diffeomorphic to S^2, if $g = 0$, and to $\mathbb{R}^2$, if $g \geq 1$.

Thus, if $g = 0$, M is diffeomorphic to the product $S^2 \times S^1$ of S^2 with S^1. (Note that every orientation preserving diffeomorphism of S^2 is isotopic to the identity. Hence, there are exactly two S^2-bundles over S^1, the product $S^2 \times S^1$ of S^2 with S^1

(i.e the mapping torus of the identity map of S^2) and the twisted product of S^2 with S^1 (i.e the mapping torus of the antipodal map of S^2). Since M is orientable and the twisted product of S^2 with S^1 is nonorientable, M can not be diffeomorphic to a twisted product of S^2 with S^1.) On the other hand, if $g \geq 1$, then $\tilde{M}$ is diffeomorphic to $\mathbb{R}^3$ and, hence, M is irreducible (i.e. every (tame) 2-sphere in M bounds a 3-ball) and aspherical (i.e. the higher homotopy groups $\pi_i(M), i > 1$, of M are zero).

Suppose now that M is a closed connected oriented 3-manifold whose product $M \times S^1$ with the circle S^1 admits a symplectic structure. From the previous observations, it follows that an affirmative answer to Taubes' question would imply that either M is diffeomorphic to $S^2 \times S^1$ or M is irreducible and aspherical. Note that the universal cover of $M \times S^1$ is diffeomorphic to $\tilde{M} \times \mathbb{R}$. It follows that $M \times S^1$ is aspherical if and only if M is aspherical.

In this paper, we shall prove the following result.

Theorem 1.1. *Let M be a closed connected oriented 3-manifold whose product $M \times S^1$ with the circle S^1 admits a symplectic structure. Then M has a unique connect sum decomposition $A \# B$ satisfying the following conditions: (i) the first Betti number $b_1(A)$ of A is at least 1, (ii) either A is diffeomorphic to $S^2 \times S^1$ or A is irreducible and aspherical, and (iii) every connected covering space of B with finite index is trivial.*

Remark 1.2. Given that $b_1(A) \geq 1$ and A is irreducible, the statement that A is aspherical is redundant. Indeed, any compact oriented irreducible 3-manifold with an infinite fundamental group is aspherical, (see Chapter IV, Section 2 of [Mc]). Since our main interest in this paper concerns asphericity, we include the adjective "aspherical" in Theorem 1.1 for emphasis. On the other hand, we include the adjective "irreducible" in Theorem 1.1 in order to achieve the uniqueness clause of this theorem.

Remark 1.3. Note that Theorem 1.1 addresses the question of the extent to which a symplectic $M \times S^1$ must be aspherical. Furthermore, the previous observations imply that an affirmative answer to Taubes' question would imply that B is a 3-sphere.

Remark 1.4. Note that the factor B in Theorem 1.1 is a homology sphere. Indeed, a manifold is an homology sphere if and only if every connected cyclic covering space of the manifold with finite index is trivial. In addition, if the fundamental group $\pi_1(B)$ of this factor B is finite, then B must be a homotopy sphere.

Remark 1.5. We recall that a group G is residually finite if every nontrivial element of G is mapped nontrivially to some finite quotient group of G. In particular, if N is a connected 3-manifold with a nontrivial residually finite fundamental group, then N has a nontrivial connected covering space with finite index. Hence, if the factor B in Theorem 1.1 has a residually finite fundamental group, it must be a homotopy sphere.

Remark 1.6. A theorem of Thurston's [Th] states that the fundamental group of a Haken manifold is residually finite. As pointed out by Hempel [H], it follows that this theorem extends to the class of all 3-manifolds whose prime factors either are virtually Haken or have finite or cyclic fundamental groups. As also pointed out by Hempel [H], it is unsolved whether this class includes all closed 3-manifolds

(cf. [Th], section 6). On the other hand, Thurston's "geometrization conjecture" implies that every closed 3-manifold lies in this class. Hence, Theorem 1.1 and Thurston's "geometrization conjecture" imply that B must be a homotopy sphere. On the other hand, as is well-known, Thurston's "geometrization conjecture" also implies the Poincare conjecture. Hence, Theorem 1.1 and Thurston's "geometrization conjecture" imply that B must be diffeomorphic to a sphere. It follows that Theorem 1.1 and Thurston's "geometrization conjecture" imply that either M is diffeomorphic to $S^2 \times S^1$ or M is aspherical and, hence, that either $M \times S^1$ is diffeomorphic to $S^2 \times S^1 \times S^1$ or $M \times S^1$ is aspherical.

Remark 1.7. In [K2], Dieter Kotschick conjectured that a general 4-manifold which is symplectic for both choices of orientation must be either ruled or aspherical. Note that if $M \times S^1$ is symplectic, then it is symplectic for both choices of orientation. Furthermore, note that $S^2 \times S^1 \times S^1$ is ruled (being an S^2 bundle over $S^1 \times S^1$). We conclude, from the previous remark, that Theorem 1.1 and Thurston's "geometrization conjecture" imply that Kotschick's conjecture holds for a general 4-manifold of the form $M \times S^1$.

Here is an outline of the paper. In Section 1, we shall prove some technical lemmas about connect sum decompositions of closed connected oriented 3-manifolds whose product with a circle admits a symplectic structure. In section 2, we shall prove the main result of the paper, Theorem 1.1.

2. Restrictions on Connect Sum Decompositions of M

In this section, we shall prove some technical lemmas about connect sum decompositions of closed connected oriented 3-manifolds whose product with a circle admits a symplectic structure. Throughout this section, M denotes a closed connected oriented 3-manifold and X denotes the product $M \times S^1$ of M with the circle S^1. We assume that S^1 is equipped with the standard orientation, and X is equipped with the corresponding product orientation.

If Y is a topological space and j is a nonnegative integer, then $H_j(Y)$ will denote the jth homology group of Y with integer coefficients, $H^j(Y)$ will denote the jth cohomology group of Y with integer coefficients, and $b_j(Y)$ will denote the jth Betti number of Y, (i.e. the rank of a maximal torsion free subgroup of $H_j(Y)$).

The intersection pairing Q is an integer valued, symmetric bilinear form on $H_2(X)$ which descends to a unimodular form on the quotient of $H_2(X)$ by its torsion subgroup. $b_2^-(X)$ denotes the rank of a maximal subgroup of $H_2(X)$ on which Q is negative definite, and $b_2^+(X)$ denotes the rank of a maximal subgroup of $H_2(X)$ on which Q is positive definite. By definition, the signature σ of Q is equal to $b_2^+(X) - b_2^-(X)$. Note that $b_2(X) = b_2^-(X) + b_2^+(X)$.

Lemma 2.1. *Let M be a closed connected oriented 3-manifold and X be the product $M \times S^1$ of M with the circle S^1. Then $b_2^-(X) = b_2^+(X) = b_1(M)$.*

Proof. By Poincare duality, $b_1(M) = b_2(M)$. On the other hand, by the Kunneth formula, $H_2(X)$ is isomorphic to the direct sum of $H_2(M) \otimes H_0(S^1)$ and $H_1(M) \otimes H_1(S^1)$. Hence, $b_2(X) = b_2(M) \times b_0(S^1) + b_1(M) \times b_1(S^1) = b_2(M) + b_1(M) = 2b_1(M)$.

Let a and b be elements of $H_2(X)$ corresponding to the "summand" $H_2(M) \otimes H_0(S^1)$ of $H_2(X)$. We may represent a and b by immersed oriented surfaces A and B in X whose images lie in $M \times \{-1\}$ and $M \times \{1\}$ respectively. Since these representatives, A and B, are disjoint, $Q(a, b) = 0$.

Let c and d be elements of $H_2(X)$ corresponding to the "summand" $H_1(M) \otimes H_1(S^1)$ of $H_2(X)$. We may represent c and d by immersed surfaces $C \times S^1$ and $D \times S^1$, where C and D are immersed oriented circles in M. We may assume that C and D are disjoint, so that these representatives, $C \times S^1$ and $D \times S^1$, of c and d are disjoint. Again, we conclude that $Q(c, d) = 0$.

Hence, the intersection pairing Q is zero on each of the two "summands" $H_2(M) \otimes H_0(S^1)$ and $H_1(M) \otimes H_1(S^1)$ of $H_2(X)$. It follows that the signature σ of Q is equal to 0. On the other hand, $\sigma = b_2^+(X) - b_2^-(X)$. Hence, $b_2^-(X) = b_2^+(X)$. It follows that $2b_1(M) = b_2(X) = b_2^-(X) + b_2^+(X) = 2b_2^+(X)$ and, hence, $b_2^+(X) = b_1(M)$.

This completes the proof of Lemma 2.1. $\qquad\qquad\qquad\qquad\qquad\square$

Lemma 2.2. *Let M be a compact connected 3-manifold. Suppose that $A\#B$ is a connect sum decomposition of M such that $b_1(A) \geq 1$ and B has a nontrivial connected covering space with finite index. Then there exists a connected covering space $\tilde{M}$ of M with finite index and an embedded 2-sphere Σ in $\tilde{M}$ such that $b_1(\tilde{M}) \geq 2$ and Σ is nonseparating, (i.e. $\tilde{M} \setminus \Sigma$ is connected).*

Proof. Since M is compact and connected, A and B are each compact and connected. By assumption, there exists a connected covering space $\tilde{B}$ of B of finite index $p \geq 2$. Since $b_1(A) \geq 1$ and A is a compact 3-manifold, $H_1(A)$ is a finitely generated abelian group with a torsion free subgroup of rank at least 1. By the classification of finitely generated abelian groups, there exists an infinite cyclic quotient $\mathbb{Z}$ of $H_1(A)$. Since $\mathbb{Z}_p$ is a quotient of $\mathbb{Z}$ and $H_1(A)$ is isomorphic to the abelianization of the fundamental group $\pi_1(A)$ of A, we obtain an epimorphism $\lambda_A : \pi_1(A) \to \mathbb{Z}_p$. By covering space theory, λ_A corresponds to a connected covering space $\tilde{A}$ of A with index p.

By assumption, M is obtained from the disjoint union of A and B by removing a ball V_A from A and a ball V_B from B and gluing the resulting complements $A \setminus V_A$ and $B \setminus V_B$ together along their boundaries ∂V_A and ∂V_B by a diffeomorphism $\phi : \partial V_A \to \partial V_B$.

Since V_A is simply connected, the preimage $\tilde{V}_A$ of V_A in $\tilde{A}$ is a disjoint union of p balls, $V_{A,j}, 1 \leq j \leq p$, in $\tilde{A}$. Likewise, the preimage $\tilde{V}_B$ of V_B in $\tilde{B}$ is a disjoint union of p balls, $V_{B,j}, 1 \leq j \leq p$, in $\tilde{B}$. Note that the diffeomorphism $\phi : \partial V_A \to \partial V_B$ lifts to a diffeomorphism $\phi_j : \partial V_{A,j} \to \partial V_{B,j}$, for each j with $1 \leq j \leq p$. Hence, we may form a compact connected 3-manifold $\tilde{M}$ by removing $\tilde{V}_A$ from $\tilde{A}$ and $\tilde{V}_B$ from $\tilde{B}$, and gluing $\partial V_{A,j}$ to $\partial V_{B,j}$ by the diffeomorphism $\phi_j : \partial V_{A,j} \to \partial V_{B,j}$, for each j with $1 \leq j \leq p$. Note that $\tilde{M}$ is a connected covering space of M with index p.

Since $\tilde{A}$ is a connected 3-manifold and $\tilde{V}_A$ is a disjoint union of p balls in $\tilde{A}$, $\tilde{A} \setminus \tilde{V}_A$ is a connected 3-manifold. Likewise, $\tilde{B} \setminus \tilde{V}_B$ is a connected 3-manifold. Note that $\partial V_{A,1}$ determines a smoothly embedded 2-sphere Σ in $\tilde{M}$. The manifold N obtained by cutting $\tilde{M}$ along Σ may be constructed by removing $\tilde{V}_A$ from $\tilde{A}$ and $\tilde{V}_B$

from $\tilde{B}$, and gluing $\partial V_{A,j}$ to $\partial V_{B,j}$ by the diffeomorphism $\phi_j : \partial V_{A,j} \to \partial V_{B,j}$, for each j with $2 \leq j \leq p$. Since $p \geq 2$, we conclude that N is a connected 3-manifold. Since $\tilde{M} \setminus \Sigma$ is equal to the interior of N, it follows that $\tilde{M} \setminus \Sigma$ is connected.

Note that $\tilde{V}_A$ determines an embedding of a disjoint union of p 2-spheres in $\tilde{M}$. We may apply the Mayer-Vietoris sequence to the corresponding decomposition of $\tilde{M}$ into two submanifolds $\tilde{A} \setminus \tilde{V}_A$ and $\tilde{B} \setminus \tilde{V}_B$. In particular, we conclude that the first Betti number $b_1(\tilde{M})$ of $\tilde{M}$ satisfies the equation $b_1(\tilde{M}) = b_1(\tilde{A}) + b_1(\tilde{B}) + p - 1$. (Here, we observe that, by the Mayer-Vietoris sequence, the removal of p disjoint balls from $\tilde{A}$ ($\tilde{B}$) does not affect $H_1(\tilde{A})$ ($H_1(\tilde{B})$).) Since $\tilde{A}$ is a covering space of A with finite index, $b_1(\tilde{A}) \geq b_1(A)$. Indeed, the covering projection $\pi : \tilde{A} \to A$ induces a homomorphism $\pi_* : H_1(\tilde{A}) \to H_1(A)$ which maps $H_1(\tilde{A})$ onto a subgroup of finite index in $H_1(A)$. Thus, since $b_1(A) \geq 1$, $b_1(\tilde{A}) \geq 1$. Since $b_1(\tilde{B}) \geq 0$ and $p \geq 2$, we conclude that $b_1(\tilde{M}) = b_1(\tilde{A}) + b_1(\tilde{B}) + p - 1 \geq 1 + 0 + 2 - 1 \geq 2$.

This completes the proof of Lemma 2.2.

$\square$

Lemma 2.3. *Let M be a closed connected oriented 3-manifold for which the product $X = M \times S^1$ of M with the circle S^1 admits a symplectic structure. Suppose that $A \# B$ is a connect sum decomposition of M such that $b_1(A) \geq 1$. Then every connected covering space of B with finite index is trivial.*

Remark 2.4. The main facts used in the proof of Lemma 2.3 are familiar: (i) the vanishing of the Seiberg-Witten invariants for 4-manifolds admitting appropriate decompositions, and (ii) the nonvanishing of certain Seiberg-Witten invariants for closed symplectic 4-manifolds.

Remark 2.5. In order to apply the familiar facts mentioned in the previous remark, we must pass to an appropriate covering space $\tilde{M}$ of M. The idea of passing to a covering space to apply these familiar facts was introduced by Dieter Kotschick in [K1], and exploited further in [KMT] and [K3]. Kotschick's "covering trick" exploits a particular covering space $\tilde{X}$ of a 4-manifold X. The covering trick which we shall use in the proof of Lemma 2.3 is similar to Kotschick's covering trick, but the covering space $\tilde{M}$ of M which we employ is not the three-dimensional analogue of the covering space $\tilde{X}$ of X exploited by Kotschick. The difference between our choice of covering spaces and Kotschick's choice of covering spaces corresponds to the fact that we appeal to a different vanishing result for Seiberg-Witten invariants than that which is invoked in Kotschick's covering trick.

Proof of Lemma 2.3. Suppose, on the contrary, that B has a nontrivial connected covering space with finite index. Let $\tilde{M}$ and Σ be as in Lemma 2.2. Orient $\tilde{M}$ by pulling back the orientation on M through the covering map $\pi : \tilde{M} \to M$. Orient the embedded 2-sphere Σ in $\tilde{M}$. Since Σ is a smooth oriented nonseparating embedded 2-sphere in the oriented 3-manifold $\tilde{M}$, we may choose a smooth embedded oriented circle γ in $\tilde{M}$ such that Σ meets γ exactly at one point x in $\tilde{M}$, this point is a point of transverse intersection of Σ with γ in $\tilde{M}$, and the sign of intersection of Σ with γ at x is positive.

Let $\tilde{X}$ denote the product $\tilde{M} \times S^1$ of $\tilde{M}$ with the circle S^1. The smoothly embedded oriented 2-sphere Σ in $\tilde{M}$ determines a smoothly embedded oriented 2-sphere $\tilde{S} = \Sigma \times \{1\}$ in the closed oriented 4-manifold $\tilde{X}$. Since $\tilde{S}$ lies in the hypersurface $\tilde{M} \times \{1\}$ in $\tilde{X}$, $\tilde{S}$ has square zero (i.e. $\tilde{Q}([\tilde{S}], [\tilde{S}]) = 0$, where $\tilde{Q}$ is

the intersection pairing on $\tilde{X}$, and $[\tilde{S}]$ is the homology class in $H_2(\tilde{X})$ represented by $\tilde{S}$). The oriented circle γ in $\tilde{M}$ determines an oriented torus $\tilde{T} = \gamma \times S^1$ in $\tilde{X}$. By our previous assumptions, $\tilde{S}$ meets $\tilde{T}$ in exactly one point, $(x, 1)$, this point is a point of transverse intersection of $\tilde{S}$ with $\tilde{T}$ in $\tilde{X}$, and the sign of intersection of $\tilde{S}$ with $\tilde{T}$ at this point is positive. Hence, $\tilde{Q}([\tilde{S}], [\tilde{T}]) = 1$. It follows that $\tilde{S}$ is *essential*, (i.e $[\tilde{S}]$ is a homology class of infinite order in $H_2(\tilde{X})$).

By Lemma 2.1 and Lemma 2.2, $b_2^+(\tilde{X}) = b_1(M) \geq 2$. Thus, $\tilde{X}$ is a closed oriented 4-manifold with $b_2^+(\tilde{X}) > 1$, and $\tilde{S}$ is an essential embedded sphere in $\tilde{X}$ of nonnegative self-intersection. Hence, by Lemma 5.1 in [FS], the Seiberg-Witten invariant $SW_{\tilde{X}}$ vanishes identically.

Since $\tilde{M}$ is a covering space of M, $\tilde{X}$ is a covering space of $X = M \times S^1$. Since, by assumption, X admits a symplectic structure, $\tilde{X}$ admits a symplectic structure. Indeed, we may pull back any symplectic structure ω on X through the covering map $\pi : \tilde{X} \to X$ to obtain a symplectic structure $\pi^*\omega$ on $\tilde{X}$. Let $\tilde{\omega}$ be a symplectic structure on $\tilde{X}$. (By assumption, $\tilde{\omega} \wedge \tilde{\omega}$ gives the orientation of $\tilde{X}$.) Then, by the Main Theorem of [T], the first Chern class of the associated almost complex structure on $\tilde{X}$ has Seiberg-Witten invariant equal to ± 1.

This is a contradiction. Hence, every connected covering space of B with finite index is trivial.

This completes the proof of Lemma 2.3.   □

3. The Main Result

In this section, we shall prove the main result of this paper, Theorem 1.1. As in the previous section, M denotes a closed connected oriented 3-manifold and X denotes the product $M \times S^1$ of M with the circle S^1. We assume that S^1 is equipped with the standard orientation, and X is equipped with the corresponding product orientation.

A 3-manifold P is *non-trivial* if it is not homeomorphic to the 3-sphere S^3. We recall that a non-trivial 3-manifold P is *prime* if there is no decomposition $P = M_1 \# M_2$ of P as a connect sum with M_1 and M_2 non-trivial. In [M], Milnor showed that each closed connected oriented 3-manifold P has a unique decomposition as a connect sum of prime factors:

Theorem 1 (Milnor) . *Every nontrivial closed connected oriented 3-manifold P is isomorphic to a connect sum $P_1 \# .... \# P_k$ of prime manifolds. The summands P_i are uniquely determined up to order and isomorphism.*

We recall that a 3-manifold P is *irreducible* if every (tame) 2-sphere in P bounds a 3-ball in P. The relationship between primitivity and irreducibility for closed connected oriented 3-manifolds may be summarized by the following results from [M]:

Lemma 1 (Milnor) . *With the exception of manifolds isomorphic to S^3 or $S^2 \times S^1$, a closed connected oriented 3-manifold is prime if and only if it is irreducible.*

Lemma 2 (Milnor) . $S^2 \times S^1$ *is prime.*

Note that S^3 is irreducible, by a theorem of Alexander [A], but not prime, since S^3 is trivial. On the other hand, $S^2 \times S^1$ is prime, by Lemma 1 (Milnor), but not irreducible, since the 2-sphere $S^2 \times \{1\}$ in $S^2 \times S^1$ does not bound a 3-ball in $S^2 \times S^1$.

We recall that a topological space T is *aspherical* if all the higher homotopy groups $\pi_i(T), i > 1$, are zero. Note that neither S^3 nor $S^2 \times S^1$ is aspherical, since $\pi_3(S^3)$ and $\pi_2(S^2 \times S^1)$ are both infinite cyclic. The relationship between irreducibility and asphericity for closed connected oriented 3-manifolds may be summarized by the following result from [M]:

Theorem 2 (Milnor) . *For every non-trivial closed connected oriented irreducible 3-manifold P, the second homotopy group $\pi_2(P)$ of P is zero. If the Poincaré hypothesis is true, then, conversely, every such manifold P with $\pi_2(P) = 0$ is irreducible.*

Suppose now that P is a closed connected oriented prime 3-manifold with first Betti number $b_1(P) \geq 1$. Since $b_1(P) \geq 1$, P is not diffeomorphic to S^3. Hence, by Lemma 1 (Milnor), P is either diffeomorphic to $S^2 \times S^1$ or P is irreducible. Since $b_1(P) \geq 1$, the fundamental group $\pi_1(P)$ of P is infinite and, hence, the universal cover $\tilde{P}$ of P is not compact. In [M], Milnor observes that if, in addition to the hypotheses on P in Theorem 2 (Milnor), the universal covering space $\tilde{P}$ of P is not compact, then $\tilde{P}$ is contractible and all the higher homotopy groups $\pi_i(P), i > 1$, are zero. In other words, in this situation, P is aspherical. Hence, we have the following consequence of the above results from [M]:

Lemma 3.1. *Suppose that P is a closed connected oriented prime 3-manifold with first Betti number $b_1(P) \geq 1$. Then either P is diffeomorphic to $S^2 \times S^1$ or P is irreducible and aspherical.*

Proof of Theorem 1.1. By assumption, $X = M \times S^1$ admits a symplectic structure, ω. We may assume that the closed 2-form ω on X represents an integral cohomology class $[\omega]$ in the second cohomology $H^2(X)$ of X. Since $\omega \cup \omega$ is a positive closed 4-form on the oriented 4-manifold X, the Poincare dual $e = PD([\omega])$ is an element of $H_2(X)$ with $Q(e, e) > 0$. It follows that $b_2^+(X) \geq 1$. Thus, by Lemma 2.1, $b_1(M) \geq 1$ and, hence, M is nontrivial. Therefore, by Theorem 1 (Milnor), there exists a connect sum decomposition $M = M_1 \# ... \# M_r$ of M into prime summands M_i, which are uniquely determined up to order and isomorphism. Note that $M = S^3 \# M_1 \# ... \# M_r$.

Since $M = M_1 \# ... \# M_r$, $b_1(M) = b_1(M_1) + ... + b_1(M_r)$. It follows, from the fact that $b_1(M) \geq 1$, that $b_1(M_i) \geq 1$ for some integer i with $1 \leq i \leq r$. We may assume, without loss of generality, that $i = r$. Let $A = M_r$ and $B = S^3 \# M_1 \# ... \# M_{r-1}$. Then $M = A \# B$, $A = M_r$ is prime, and $b_1(A) = b_1(M_r) \geq 1$. By Lemma 3.1, either A is diffeomorphic to $S^2 \times S^1$ or A is irreducible and aspherical. By Lemma 2.3, every connected covering space of B with finite index is trivial.

This proves the existence of a connect sum decomposition of M with the stipulated properties.

Suppose that $M = C \# D$ is a connect sum decomposition of M where (i) $b_1(C) \geq 1$, (ii) either C is diffeomorphic to $S^2 \times S^1$ or C is irreducible and aspherical, and (iii) every connected covering space of D with finite index is trivial.

Since either C is diffeomorphic to $S^2 \times S^1$ or C is irreducible and aspherical, it follows from Lemma 1 (Milnor) and Lemma 2 (Milnor) that C is prime. By Theorem 1 (Milnor), on the other hand, there exists a connect sum decomposition $D = S^3 \# D_1 \# ... \# D_s$ of D into prime summands D_i. (Here, we allow for the possibility that D is trivial, (i.e. $s = 0$).) It follows that $M = C \# D_1 \# ... \# D_s$ is a decomposition of M into prime summands, $C, D_1, ..., D_s$. By the uniqueness clause

of Theorem 1 (Milnor), the factors $C, D_1, ..., D_s$ must be isomorphic to the factors $M_1, ..., M_r$, respectively, up to a reordering of these factors.

By Lemma 2.3, every connected covering space of B with finite index is trivial. As pointed out in Remark 1.4, this implies that B is a homology sphere. In particular, $b_1(B) = 0$. On the other hand, since $B = S^3 \# M_1 \# ... \# M_{r-1}$ and $b_1(S^3) = 0$, $b_1(B) = b_1(M_1) + ... + b_1(M_{r-1})$. It follows that $b_1(M_i) = 0$ for $1 \leq i < r$. Thus, M_r is the unique prime factor of M with positive first Betti number. Since C is such a prime factor of M, we conclude that $C = M_r = A$. Likewise, it follows that $D = S^3 \# D_1 \# ... \# D_s = S^3 \# M_1 \# ... \# M_{r-1} = B$.

This proves the uniqueness of a connect sum decomposition of M with the stipulated properties.

This completes the proof of Theorem 1.1.   □

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