WEIERSTRASS POINTS AND $\mathbf{Z}_2$ HOMOLOGY

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0. Introduction

In a recent paper [Lu], Lustig established a beautiful connection between the 6 Weierstrass points on a Riemann surface M_2 of genus 2 and intersection points of closed geodesics for the associated hyperbolic metric. As a consequence, he was able to construct an action of the mapping class group $Out(\pi_1 M_2)$ of M_2 on the Weierstrass points of M_2 which afforded an epimorphism $Out(\pi_1 M_2) \to S_6$ ([Lu], Lemma 3.4). (Here S_6 denotes the symmetric group on 6 elements.) Furthermore, he showed that the stabilizer in $Out(\pi_1 M_2)$ of a single Weierstrass point P acts naturally on $\pi_1 M_2$ affording a virtual splitting of $Aut(\pi_1 M_2) \to Out(\pi_1 M_2)$ ([Lu], Theorem 3.5). Our discussion in this paper begins with the observation that these two results of Lustig's are direct consequences of the work of Birman and Hilden ([B-H]) on equivariant homotopies for surface homeomorphisms.

It is a well-known fact of finite group theory that there is an exceptional isomorphism $S_6 \to Sp_4(\mathbf{Z}_2)$ ([O]). On the other hand, it is a well-known fact of surface topology that $Out(\pi_1 M_2)$ acts on $H_1(M_2, \mathbf{Z}_2)$ affording an epimorphism $Out(\pi_1 M_2) \to Sp_4(\mathbf{Z}_2)$. (In this context $Sp_4(\mathbf{Z}_2)$ arises as the automorphisms of $H_1(M_2, \mathbf{Z}_2)$ which preserve the Z_2-valued intersection pairing on $H_1(M_2, \mathbf{Z}_2)$.) In this paper, we show that the exceptional isomorphism $S_6 \to Sp_4(\mathbf{Z}_2)$ of finite group theory arises from a natural connection between the Weierstrass points on M_2 and $H_1(M_2, \mathbf{Z}_2)$. As a consequence, we show that the exceptional isomorphism $S_6 \to Sp_4(\mathbf{Z}_2)$ identifies Lustig's representation $Out(\pi_1 M_2) \to S_6$ with the Z_2 symplectic representation $Out(\pi_1 M_2) \to Sp_4(\mathbf{Z}_2)$.

Here is an outline of the paper. In section 1, using the work of Birman and Hilden referred to above, we construct an action of $Out(\pi_1 M_2)$ on the set of Weierstrass points of M_2 and a virtual splitting of $Aut(\pi_1 M_2) \to Out(\pi_1 M_2)$. In section 2, we develop the connection between the Weierstrass points of M_2 and $H_1(M_2, \mathbf{Z}_2)$ and the corresponding actions of $Out(\pi_1 M_2)$. We identify the kernel of the action of $Out(\pi_1 M_2)$ on the set of Weierstrass points of M_2 given in section 1. In addition, we give an independent proof of Lustig's condition for simple closed curves on M_2 ([Lu], Theorem 3.2). Finally, in section 3, we show that our action and virtual splitting agree with those constructed by Lustig. In addition, we give an independent proof

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of Lustig's result relating intersection points of base pairs and Weierstrass points on M_2 ([Lu], Theorems 2.3 and 2.4).

1. An action and a virtual splitting

Let M_2 be a closed Riemann surface of genus 2. Let W denote the set of 6 Weierstrass points of M_2. Let $i : M_2 \to M_2$ be the hyperelliptic involution. The set of fixed points of i is equal to W ([F-K], pp. 101 - 102). The action of i on M_2 affords a 2-fold branched covering map $q : M_2 \to S^2$ branched over the 6 points of $q(W)$ in S^2. (See Figure 1.)

Figure 1

As a 2-fold branched covering of S^2 branched over $q(W)$, q is classified by a homomorphism $\lambda : H_1(S^2 \setminus q(W)) \to \mathbf{Z}_2$. For each Weierstrass point P of M_2, let β_P be a small loop in $S^2 \setminus q(W)$ around the point $q(P)$. $H_1(S^2 \setminus q(W), \mathbf{Z})$ is generated by the homology classes of the loops β_P. Let α_P be the preimage of β_P in $M_2 \setminus W$. α_P is a small loop in $M_2 \setminus W$ around the point P. (See Figure 2.) Since P is an isolated fixed point of the orientation preserving hyperelliptic involution i, the restriction of q to α_P is a two fold covering map $q| : \alpha_P \to \beta_P$. Hence, λ assigns 1 to the homology class of each loop β_P.

Suppose that $g : M_2 \to M_2$ is a homeomorphism of M_2. We say that g *preserves the fibers of* q if $q(x) = q(y)$ implies that $q(g(x)) = q(g(y))$. If $h : S^2 \to S^2$ is a homeomorphism of S^2 for which $h \circ q = q \circ g$, we say that g is *a lift of* h. It is easy to see that the following are equivalent:

- g preserves the fibers of q,
- g is the lift of a homeomorphism of S^2,
- g commutes with i.

FIGURE 2

If g commutes with i, then g must preserve the fixed point set of i. Hence, if $Homeo(M_2, i)$ denotes the group of homeomorphisms of M_2 which commute with i, $Homeo(M_2, i)$ acts on W. Since we have chosen a labeling of the points of W, $W = \{P_1, ..., P_6\}$, this action affords a representation $\rho : Homeo(M_2, i) \to S_6$.

Lemma 1.1. *The representation $\rho : Homeo(M_2, i) \to S_6$ is surjective.*

Proof. Let $\sigma \in S_6$. Let h be a homeomorphism of S^2 such that $h(q(P_i)) = q(P_{\sigma(i)})$. Let $(h|)_*$ be the automorphism of $H_1(S^2 \setminus q(W), \mathbf{Z})$ induced by the restriction $h| : S^2 \setminus q(W) \to S^2 \setminus q(W)$. Clearly, $(h|)_*$ maps the homology class of a small loop around $q(P_i)$ to the homology class of a small loop around $q(P_{\sigma(i)})$. Thus, from the description of λ given above, we conclude that $\lambda \circ (h|)_* = \lambda$. It follows from elementary covering space theory that h lifts to a homeomorphism g of M_2 such that $g(P_i) = P_{\sigma(i)}$. Since g is a lift of a homeomorphism of S^2, $g \in Homeo(M_2, i)$. □

Let Γ_2 be the full (or extended) mapping class group of M_2 and Γ_2^+ be the mapping class group of M_2. Γ_2^+ is the subgroup of index 2 in Γ_2 consisting of the mapping classes of orientation preserving homeomorphisms of M_2. As observed in the proof of Theorem 4.8 of [B], there exists a collection of twist maps $g_i : M_2 \to M_2, i = 1, ..., 5$ such that:

- g_i is the lift of a homeomorphism of S^2 for $i = 1, ..., 5$,
- Γ_2^+ is generated by the isotopy classes of $g_i, i = 1, ..., 5$.

Let h_0 be an orientation reversing homeomorphism of S^2 which fixes each point of $q(W)$. By the proof of Lemma 1.1, h_0 lifts to a homeomorphism g_0 of M_2 which fixes each point of W. Since h_0 is orientation reversing, g_0 is orientation reversing. Hence, Γ_2 is generated by the isotopy classes of

$g_i, i = 0, ..., 5$. Since g_i is a lift of a homeomorphism of S^2, g_i is an element of $Homeo(M_2, i)$, $i = 0, ..., 5$. Hence, we have an epimorphism:

Lemma 1.2. *The natural homomorphism $\eta : Homeo(M_2, i) \to \Gamma_2$ is surjective.*

Proposition 1.1. *There exists a unique representation r such that the following diagram commutes:*

$$\begin{array}{ccc} Homeo(M_2, i) & \xrightarrow{\eta} & \Gamma_2 \\ & \searrow^{\rho} & \downarrow r \\ & & S_6. \end{array}$$

The associated action of Γ_2 on the set of Weierstrass points of M_2 is given by the rule $\tau \cdot P = g(P)$ for every Weierstrass point P of M_2, every mapping class τ in Γ_2 and every homeomorphism $g \in Homeo(M_2, i)$ representing τ. Moreover, r is surjective.

Proof. The second and third statements follow immediately from the first statement and Lemma 1.1. Suppose that h is in the kernel of η. We must show that h is in the kernel of ρ. By our assumption and the previous observations, h is a homeomorphism of M_2 which respects the fibers of q and is isotopic to the identity. By Theorem 4.7 of [B], there is an isotopy h_t between $h = h_0$ and $id = h_1$ such that for each $t \in [0, 1]$ the map h_t is fiber-preserving. By the previous observations, each homeomorphism h_t acts on the set of Weierstrass points of M_2. Since this is a discrete set of points, it follows that $h_t(P) = h_0(P)$ for all $t \in [0, 1]$ and all $P \in W$. Since $h_1 = h$ and $h_0 = id$, we conclude that h is in the kernel of ρ. □

By our previous discussion, Γ_2 is generated by Γ_2^+ and the mapping class of an orientation reversing homeomorphism g_0 of M_2 which fixes each Weierstrass point of M_2. Since Γ_2^+ is generated by the mapping classes of Dehn twists about nonseparating simple closed curves, r is completely determined by the action of such classes. We now describe the action of these classes.

Lemma 1.3. *Let c be an isotopy class of unoriented nonseparating simple closed curves on M_2. There exists a nonseparating simple closed curve $\gamma \in c$ such that $i(\gamma) = \gamma$.*

Proof. This is an easy consequence of Theorem 3.2 of [Lu]. We now give an independent proof which illustrates the nature of our arguments in this paper.

Let P and Q be a pair of distinct Weierstrass points. Let J be an embedded arc in S^2 such that J meets $q(W)$ precisely at its endpoints $q(P)$ and $q(Q)$. The preimage $\gamma_0 = q^{-1}(J)$ in M_2 is a nonseparating simple closed curve on M_2 and $i(\gamma_0) = \gamma_0$. Since γ_0 is a nonseparating simple closed curve on M_2, there exists a homeomorphism h such that $h(\gamma_0)$ represents c. By Lemma 1.2, there exists a homeomorphism h' such that h is isotopic to h' and $i \circ h' = h' \circ i$. Let $\gamma = h'(\gamma_0)$. Since γ_0 is a nonseparating simple closed curve, γ is a nonseparating simple closed curve. Since h' is isotopic to h and $h(\gamma_0) \in c$, $\gamma \in c$. Finally, since $i(\gamma_0) = \gamma_0$, $i(\gamma) = i(h(\gamma_0)) = h(i(\gamma_0)) = h(\gamma_0) = \gamma$. □

Lemma 1.4. *Let γ be a nonseparating simple closed curve on M_2 such that $i(\gamma)$ is equal to γ. Then γ contains exactly two Weierstrass points of M_2.*

Proof. The restriction of i to γ is an involution $i|$ of a circle. Since i has only finitely many fixed points, $i|$ is a nontrivial involution. If $i|$ is orientation preserving, then $i|$ has no fixed points. On the other hand, if $i|$ is orientation reversing, then $i|$ has exactly two fixed points. It suffices, therefore, to show that $i|$ has at least one fixed point.

Suppose that $i|$ has no fixed points. Then the restriction of q to γ gives a two fold covering $q| : \gamma \to q(\gamma)$ and $\gamma = q^{-1}(q(\gamma))$. The image $q(\gamma)$ is an embedded simple closed curve in S^2. Hence, $q(\gamma)$ bounds a disc D in S^2. Since $\gamma = q^{-1}(q(\gamma))$, γ bounds the surface $q^{-1}(D)$ in M_2. Since γ is nonseparating, this is impossible. □

Proposition 1.2. *Let γ be a nonseparating simple closed curve on M_2 such that $i(\gamma) = \gamma$ and P and Q be the two Weierstrass points of M_2 on γ. Let $\tau_\gamma \in \Gamma_2$ be the mapping class of a Dehn twist about γ. Then the action of τ_γ on the set of Weierstrass points of M_2 is given by the transposition of P and Q.*

Proof. Let $J = q(\gamma)$ be the image of γ in S^2. Since the restriction of i to γ is an involution with two fixed points P and Q, J is an embedded arc in S^2 joining $q(P)$ to $q(Q)$ and $\gamma = q^{-1}(J)$. Since P and Q are the only Weierstrass points of M_2 on γ, $q(P)$ and $q(Q)$ are the only points of $q(W)$ on J. Let D be a regular neighborhood of J in S^2 such that P and Q are the only Weierstrass points of M_2 in $q^{-1}(D)$. Let h be a homeomorphism of S^2 which fixes $S^2 \setminus D$ pointwise and permutes $q(P)$ and $q(Q)$. We assume that the restriction of h to D represents the standard generator of the braid group on two strings. h lifts to a homeomorphism g of M_2 which represents τ_γ and permutes P and Q. Since g is a lift of a homeomorphism of M_2, $g \in Homeo(M_2, i)$. Hence, by Proposition 1.1, the action of τ_γ on W is equal to the action of g on W. □

Lemma 1.5. *Let c be an isotopy class of unoriented nontrivial separating simple closed curves on M_2. There exists a nontrivial separating simple closed curve $\gamma \in c$ such that $i(\gamma) = \gamma$.*

Proof. This is an easy consequence of Theorem 3.2 of [Lu]. We give an independent proof.

Let D be a disc in S^2 such that D meets $q(W)$ in precisely 3 points none of which lie on the boundary β of D. The preimage $\gamma_0 = q^{-1}(\beta)$ is a simple closed curve in M_2 bounding the preimage $T = q^{-1}(D)$. Since T is a two-fold branched cover of D branched over 3 points, T is a torus with one hole. Hence, γ_0 separates M_2 into two tori with one hole each. Thus γ_0 is a nontrivial separating simple closed curve on M_2. Moreover, since $\gamma_0 = q^{-1}(\beta)$, $i(\gamma_0) = \gamma_0$. Since γ_0 is a nontrivial separating simple closed curve on M_2, there exists a homeomorphism h such that $h(\gamma_0)$ represents c. The result follows as in the proof of Lemma 1.3. □

Lemma 1.6. *Let γ be a nontrivial separating simple closed curve on M_2 such that $i(\gamma)$ is equal to γ. Then γ contains no Weierstrass points of M_2.*

Proof. The restriction of i to γ is an involution $i|$ of a circle. Since i has only finitely many fixed points, $i|$ is a nontrivial involution. If $i|$ is orientation preserving, then $i|$ has no fixed points. On the other hand, if $i|$ is orientation reversing, then $i|$ has exactly two fixed points. It suffices, therefore, to show that $i|$ has no fixed points.

Suppose that $i|$ has a fixed point. Then $i|$ has two fixed points P and Q and the image $q(\gamma)$ is an embedded arc J which meets $q(W)$ precisely at its endpoints $q(P)$ and $q(Q)$ and $\gamma = q^{-1}(J)$. This is impossible, since it implies that γ is a nonseparating simple closed curve in M_2. □

We are now able to give an independent proof of Lustig's criterion for simple closed curves.

Theorem 1.1 (Lustig). *Consider the presentation of $\pi_1 M_2$ arising from an appropriate edge pairing of an octagon $\pi_1 M_2 =< a, b, c, d | abcda^{-1}b^{-1}c^{-1}d^{-1} >$ and the automorphism $j : \pi_1 M_2 \to \pi_1 M_2$ which maps each of the generators a, b, c and d to its inverse. Let $w \in \pi_1 M_2$ be such that the homotopy class $[w]$ contains a simple closed curve C on M_2. If C is separating, it follows that $j(w)$ is conjugate to w. If C is nonseparating, then $j(w)$ is conjugate to w^{-1}.*

Proof. Let $P_1, ..., P_6$ be the Weierstrass points of M_2. Let $J_i, i = 1, ..., 4$ be a collection of embedded arcs in S^2 with the following properties:

- J_i and J_j meet precisely at $q(P_1)$ for each distinct pair i and j,
- J_i meets $q(W)$ precisely at its endpoints $q(P_1)$ and $q(P_{i+1})$,
- the indexing of these arcs agrees with their cyclic ordering around $q(P_1)$.

Let $\gamma_j = q^{-1}(J_j)$ so that γ_j is a nonseparating simple closed curve on M_2 such that $q(\gamma_j) = \gamma_j$ and P_1 and P_{j+1} are the two Weierstrass points of M_2 on γ_j. (See Figure 3.) By our previous remarks, the restriction of i to γ_j is an orientation reversing involution with fixed points P_1 and P_{j+1}. Hence, if x_j denotes the element of $\pi_1(M_2, P_1)$ represented by γ_j, $i_*(x_j) = x_j^{-1}$.

If we orient the curves γ_j, we see that the cyclic ordering of the curves γ_j around P_1 is $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_1^{-1}, \gamma_2^{-1}, \gamma_3^{-1}, \gamma_4^{-1}$. If we cut M_2 open along the curves γ_j we obtain a surface F which is a 2-fold branched cover over the complement D of the arcs J_j in S^2. Since D is a disc with one branch

FIGURE 3

point $q(P_6)$, F is a disc. On the other hand, our observation regarding the cyclic ordering implies that the boundary of F is represented by the word $\gamma_1\gamma_4\gamma_3^{-1}\gamma_2\gamma_1^{-1}\gamma_4^{-1}\gamma_3\gamma_2^{-1}$. (See Figure 4.) Hence, since M_2 is obtained from F by the obvious edge pairing, we obtain the presentation of $\pi_1 M_2$ given in the theorem provided we let $a = x_1$, $b = x_4$, $c = x_3^{-1}$ and $d = x_2$. Moreover, the hyperelliptic involution i induces the given automorphism j.

FIGURE 4

Suppose that $[w]$ contains a separating simple closed curve γ. We may assume that γ is nontrivial. By Lemma 1.5, we can assume that $i(\gamma) = \gamma$.

Hence, by Lemma 1.6, the restriction of i to γ has no fixed points. Therefore, i must preserve the orientation of γ. Thus $j(w)$ is conjugate to w.

Suppose, on the other hand, that $[w]$ contains a nonseparating simple closed curve γ. By Lemma 1.3, we can assume that $i(\gamma) = \gamma$. Hence, by Lemma 1.4, the restriction of i to γ has exactly two fixed points. Therefore, i must reverse the orientation of γ. Thus $j(w)$ is conjugate to w^{-1}. □

Let P be a Weierstrass point on M_2. Let $Homeo(M_2, i, P)$ be the stabilizer in $Homeo(M_2, i)$ of P with respect to the representation ρ. Since $Homeo(M_2, i, P)$ is a group of homeomorphisms of the pointed space (M_2, P), we have a natural action of $Homeo(M_2, i, P)$ on $\pi_1(M_2, P)$. This action affords a representation $\sigma : Homeo(M_2, i, P) \to Aut(\pi_1 M_2)$. Let $\Gamma_2(P)$ be the stabilizer in Γ_2 of P with respect to the representation r. By Lemma 1.2 and Proposition 1.1, η restricts to an epimorphism $\eta| : Homeo(M_2, i, P) \to \Gamma_2(P)$.

Theorem 1.2. *There exists a unique representation s such that the following diagram commutes:*

$$\begin{array}{ccc} Homeo(M_2, i, P) & \xrightarrow{\eta|} & \Gamma_2(P) \\ & \searrow^{\sigma} & \downarrow^{s} \\ & & Aut(\pi_1(M_2, P)). \end{array}$$

The representation s is given by the rule $s(\tau) = g_*$ for any mapping class $\tau \in \Gamma_2$ and any homeomorphism $g \in Homeo(M_2, i, P)$ representing τ. Moreover, s is a virtual splitting of the natural homomorphism $Aut(\pi_1(M_2, P)) \to Out(\pi_1(M_2, P))$.

Proof. The second and third statements are immediate consequences of the first statement, the surjectivity of $\eta|$ and the definition of σ. Suppose that h is in the kernel of $\eta|$. We must show that h is in the kernel of σ. By our assumption and the previous observations, h is a homeomorphism of M_2 which respects the fibers of q and is isotopic to the identity. By Theorem 4.7 of [B], there is an isotopy h_t between $h = h_0$ and $id = h_1$ such that for each $t \in [0, 1]$ the map h_t is fiber-preserving. By the previous observations, each homeomorphism h_t acts on the set of Weierstrass points of M_2. Since this is a discrete set of points, it follows that $h_t(P) = h_0(P)$ for all $t \in [0, 1]$. Thus, h is isotopic to id relative to P and, consequently, the action of h on $\pi_1(M_2, P)$ agrees with that of id. Hence, h is in the kernel of σ. □

Proposition 1.3. *Let $\tau \in \Gamma_2(P)$. Then $s(\tau)$ is the unique automorphism ϕ in the outer automorphism class τ such that $\phi \circ i_* = i_* \circ \phi$.*

Proof. Let $h \in Homeo(M_2, i, P)$ represent the mapping class τ. Then $i \circ h = h \circ i$ and $h(P) = P$. By Theorem 1.2, $s(\tau) = h_*$. Since $i \circ h = h \circ i$, it follows that $h_* \circ i_* = i_* \circ h_*$. It remains to prove the uniqueness statement.

Suppose that $\phi \in Aut(\pi_1(M_2, P))$ is a representative of the outer automorphism class τ and $\phi \circ i_* = i_* \circ \phi$. Since ϕ and h_* represent the same outer automorphism class τ, $\phi = \chi^c \circ h_*$ where χ^c denotes the inner automorphism of $\pi_1(M_2, P)$ corresponding to an element c of $\pi_1(M_2, P)$. Since h_* and ϕ both commute with i_*, we conclude that χ^c commutes with i_*. Since the center of $\pi_1(M_2, P)$ is trivial, this implies that $i_*(c) = c$. Hence, by Lemma 1.1 of [B-H], we conclude that c is the identity element of $\pi_1(M_2, P)$. Hence, $\phi = h_*$. □

2. Weierstrass points and Z_2-homology

The restriction $q|$ of q to $M_2 \setminus W$ and $S^2 \setminus q(W)$ induces a homomorphism $(q|)_* : H_1(M_2 \setminus W, \mathbf{Z}_2) \to H_1(S^2 \setminus q(W), \mathbf{Z}_2)$. Since M_2 is obtained from $M_2 \setminus W$ by replacing the points of W, the inclusion $inc : M_2 \setminus W \to M_2$ induces an epimorphism $inc_* : H_1(M_2 \setminus W, \mathbf{Z}_2) \to H_1(M_2, \mathbf{Z}_2)$. For each Weierstrass point P of M_2, let $\alpha_P \subset M_2 \setminus W$ and $\beta_P \subset S^2 \setminus q(W)$ be small loops around P and $q(P)$ respectively as defined in section 1 and depicted in Figure 2. $H_1(S^2 \setminus q(W), \mathbf{Z}_2)$ is generated by the homology classes of the loops β_P. Indeed, $H_1(S^2 \setminus q(W), \mathbf{Z}_2)$ is naturally isomorphic to the quotient of the free Z_2 module on the loops β_P by the single relation:

$$\sum_{P \in W} \beta_P = 0. \tag{2.1}$$

The kernel of inc_* is generated by the homology classes of the loops α_P. On the other hand, since P is an isolated fixed point of the orientation preserving hyperelliptic involution i, we conclude that $q_*([\alpha_P]) = 2[\beta_P] = 0 \in H_1(S^2 \setminus q(W), \mathbf{Z}_2)$. Hence, we have the following lemma.

Lemma 2.1. *There exists a unique homomorphism ω such that the following diagram commutes:*

$$\begin{array}{ccc} H_1(M_2 \setminus W, \mathbf{Z}_2) & \xrightarrow{inc_*} & H_1(M_2, \mathbf{Z}_2) \\ & \searrow^{(q|)_*} & \downarrow \omega \\ & & H_1(S^2 \setminus q(W), \mathbf{Z}_2). \end{array}$$

The homomorphism ω is given by the rule $\omega(v) = (q|)_*(v')$ for every homology class $v \in H_1(M_2, \mathbf{Z}_2)$ and $v' \in H_1(M_2 \setminus W, \mathbf{Z}_2)$ such that $inc_*(v') = v$.

Lemma 2.2. *Let γ be a nonseparating simple closed curve such that $i(\gamma) = \gamma$ and v be the homology class of γ in $H_1(M_2, \mathbf{Z}_2)$. Let P and Q be the two Weierstrass points of M_2 on γ. Then $\omega(v) = \beta_P + \beta_Q$.*

Proof. The image $J = q(\gamma)$ is an embedded arc in S^2 which meets $q(W)$ precisely at its endpoints $q(P)$ and $q(Q)$. Let δ be the boundary of a regular neighborhood D of J. We assume that $q(P)$ and $q(Q)$ are the only points of $q(W)$ in D. In particular, this implies that δ represents the homology class $\beta_P + \beta_Q$. The preimage $A = q^{-1}(D)$ is a regular neighborhood of γ and P and Q are the only points of W in A. Let γ' be one of the boundary components of A. Then $\gamma' \in v$, $\gamma' \subset M_2 \setminus W$ and $(q|)_*([\gamma']) = [\delta]$. □

Lemma 2.3. *Let v be a nontrivial homology class in $H_1(M_2, \mathbf{Z}_2)$. There exists a unique pair of distinct Weierstrass points P and Q such that $\omega(v) = \beta_P + \beta_Q$.*

Proof. Suppose that v is a nontrivial element of $H_1(M_2, \mathbf{Z}_2)$. Then there is a nonseparating simple closed curve γ representing v. By Lemma 1.3, we can assume that $i(\gamma) = \gamma$. Thus, by Lemma 1.4, there are precisely two Weierstrass points P and Q on γ. Hence, by Lemma 2.2, $\omega(v) = \beta_P + \beta_Q$. This proves the existence statement. The uniqueness follows from the previous description of $H_1(S^2 \setminus q(W), \mathbf{Z}_2)$ in terms of the relation 2.1. □

Lemma 2.4. *The homomorphism $\omega : H_1(M_2, \mathbf{Z}_2) \to H_1(S^2 \setminus q(W), \mathbf{Z}_2)$ is injective.*

Proof. Suppose that v is a nontrivial element of $H_1(M_2, \mathbf{Z}_2)$. Then, by Lemma 2.3, there are precisely two Weierstrass points P and Q, such that $\omega(v) = \beta_P + \beta_Q$. By the previous description of $H_1(S^2 \setminus q(W), \mathbf{Z}_2)$ in terms of the relation 2.1, we conclude that $\beta_P + \beta_Q \neq 0$. □

Lemma 2.5. *Let P and Q be a distinct pair of Weierstrass points of M_2. Then there exists a unique nontrivial homology class v in $H_1(M_2, \mathbf{Z}_2)$ such that $\omega(v) = \beta_P + \beta_Q$.*

Proof. The uniqueness follows from Lemma 2.4. Let J be an embedded arc in S^2 meeting $q(W)$ precisely at its endpoints $q(P)$ and $q(Q)$ and let $\gamma = q^{-1}(J)$. Then γ is a nonseparating simple closed curve in M_2, $i(\gamma) = \gamma$ and P and Q are the two Weierstrass points of M_2 on γ. Let v be the homology class of γ. By Lemma 2.2, $\omega(v) = \beta_P + \beta_Q$. □

Henceforth, if v is a nontrivial homology class in $H_1(M_2, \mathbf{Z}_2)$ and P and Q are the unique pair of Weierstrass points such that $\omega(v) = \beta_P + \beta_Q$, we say that *$P$ and Q are the two Weierstrass points of M_2 on v*. Lemmas 2.3 and 2.5 establish a bijection:

$$\Omega : H_1(M_2, \mathbf{Z}_2)^* \to S_2^*(W)$$

between the set $H_1(M_2, \mathbf{Z}_2)^*$ of nontrivial homology classes in $H_1(M_2, \mathbf{Z}_2)$ and the set $S_2^*(W)$ of pairs of distinct Weierstrass points of M_2. $\Omega(v) = \{P, Q\}$ if and only if $\omega(v) = \beta_P + \beta_Q$. The naturality of Ω is expressed in the following lemma.

Lemma 2.6. *Let $v \in H_1(M_2, \mathbf{Z}_2)^*$ and $\tau \in \Gamma_2$. If $\Omega(v) = \{P, Q\}$, then $\Omega(\tau \cdot v) = \{\tau \cdot P, \tau \cdot Q\}$.*

Proof. Let $g \in Homeo(M_2, i)$ represent the mapping class τ. Since g commutes with i, $g(W) = W$. Moreover, there exists a homeomorphism h of

S^2 such that $h \circ q = q \circ g$. Hence, $h(q(W)) = q(g(W)) = q(W)$. Hence, g restricts to a homeomorphism $g|$ of $M_2 \setminus W$ and h restricts to a homeomorphism $h|$ of $S^2 \setminus q(W)$. Since $h \circ q = q \circ g$, $h| \circ q| = q| \circ g|$, where $q|$ is the restriction of q to $M_2 \setminus W$ and $S^2 \setminus q(W)$. Applying $H_1(\_, \mathbf{Z}_2)$, we conclude that $(q|)_* \circ (g|)_* = (h|)_* \circ (q|)_*$. From the definition of ω, it follows that $\omega \circ (g)_* = (h|)_* \circ \omega$.

By the definition of the action of Γ_2 on W, $\tau \cdot P = g(P)$ and $\tau \cdot Q = g(Q)$. On the other hand, $\tau \cdot v = g_*(v)$. Since $h \circ q = q \circ g$, $h(q(P)) = q(g(P))$ and $h(q(Q)) = q(g(Q))$. Hence, $(h|)_*(\beta_P) = \beta_{g(P)}$ and $(h|)_*(\beta_Q) = \beta_{g(Q)}$. By assumption, $\omega(v) = \beta_P + \beta_Q$. Since $\omega \circ (g)_* = (h|)_* \circ \omega$, we conclude that:

$$\omega(g_*(v)) = (h|)_*(\omega(v)) = (h|)_*(\beta_P + \beta_Q) = \beta_{g(P)} + \beta_{g(Q)}.$$

□

Let $<,>$ be the $\mathbf{Z}_2$-valued intersection form on $H_1(M_2, \mathbf{Z}_2)$.

Lemma 2.7. *Let v_1 and v_2 be two nontrivial homology classes in $H_1(M_2, \mathbf{Z}_2)$ and P_i and Q_i be the two Weierstrass points of M_2 on v_i, $i = 1, 2$. Then $< v_1, v_2 >$ is equal to the congruence class modulo 2 of the number of points in $\{P_1, Q_1\} \cap \{P_2, Q_2\}$.*

Proof. Let J_j be an embedded arc in S^2 joining $q(P_j)$ to $q(Q_j)$. We may assume that $J_1 \cap J_2 = \{q(P_1), q(Q_1)\} \cap \{q(P_2), q(Q_2)\}$. Let $\gamma_j = q^{-1}(J_j)$. Then γ_j is a simple closed curve in M_2 such that $i(\gamma_j) = \gamma_j$ and P_j and Q_j are the two Weierstrass points of M_2 on γ_j. By choosing the arcs J_i carefully, we may assume that γ_1 and γ_2 are transverse. Hence, $< v_1, v_2 >$ is equal to the congruence class modulo 2 of the number of points in $\gamma_1 \cap \gamma_2$. On the other hand, since $J_1 \cap J_2$ is equal to $\{q(P_1), q(Q_1)\} \cap \{q(P_2), q(Q_2)\}$, $\gamma_1 \cap \gamma_2 = \{P_1, Q_1\} \cap \{P_2, Q_2\}$. □

Let v and w be two nontrivial homology classes in $H_1(M_2, \mathbf{Z}_2)$. We say that v and w are $\mathbf{Z}_2$ *disjoint* if $< v, w >= 0$. By Lemma 2.7 and the fact that Ω is injective, v and w are $\mathbf{Z}_2$ *disjoint* if and only if they are equal or have no common Weierstrass point. We say that $\{v, w\}$ is a $\mathbf{Z}_2$ *base pair* if $< v, w >= 1$. Again, by Lemma 2.7, $\{v, w\}$ is a $\mathbf{Z}_2$ base pair if and only if there is exactly one common Weierstrass point on v and w. The following result is an immediate consequence of these observations.

Proposition 2.1. *Let $\{v_1, w_1\}$ and $\{v_2, w_2\}$ be $\mathbf{Z}_2$ base pairs on M_2 which are $\mathbf{Z}_2$ disjoint from one another. Then the Weierstrass points of M_2 consist of the following 6 points:*

- *the common Weierstrass points P_i of v_i and w_i, $i = 1, 2$;*
- *the remaining Weierstrass points on the 4 homology classes v_1, w_1, v_2 and w_2.*

Remark 2.1. A result of W. B. R. Lickorish [Li] implies that Γ_2^+ is generated by the mapping classes of Dehn twists $\tau_i, i = 1, ..., 5$ about any system of nonseparating simple closed curves $\gamma_i, i = 1, .., 5$ with the following properties:

- γ_i and γ_j are transverse for all $i \neq j$,
- γ_i and γ_{i+1} meet in precisely one point,
- γ_i and γ_j are disjoint whenever $|i - j| \geq 2$.

(A result of S. Humphries [H] implies that this system of Dehn twists is the smallest system of Dehn twists generating Γ_2^+.) By an argument similar to the proof of Lemma 1.3, we may assume that $i(\gamma_j) = \gamma_j, j = 1, ..., 5$. Hence, each of these curves contain exactly two Weierstrass points. The above results imply that we may label the Weierstrass points $P_1, ..., P_6$ such that P_i and P_{i+1} are the Weierstrass points of M_2 on γ_i. Hence, by Proposition 1.2, τ_i is sent to the i-th standard generator of S_6, the transposition interchanging P_i and P_{i+1}. Hence, we have a complete description of the representation r.

The $\mathbf{Z}_2$-valued intersection form $<, >$ is a $\mathbf{Z}_2$ symplectic form on $H_1(M_2, \mathbf{Z}_2)$. We recall the fact that the action of homeomorphisms on homology respects this form and induces an epimorphism $\psi : \Gamma_2 \to Sp_4(\mathbf{Z}_2)$.

Theorem 2.1. *There exists a unique isomorphism μ such that the following diagram commutes:*

$$\begin{array}{ccc} \Gamma_2 & \xrightarrow{r} & S_6 \\ & \searrow^{\psi} & \downarrow \mu \\ & & Sp_4(\mathbf{Z}_2). \end{array}$$

where $r : \Gamma_2 \to S_6$ is the representation of Proposition 1.1 and ψ is the standard $\mathbf{Z}_2$ symplectic representation $\Gamma_2 \to Sp_4(\mathbf{Z}_2)$.

Proof. Since r and ψ are both epimorphisms, it suffices to show that they have the same kernel. The action of Γ_2 on W associated to the representation r induces an action of Γ_2 on $S_2^*(W)$. Let r^* denote the representation of Γ_2 associated to this action. Likewise, the action of Γ_2 on $H_1(M_2, \mathbf{Z}_2)$ associated to the representation ψ induces an action of Γ_2 on $H_1(M_2, \mathbf{Z}_2)^*$ with an associated representation ψ^*. Clearly, the kernel of ψ^* is equal to the kernel of ψ. On the other hand, by Lemma 2.6, Ω is a Γ_2 equivariant bijection of the Γ_2 sets $H_1(M_2, \mathbf{Z}_2)^*$ and $S_2^*(W)$. Thus, the kernel of r^* is equal to the kernel of ψ^*. Hence, it suffices to show that the kernel of r^* is equal to the kernel of r.

Clearly, the kernel of r is contained in the kernel of r^*. Suppose, on the other hand, that τ is an element of Γ_2 contained in the kernel of r^*. Let P be a Weierstrass point of M_2. We must show that $\tau \cdot P = P$. Suppose, that $\tau \cdot P \neq P$. Let Q be a Weierstrass point of M_2 with $P \neq Q$. Since τ is in the kernel of r^*, $\{\tau \cdot P, \tau \cdot Q\} = \{P, Q\}$. Since $\tau \cdot P \neq P$, we conclude that $\tau \cdot P = Q$. This last identity holds for every Weierstrass point Q of M_2 with $P \neq Q$. Since there are more than two Weierstrass points on M_2, this is impossible. ☐

Corollary 2.1. *The subgroup of* Γ_2^+ *that acts trivially on the set of Weierstrass points of* M_2 *via the representation* $r : \Gamma_2 \to S_6$ *of Proposition 1.1 is equal to the subgroup of* Γ_2 *generated by all squares of Dehn twists on simple closed curves in* M_2.

Proof. Let Γ_g^+ be the mapping class group of a closed Riemann surface M_g of genus g. By Theorem 8 of [W], the subgroup $\Gamma_g^+[2]$ of Γ_g^+ which acts trivially on $H_1(M_g, \mathbf{Z}_2)$ is equal to the subgroup of Γ_g^+ generated by all squares of Dehn twists on simple closed curves in M_g. Hence, the corollary follows immediately from Theorem 2.1. □

Remark 2.2. In the classical argument, (Theorem 3.1.5 of [O]), an isomorphism from $Sp_4(\mathbf{Z}_2)$ to S_6 is established by considering *configurations* in V, where V is a 4-dimensional regular alternating space over $\mathbf{Z}_2$. By definition, a configuration is any subset C of 5 elements in V with the property that no two distinct elements of C are orthogonal (with respect to the alternating form). It is shown that there are precisely 6 configurations in V and that $Sp_4(\mathbf{Z}_2)$ acts effectively on the set of configurations in V. Hence, there is a monomorphism $\nu : Sp_4(\mathbf{Z}_2) \to S_6$. Since $Sp_4(\mathbf{Z}_2)$ and S_6 have the same order, one concludes that ν is an isomorphism.

Now in our context, V is equal to $H_1(M_2, \mathbf{Z}_2)$ equipped with the intersection form $<,>$. Configurations in V are naturally identified with Weierstrass points as follows. Let P be a Weierstrass point of M_2. Let:

$$C_P = \{\Omega^{-1}(\{P, Q\}) | Q \in W \setminus \{P\}\}.$$

Lemma 2.7 implies that C_P is a configuration. It is easy to see that the correspondence $P \mapsto C_P$ defines a bijection between the set of Weierstrass points of M_2 and the set of configurations in $H_1(M_2, \mathbf{Z}_2)$. The correspondence $P \mapsto C_P$ is in some sense "dual" to our correspondence Ω. From this duality, we see that the isomorphism μ of Theorem 2.1 is the inverse of the isomorphism ν constructed by the classical argument with configurations.

This duality can be understood as follows. Let G denote the full graph on 6 vertices (i.e. the 1 skeleton of a 5 -simplex). G has two interpretations relevant to our discussion. In the first interpretation, the vertices of G correspond to the Weierstrass points of M_2 and the edges of G correspond to the distinct pairs of Weierstrass points of M_2. The vertices of an edge of G correspond to the two Weierstrass points of the corresponding pair. In the second interpretation, the vertices of G correspond to the configurations and the edges of G correspond to the nontrivial homology classes in $H_1(M_2, \mathbf{Z}_2)$. The vertices of an edge of G are the two configurations containing the corresponding homology class. From this point of view, the correspondence $P \mapsto C_P$ is a "vertex" isomorphism and the correspondence Ω is the associated "edge" isomorphism.

3. Identification with Lustig's action and virtual splitting

Let d be the unique hyperbolic metric associated to the Riemann surface M_2. Since the hyperelliptic involution i is a conformal automorphism of M_2 ([F-K]), it is an isometry of d. We recall that there exists a unique simple closed hyperbolic geodesic of the hyperbolic surface (M_2, d) in the isotopy class of any nontrivial simple closed curve on M_2.

Lemma 3.1. *Let c be an isotopy class of a nonseparating simple closed curve on M_2 and γ be the unique hyperbolic geodesic of (M_2, d) in c. Then $i(\gamma) = \gamma$.*

Proof. This is a consequence of Theorem 2.3 of [Lu]. We give an independent argument.

By Lemma 1.3, there exists a nonseparating simple closed curve $\gamma' \in c$ such that $i(\gamma') = \gamma'$. Thus i preserves the isotopy class c. Since i is an isometry, $i(\gamma)$ is a geodesic. By the uniqueness of the geodesic in a given isotopy class, therefore, $i(\gamma) = \gamma$. □

In order to state our next theorem, we recall the following notions from [Lu]. A geodesic base pair on a Riemann surface M is a pair of simple closed geodesics on M which meet in exactly one point. A pair of points P and Q on a closed geodesic γ on M are antipodes on γ if P and Q separate γ into two geodesic segments of equal hyperbolic length.

Theorem 3.1 (Lustig)**.** *The Weierstrass points of M_2 coincide for any two disjoint geodesic base pairs on M_2 with the two intersection points and the four antipodes.*

Proof. This is essentially Theorems 2.3 and 2.4 of [Lu]. We give an independent proof.

By Lemmas 1.4 and 3.1, we see that there are exactly two Weierstrass points on each of the geodesics in the given base pairs. By Lemma 2.2, these two points are the Weierstrass points on the corresponding nontrivial homology classes in $H_1(M_2, \mathbf{Z}_2)$. These homology classes form a pair of $\mathbf{Z}_2$ disjoint $\mathbf{Z}_2$ base pairs on M_2. Hence, the result follows immediately from Proposition 2.1. □

Theorem 3.2. *The representation $r : \Gamma_2 \to S_6$ of Proposition 1.1 is equal to the induced map $p : Out(\pi_1 M_2) \to S_6$ of Lemma* 3.4 *of* [*Lu*].

Proof. Let $g \in Homeo(M_2, i)$ represent an element τ of Γ_2 and let P be a Weierstrass point of M_2. Let γ_1 and γ_2 be a base pair of geodesics of (M_2, d) such that $P = \gamma_1 \cap \gamma_2$. Let Q_j be the Weierstrass point of M_2 such that P and Q_j are the two Weierstrass points of M_2 on γ_j, $j = 1, 2$. Let γ'_j be the unique hyperbolic geodesic in the isotopy class of $g(\gamma_j)$, $j = 1, 2$. We must show that $g(P)$ is the common Weierstrass point of γ'_1 and γ'_2.

Since $g \in Homeo(M_2, i)$, $i(g(\gamma_j)) = g(\gamma_j)$ and $g(P)$ and $g(Q_j)$ are the two Weierstrass points of M_2 on $i(\gamma_j)$, $j = 1, 2$. On the other hand, by Lemma

3.1, $i(\gamma_j') = \gamma_j'$, $j = 1, 2$. Since γ_j' is isotopic to $g(\gamma_j)$ and γ_j is a nonseparating simple closed curve, γ_j' and $g(\gamma_j)$ represent the same nontrivial homology class in $H_1(M_2, \mathbf{Z}_2)$. Hence, by Lemmas 2.2 and 2.3, the Weierstrass points of γ_j' and $g(\gamma_j)$ coincide. Thus, $g(P)$ is a common Weierstrass point of γ_1' and γ_2'. □

Remark 3.1. From Theorem 3.2 and Corollary 2.1, we see that the subgroup of Γ_2^+ which acts trivially on the set of Weierstrass points of M_2 via the induced map $p : Out(\pi_1 M_2) \to S_6$ of Lemma 3.4 of [Lu] is equal to the subgroup of Γ_2 generated by all squares of Dehn twists on simple closed curves in M_2.

The previous lemma implies that $\Gamma_2(P) = p^{-1}(Stab(P))$. By Theorem 3.5 of [Lu], there is a subgroup $0S(M_2$ of $Aut(\pi_1 M_2)$ which maps isomorphically to $p^{-1}(Stab(P))$ via the natural homomorphism $Aut(\pi_1 M_2) \to Out(\pi_1 M_2)$. Since $p^{-1}(Stab(P))$ has finite index in $Out(\pi_1 M_2)$, the inverse of $OS(M_2) \to p^{-1}(Stab(P))$ is a virtual splitting of $Aut(\pi_1 M_2) \to Out(\pi_1 M_2)$.

Theorem 3.3. *The representation $s : \Gamma_2(P) \to Aut(\pi_1 M_2)$ of Theorem 1.2 corresponds to the virtual splitting of Theorem* 3.5 *of* [*Lu*].

Proof. As observed in the proof of Proposition 3.6 of [Lu], the factor $OS(M_2)$ of the splitting of Theorem 3.5 of [Lu] can be characterized precisely as follows. Let $\phi \in Aut(\pi_1(M_2, P))$. Then $\phi \in OS(M_2)$ if and only if $i_*(\phi(x)) = \phi(x^{-1})$ for each element x of a specified set $\{a, b, c, d\}$ of generators of $\pi_1(M_2, P)$. But $i_*(x) = x^{-1}$ for each element x of this set of generators. Hence, $\phi \in OS(M_2)$ if and only if $i_* \circ \phi = \phi \circ i_*$.

Thus, by Proposition 1.3, $s(\Gamma_2(P)) \subset OS(M_2)$. By Theorem 3.5 of [Lu], $OS(M_2)$ is mapped isomorphically to $p^{-1}(Stab(P)) = \Gamma_2(P)$ by the natural homomorphism $Aut(\pi_1(M_2, P)) \to Out(\pi_1(M_2, P))$. Since s is a virtual splitting of this homomorphism on the subgroup of finite index $\Gamma_2(P)$, we conclude that $s(\Gamma_2(P))$ is equal to $OS(M_2)$. □

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