Surface mapping class groups are ultrahopfian

By MUSTAFA KORKMAZ

Department of Mathematics, Middle East Technical University, Ankara, 06531, Turkey
email: korkmaz@arf.math.metu.edu.tr

and JOHN D. MCCARTHY

Department of Mathematics, Michigan State University, East Lansing, MI 48824
email: mccarthy@math.msu.edu

(15 June 1997)

Introduction

Let S denote a compact, connected, orientable surface with genus g and h boundary components. We refer to S as a surface of genus g with h holes. Let $\mathcal{M}_S$ denote the mapping class group of S, the group of isotopy classes of orientation-preserving homeomorphisms $S \to S$.

Let G be a group. G is hopfian if every homomorphism from G onto itself is an automorphism. G is residually finite if for every $g \in G$ with $g \neq 1$ there exists a normal subgroup of finite index in G which does not contain g. Every finitely generated residually finite group is hopfian ([12], [11]). A group G is hyperhopfian ([2],[3]) if every homomorphism $\psi : G \to G$ with $\psi(G)$ normal in G and $G/\psi(G)$ cyclic is an automorphism. As observed in [14], examples of hopfian groups which are not hyperhopfian are afforded by the fundamental groups of torus knots.

By a result of Grossman [5], $\mathcal{M}_S$ is residually finite. Since $\mathcal{M}_S$ is also finitely generated, it is hopfian. It is a natural question to ask whether $\mathcal{M}_S$ is hyperhopfian. In this paper, we shall answer a more general question. We say that a group G is ultrahopfian if every homomorphism $\psi : G \to G$ with $\psi(G)$ normal in G and $G/\psi(G)$ abelian is an automorphism. Note that an ultrahopfian group is hyperhopfian. We shall prove the following result.

Theorem A. *Let S denote a connected orientable surface of genus g with h holes. Suppose that S is not an annulus, a sphere with four holes or a torus with at least two holes. Then $\mathcal{M}_S$ is ultrahopfian.*

Note that by combining this with the description of automorphisms of $\mathcal{M}_S$ given in [6] and [13], we have a complete description of all homomorphisms $\psi : G \to G$ with $\psi(G)$ normal in G and $G/\psi(G)$ abelian, where G is the mapping class group of any surface S as in Theorem A.

Theorem A is false if S is an annulus. In this case, $\mathcal{M}_S$ is a cyclic group of order two. Hence, the trivial homomorphism $\mathcal{M}_S \to \mathcal{M}_S$ demonstrates that $\mathcal{M}_S$ is not even hyperhopfian. We do not know whether $\mathcal{M}_S$ is ultrahopfian, or even hyperhopfian, in the remaining cases not covered by this theorem, (i.e. a sphere with four holes or a torus with at least two holes).

Here is an outline of the paper. In Section 1, we define the notions of bridges and circles on S and the corresponding notions of dual equivalence for each of these objects. We prove that all bridges on S are dually equivalent, provided $h \geq 3$, and all nonseparating circles on S are dually equivalent, provided $g \geq 1$. In Section 2, we discuss Dehn twists and elementary braids on S. We show that $\mathcal{M}_S$ and certain subgroups of $\mathcal{M}_S$ are generated by appropriate Dehn twists and elementary braids and we describe some useful relations between these generators. In Section 3, we compute the first homology $H_1(\mathcal{M}_S)$ of $\mathcal{M}_S$. Except for Theorem 3.13, the results of this section are not essential for the proof of Theorem A. On the other hand, these results are a natural development of the previous discussion, and serve as a "warmup" for the results of the next section. In Section 4, using the results of Section 1, we prove that certain derived subgroups of $\mathcal{M}_S$ are perfect subgroups of finite index in $\mathcal{M}_S$, provided $g \neq 1$. In most cases, we establish this property for the commutator subgroup of $\mathcal{M}_S$. The exceptional cases occur when $g \geq 2$ and $h = 2$, 3 or 4. In Section 5, we prove the main result of this paper, Theorem A.

1. *Bridges and circles*

In this section, S denotes a surface of genus g with h holes.

We say that a properly embedded arc a in S is a bridge if a joins distinct boundary components A and B of S. Note that S admits a bridge if and only if $h \geq 2$.

We say that two bridges a and b are dual if a and b are disjoint and there exists three distinct boundary components of S, A, B and C, such that a joins A to B and b joins B to C. Note that S admits a pair of dual bridges if and only if $h \geq 3$.

Let a and b be bridges on S. We shall say that a is dually equivalent to b if there exists a sequence of bridges $a_1, a_2, ..., a_n$ on S such that a_1 is isotopic to a, a_i and a_{i+1} are dual bridges, for $1 \leq i < n$, and a_n is isotopic to b.

THEOREM 1.1. *Let S be a surface of genus g with h holes, where $h \geq 3$. Suppose that a and b are bridges on S. Then a and b are dually equivalent.*

Proof: We may assume that a and b are transverse and do not meet at their endpoints. We shall prove the result by induction on the number of points of $a \cap b$.

Suppose that a joins the boundary components A and B of S and b joins the boundary components C and D of S.

Suppose that a and b are disjoint. There are three possibilities to consider, (i) $\{A, B\} = \{C, D\}$, (ii) $\{A, B\}$ and $\{C, D\}$ share exactly one element, and (iii) $\{A, B\} \cap \{C, D\} = \emptyset$.

In the first case, since $h \geq 3$, there exists a boundary component E of S which is not equal to A or B. We may construct a bridge c from A to E which is disjoint from both a and b. It follows that a is dual to c and c is dual to b. Hence, a and b are dually equivalent.

In the second case, a and b are dual. Hence, again, a and b are dually equivalent.

Finally, in the third case, we may construct a bridge c from A to C which is disjoint from a and b. It follows that a is dual to c and c is dual to b. Hence, a and b are dually equivalent.

Now suppose that the result holds for bridges which meet in at most n points, where n is a nonnegative integer. Suppose that a and b meet in $n+1$ points. Choose an orientation for the surface S and for the bridge a. Beginning at the initial endpoint of a travel along the right side of a to the first point x of intersection of a with b. Orient b so that the orientation of S at x is the sum of the orientations of b and a at x. Now continue along the right side of b to the terminal point of b. In this manner, we construct a properly embedded arc c disjoint from b, meeting a in at most n points and joining the initial boundary component of a to the terminal boundary component of b. In a similar manner, by traveling along the left side of a to x and continuing along the left side of b (with respect to the opposite orientation on b), we construct a properly embedded arc d disjoint from b, meeting a in at most n points and joining the initial boundary component of a to the initial boundary component of b. Since b is a bridge, the initial boundary component of b is not equal to the terminal boundary component of b. Hence, at least one of c and d is a bridge on S.

It follows that there exists a bridge e on S such that a meets e in at most n points and e is disjoint from b. By induction, a is dually equivalent to e and e is dually equivalent to b. Since dual equivalence is an equivalence relation, a is dually equivalent to b. □

A circle a on S is a closed, connected, one-dimensional submanifold of S embedded in the interior of S. S_a denotes the surface obtained by cutting S along a. We say that a is separating if S_a is disconnected. Note that S_a has one component if a is nonseparating and two components if a is separating. We say that a is trivial if a is separating and one of the two components of S_a is either a disc or an annulus. Let k be an integer such that $0 \leq k \leq h$. We shall say that a separating circle a on S is k-separating if one of the two components of S_a is a sphere with $k+1$ holes. Note that every nonseparating circle on S is nontrivial.

We say that two circles a and b on S are dual if they are transverse and meet at exactly one point. Note that dual circles are nonseparating. Hence, S admits a pair of dual circles if and only if $g \geq 1$.

Let a and b be nonseparating circles on S. We shall say that a is dually equivalent to b if there exists a sequence of nonseparating circles $a_1, a_2, ..., a_n$ on S such that a_1 is isotopic to a, a_i and a_{i+1} are dual circles for $1 \leq i < n$, and a_n is isotopic to b.

THEOREM 1.2. *Let S be a surface of positive genus g with h holes. Suppose that a and b are nonseparating circles on S. Then a and b are dually equivalent.*

Proof: First, we prove the result for closed surfaces.

Suppose that S is a torus. We may identify S with the quotient of $\mathbb{R}^2$ by the lattice $\mathbb{Z}^2$. Moreover, we may equip S with the euclidean metric induced by the natural map from $\mathbb{R}^2$ to this quotient. Then every nonseparating circle on S is isotopic to a simple closed euclidean geodesic on S, the image of a line of slope q/p for some pair of relatively prime integers p and q. We shall denote such a geodesic by $[p, q]$. It suffices to show that $[1, 0]$ and $[p, q]$ are dually equivalent. Note that $[p, q] = [-p, -q]$. Hence, we may assume that $q \geq 0$. We shall prove that $[1, 0]$ and $[p, q]$ are dually equivalent by induction on q.

Suppose that $q = 0$. Since $[p, q] = [-p, -q]$, we may assume that $p \geq 0$. Since p and q are relatively prime, it follows that $p = 1$. Hence, in this case, $[p, q] = [1, 0]$. Hence, $[1, 0]$ is dually equivalent to $[p, q]$.

Let n be a nonnegative integer and suppose that the result is true when q is less than or equal to n. Suppose that $q = n + 1$. Each element of the mapping class group of S is isotopic to an orientation-preserving affine map of S induced by the action of an element A of $SL(2, \mathbb{Z})$ on $\mathbb{R}^2$. We shall denote this induced affine map of S by A. We may choose such an orientation-preserving affine map A which maps $[1, 0]$ to $[1, 0]$ and $[0, 1]$ to $[m, 1]$, where m is an integer. A maps $[p, q]$ to $[p + mq, q]$. Since A is a homeomorphism, it suffices to show that $[1, 0]$ is dually equivalent to $[p + mq, q]$. By choosing m appropriately, we may assume that $0 < p + mq \leq q$. Hence, we may assume that $0 < p \leq q$. Note that $[1, 0]$ is dual to $[1, 1]$. Hence, it suffices to prove that $[1, 1]$ is dually equivalent to $[p, q]$.

Now consider an orientation-preserving affine map B that sends $[1, 0]$ to $[1, -1]$ and $[0, 1]$ to $[0, 1]$. Note that B sends $[1, 1]$ to $[1, 0]$ and $[p, q]$ to $[p, q - p]$. Since $0 < p \leq q$, we conclude that $0 \leq q - p < q$. Hence, by our induction hypothesis, $[1, 0]$ is dually equivalent to $[p, q - p]$. Note that dual equivalence is preserved under the action of homeomorphisms on pairs of nonseparating circles. Since B^{-1} is a homeomorphism of S, we conclude that $[1, 1]$ is dually equivalent to $[p, q]$. Hence, $[1, 0]$ is dually equivalent to $[p, q]$.

This completes the induction and, hence, establishes the result for a torus.

Now suppose that S is a closed surface of genus $g \geq 2$.

By a result of Ivanov ([7]), there exists a sequence $a_1, a_2, ..., a_n$ of nontrivial circles on S such that a_1 is isotopic to a, a_i is disjoint from a_{i+1} and a_n is isotopic to b. Choose such a sequence of shortest length n. Suppose that some element a_i in this sequence is a separating circle on S. Since a and b are nonseparating circles, $1 < i < n$. Since S is a closed surface of genus ≥ 2 and a_i is nontrivial, S_{a_i} has two components, each a surface of positive genus with one hole. Since a_i is disjoint from a_{i-1} and a_{i+1}, a_{i-1} and a_{i+1} each lie in one of the components of S_{a_i}. If a_{i-1} and a_{i+1} lie in different components of S_{a_i}, then we may shorten the sequence $a_1, a_2, ..., a_n$ of consecutively disjoint nontrivial circles on S by deleting a_i. This contradicts our assumption about choosing a shortest such sequence. Hence, a_{i-1} and a_{i+1} lie in the same component of S_{a_i}. Since the other component of S_{a_i} has positive genus, we may choose a nonseparating circle c in this other component. We may then replace the term a_i in the sequence $a_1, a_2, ..., a_n$ of consecutively disjoint nontrivial circles on S by the nonseparating circle c on S, without affecting the length of this sequence. It follows, by induction on the number of separating circles in the sequence $a_1, a_2, ..., a_n$ of consecutively disjoint nontrivial circles on S, that we may choose such a sequence so that each circle a_i is nonseparating.

Now for each integer i with $1 \leq i < n$, a_i and a_{i+1} are a pair of disjoint nonseparating circles on S. It follows that we may choose a circle c_i on S so that c_i is dual to a_i and a_{i+1}. In this way, we may construct a sequence $a_1, c_1, a_2, c_2, ..., a_{n-1}, c_{n-1}, a_n$ of consecutively dual circles on S, beginning with a and ending with b. In other words, a is dually equivalent to b.

This completes the proof for closed surfaces of genus $g \geq 2$.

Hence, we have established the result for all closed surfaces of positive genus.

Now suppose that S is a surface of positive genus with h holes, where $h > 0$. We may assume that a and b are in minimal position, (i.e. that a and b are transverse and have the least number of intersections among all pairs of circles a' and b' isotopic to a and b respectively). We may also assume that a is not isotopic to b.

Consider the closed surface T of genus g associated to S by capping off each boundary component of S with a disc. We may consider a and b as circles on T. Suppose that a and b are in minimal position on T. Then we may choose a metric of constant curvature on T, (euclidean if $g = 1$ and hyperbolic if $g \geq 2$), such that a and b are geodesic. By the result for closed surfaces, there exists a sequence $a_1, ..., a_n$ of consecutively dual circles on T such that a_1 is isotopic to a and a_n is isotopic to b. We may replace each of the circles a_i in this sequence, where $1 < i < n$, by any geodesic in its isotopy class. Since geodesics in distinct isotopy classes of circles are in minimal position, these geodesics are still consecutively dual. Hence, we may assume that the sequence $a_1, ..., a_n$ of consecutively dual circles on T begins with a and ends with b. Since the holes of S are disjoint from a and b, there exists a homeomorphism $F : T \to T$ which is isotopic to the identity, fixes $a \cup b$ and maps $a_1 \cup a_2 \cup ... \cup a_n$ into S. Applying this homeomorphism F to the sequence $a_1, ..., a_n$ of consecutively dual circles on T beginning with a and ending with b, we obtain a sequence of consecutively dual circles on S beginning with a and ending with b. Hence, a is dually equivalent to b.

We shall now complete the proof by induction on the number n of points of $a \cap b$. If a and b are disjoint, then a and b are in minimal position in T and the result follows by the argument in the previous paragraph. Hence, we may assume that $n > 0$. Likewise, we may assume that a and b are not in minimal position on T. Then there must exist an embedded sphere D with $k + 1$ holes in S, where $k > 0$, k of which correspond to holes of S and 1 of which is equal to the union $c \cup d$ of an arc c of a and an arc d of b which meet $a \cap b$ precisely at their common endpoints P and Q. We shall refer to D as a "bigon" with holes in the complement of $a \cup b$. Orient a and b so that P is the first endpoint of both c and d. Equip S with the orientation determined by the sum of the orientations of a and b at the point P. Beginning at a point x near Q, on the right side of a and b, travel along the right side of b to the vicinity of P. Then take a right turn and travel along the right side of a to the point x. In this way, we construct an embedded circle e such that e is disjoint from b, e meets a in $n - 2$ points, and e is isotopic to b on T. From the construction of e, we may construct a circle f on S which is dual to e and b. Hence, e is dually equivalent to b. It suffices, therefore, to prove that a is dually equivalent to e. Since isotopic nonseparating simple closed curves are dually equivalent, we may assume that a and e are in minimal position. Note that this assumption does not affect the fact that e intersects a in less than n points. Hence, by the induction hypothesis, a is dually equivalent to e. □

2. *Generators and relations for certain subgroups of* $\mathcal{M}_S$

In this section, S denotes a surface of genus g with h holes; $\mathcal{M}_S$ denotes the mapping class group of S, the group of isotopy classes of orientation-preserving homeomorphisms $S \to S$; $[\mathcal{M}_S, \mathcal{M}_S]$ denotes the commutator subgroup of $\mathcal{M}_S$; and $\mathcal{P}M_S$ denotes the pure mapping class group of S, the subgroup of $\mathcal{M}_S$ consisting of isotopy classes of orientation-preserving homeomorphisms $S \to S$ which preserve each boundary component of S. We shall give generators for $\mathcal{M}_S$, $[\mathcal{M}_S, \mathcal{M}_S]$ and $\mathcal{P}M_S$ and describe some useful relations satisfied by these generators. We assume that S is oriented.

We recall that a pair of pants is a sphere with three holes. Let $F : S \to S$ be a homeomorphism such that F is supported on a pair of pants P in S, exactly two of the boundary components of P are boundary components of S, and F interchanges these two boundary components of P. Let c denote the boundary component of P which is preserved by F. Note that c is a circle on S. Moreover, F^2 is isotopic to a power of a right or left Dehn twist about c. We shall say that F is a right (left) elementary braid supported on P if F^2 is isotopic to a right (left) Dehn twist about c.

Recall that two bridges or circles, a and b, on S are topologically equivalent if there exists an (orientation-preserving) homeomorphism $F : S \to S$ such that $F(a) = b$.

If a is a bridge on S joining the boundary components A and B of S, then a regular neighborhood of $a \cup A \cup B$ in S is a pair of pants P as above. We denote by σ_a the isotopy class of a right elementary braid supported on P. σ_a depends only upon the isotopy class of a in S. We shall refer to σ_a as a right elementary braid. If f is the isotopy class of an orientation-preserving homeomorphism $F : S \to S$, then $f\sigma_a f^{-1} = \sigma_{F(a)}$. Since any two bridges on S are topologically equivalent, it follows that right elementary braids are all conjugate in $\mathcal{M}_S$. In particular, they represent the same class ψ in the abelianization $H_1(\mathcal{M}_S)$ of $\mathcal{M}_S$.

For each circle c on S, t_c denotes the isotopy class of a right Dehn twist about c. t_c depends only upon the isotopy class of c in S. We shall refer to t_c as a right Dehn twist about c. If f is the isotopy class of an orientation-preserving homeomorphism $F : S \to S$, then $ft_c f^{-1} = t_{F(c)}$. Since any two nonseparating circles on S are topologically equivalent, it follows that right Dehn twists about nonseparating circles on S are all conjugate in $\mathcal{M}_S$. In particular, they represent the same class τ in $H_1(\mathcal{M}_S)$.

There are two well-known "braid" relations between right elementary braids on S and right Dehn twists about nonseparating circles on S. First, if a and b are dual bridges, then:

$$\sigma_a \sigma_b \sigma_a = \sigma_b \sigma_a \sigma_b. \tag{2.1}$$

Secondly, if a and b are dual circles, then:

$$t_a t_b t_a = t_b t_a t_b. \tag{2.2}$$

Suppose that S is a sphere with h holes. Then we have the following well-known result concerning generators for $\mathcal{M}_S$.

THEOREM 2.1. *Let S be a sphere with h holes. If $h = 0$ or 1, then $\mathcal{M}_S$ is trivial. If $h = 2$, then $\mathcal{M}_S$ is a cyclic group of order two generated by a single right elementary braid. If $h \geq 3$, then $\mathcal{M}_S$ is generated by a finite collection of right elementary braids in $\mathcal{M}_S$.*

THEOREM 2.2. *Let S be a sphere with h holes, where $h \geq 3$. Then the commutator subgroup of $\mathcal{M}_S$ is generated by the collection of elements of the form $\sigma_a \sigma_b^{-1}$, where a and b are dual bridges on S.*

Proof: Let Σ be the subgroup of $\mathcal{M}_S$ generated by the collection of elements of the form $\sigma_a \sigma_b^{-1}$, where a and b are dual bridges on S.

Suppose that a and b are dual bridges on S and $F : S \to S$ is an orientation-preserving homeomorphism. Let $c = F(a)$ and $d = F(b)$. Then c and d are dual bridges on S. Let f denote the isotopy class of F. Then $f(\sigma_a \sigma_b^{-1})f^{-1} = \sigma_c \sigma_d^{-1}$. It follows that Σ is a normal subgroup of $\mathcal{M}_S$. (Moreover, since any two pairs of dual bridges are topologically equivalent, Σ is normally generated by any element of the form $\sigma_a \sigma_b^{-1}$, where a and b are dual bridges on S.)

Let A denote the quotient of $\mathcal{M}_S$ by the normal subgroup Σ. Suppose that a and b are dual bridges on S. Then the element $\sigma_a \sigma_b^{-1}$ lies in Σ. Hence, the classes in A of σ_a and σ_b are equal.

Let a and b be dual bridges on S. Since right elementary braids are all conjugate in $\mathcal{M}_S$, the element $\sigma_a \sigma_b^{-1}$ lies in the commutator subgroup $[\mathcal{M}_S, \mathcal{M}_S]$ of $\mathcal{M}_S$. Indeed, $\sigma_b^{-1} = f \sigma_a^{-1} f^{-1}$ for some f in $\mathcal{M}_S$. It follows that Σ is contained in $[\mathcal{M}_S, \mathcal{M}_S]$. Hence, it remains only to show that A is abelian. Indeed, we shall show that A is cyclic.

Since, by Theorem 2.1, $\mathcal{M}_S$ is generated by right elementary braids, it suffices to show that the classes in A of any two right elementary braids σ_a and σ_b are equal. We have already observed this to be true when a and b are dual bridges. Since, by Theorem 1.1, any two bridges on S are dually equivalent (provided $h \geq 3$), the result follows. □

Now suppose that the genus of S is positive. Then we have the following well-known result concerning generators for $\mathcal{P}M_S$ and $\mathcal{M}_S$.

THEOREM 2.3. *Let S be a surface of positive genus g with h holes. Then $\mathcal{P}M_S$ is generated by a finite collection of right Dehn twists about nonseparating circles on S.*

THEOREM 2.4. *Let S be a surface of positive genus g with h holes. If $h = 0$ or 1, then $\mathcal{M}_S$ is generated by a finite collection of right Dehn twists about nonseparating circles on S. If $h = 2$, then $\mathcal{M}_S$ is generated by a finite collection of right Dehn twists about nonseparating circles on S and one right elementary braid on S. If $h \geq 3$, then $\mathcal{M}_S$ is generated by a finite collection consisting of right Dehn twists about nonseparating circles on S and right elementary braids on S.*

Note that $\mathcal{P}M_S = \mathcal{M}_S$ if $h = 0$ or 1.

THEOREM 2.5. *Let S be a surface of positive genus with h holes. Then the commutator subgroup of $\mathcal{P}M_S$ is generated by the collection of elements of the form $t_a t_b^{-1}$, where a and b are dual circles on S.*

Proof: Let T be the subgroup of $\mathcal{P}M_S$ generated by the collection of elements of the form $t_a t_b^{-1}$, where a and b are dual circles on S.

Suppose that a and b are dual circles on S and $F : S \to S$ is an orientation-preserving homeomorphism. Let $c = F(a)$ and $d = F(b)$. Then c and d are dual circles on S.

Let f denote the isotopy class of F. Then $f(t_a t_b^{-1})f^{-1} = t_c t_d^{-1}$. It follows that T is a normal subgroup of $\mathcal{M}_S$ and, hence, of $\mathcal{P}M_S$. (Moreover, since any two pairs of dual circles are topologically equivalent by an orientation-preserving homeomorphism preserving each boundary component of S, T is normally generated by any element of the form $t_a t_b^{-1}$, where a and b are dual circles on S.)

Let B denote the quotient of $\mathcal{P}M_S$ by the normal subgroup T. Suppose that a and b are dual circles on S. By the definition of T, the element $t_a t_b^{-1}$ lies in T. Hence, the classes in B of t_a and t_b are equal.

Let a and b be dual circles on S. Since right Dehn twists about nonseparating circles on S are all conjugate in $\mathcal{P}M_S$, the element $t_a t_b^{-1}$ lies in the commutator subgroup $[\mathcal{P}M_S, \mathcal{P}M_S]$ of $\mathcal{P}M_S$. It follows that T is contained in $[\mathcal{P}M_S, \mathcal{P}M_S]$. Hence, it remains only to show that B is abelian. Indeed, we shall show that B is cyclic.

Since $\mathcal{P}M_S$ is generated by right Dehn twists about nonseparating circles on S, it suffices to show that the classes in B of any two right Dehn twists t_a and t_b about nonseparating circles a and b on S are equal. We have already observed this to be true when a and b are dual circles. Since, by Theorem 1.2, any two nonseparating circles on S are dually equivalent, the result follows. □

As a consequence of Theorem 2.5, we obtain the following result.

THEOREM 2.6. *Let S be a surface of positive genus with h holes, where $h = 0$ or 1. Then the commutator subgroup of $\mathcal{M}_S$ is generated by the collection of elements of the form $t_a t_b^{-1}$, where a and b are dual circles on S.*

Proof: In the case where $h = 0$ or 1, $\mathcal{M}_S = \mathcal{P}M_S$. □

The argument in the proof of Theorem 2.5 may be adapted slightly further to prove the following result.

THEOREM 2.7. *Let S be a surface of positive genus with 2 holes. Then the commutator subgroup of $\mathcal{M}_S$ is generated by the collection of elements of the form $t_a t_b^{-1}$, where a and b are dual circles on S.*

Proof: Let T be defined as in the proof of Theorem 2.5. By the proof of Theorem 2.5, T is a normal subgroup of $\mathcal{M}_S$. Let C be the quotient of $\mathcal{M}_S$ by its normal subgroup T. Since S has 2 holes, it follows that $\mathcal{M}_S$ is generated by $\mathcal{P}M_S$ and the isotopy class of a right elementary braid σ_b, where b is any bridge on S. It follows, from the proof of Theorem 2.5, that C is generated by the class τ of a right Dehn twist t_a about any nonseparating circle a on S and the class ψ of σ_b. We may choose a nonseparating circle a on S such that a and b are disjoint. Then t_a and σ_b commute. Hence, τ and ψ commute. It follows that C is abelian. The result follows as in the proof of Theorem 2.5. □

Suppose now that $h \geq 3$.

THEOREM 2.8. *Let S be a surface of positive genus with h holes, where $h \geq 3$. Then the commutator subgroup of $\mathcal{M}_S$ is generated by the collection consisting of all elements of the form $t_a t_b^{-1}$, where a and b are dual circles on S and all elements of the form $\sigma_c \sigma_d^{-1}$, where c and d are dual bridges on S.*

Proof: Let U be the subgroup of $\mathcal{M}_S$ generated by the collection of all elements of the form $t_a t_b^{-1}$, where a and b are dual circles on S and all elements of the form $\sigma_c \sigma_d^{-1}$, where c and d are dual bridges on S.

As in the previous proofs, it follows that U is a normal subgroup of $\mathcal{M}_S$. (Moreover, as in the previous proofs, it follows that U is normally generated by two elements, any element of the form $t_a t_b^{-1}$, where a and b are dual circles on S and any element of the form $\sigma_c \sigma_d^{-1}$, where c and d are dual bridges on S.)

Let C denote the quotient of $\mathcal{M}_S$ by the normal subgroup U. By the usual argument, it suffices to show that C is abelian. On the other hand, by the usual argument and Theorem 2.4, it follows that C is generated by two elements, the class of a right Dehn twist t_a about any nonseparating circle a on S and the class of a right elementary braid σ_b, where b is any bridge on S. Since we may choose a and b to be disjoint, the result follows as in the proof of Theorem 2.7. $\square$

In addition to the above braid relations, (2.1) and (2.2), for dual bridges and dual circles on S, we shall require the following relations between right elementary braids and right Dehn twists about circles on S.

Suppose that $h \geq 5$. Let $A_1, ..., A_5$ be five distinct boundary components of S. Let a, b and c be three disjoint bridges on S such that a joins A_1 to A_2, b joins A_2 to A_3, and c joins A_4 to A_5. (Note that a and b are dual bridges.) Then we have the following reformulation of the braid relation (2.1) for dual bridges:

$$\sigma_a \sigma_b^{-1} = [\sigma_c \sigma_b^{-1}, \sigma_c \sigma_a^{-1}], \tag{2.3}$$

where $[x, y] = xyx^{-1}y^{-1}$.

This relation is actually equivalent to the braid relation (2.1) for dual bridges on S. Since c and the boundary components of S joined by c are disjoint from a and b, σ_c commutes with σ_a and σ_b. Hence, the relation (2.3) is equivalent to the relation $\sigma_a \sigma_b^{-1} = [\sigma_b^{-1}, \sigma_a^{-1}]$. Expanding the commutator, we see that this last relation is equivalent to the braid relation (2.1) for dual bridges.

Suppose that $g \geq 2$. Let a, b and c be three nonseparating circles on S such that a and b are dual circles and c is disjoint from a and b. Then we have the following reformulation of the braid relation (2.2) for dual circles:

$$t_a t_b^{-1} = [t_c t_b^{-1}, t_c t_a^{-1}]. \tag{2.4}$$

This relation is actually equivalent to the braid relation (2.2) for dual circles on S. The argument is the same as that given for the reformulated braid relation (2.3) for dual bridges on S.

Consider a sphere S_0 with four holes embedded in S so that the boundary components are circles d_0, d_1, d_2 and d_3 on S. Let b_{12}, b_{13} and b_{23} be three disjoint bridges on S_0 such that b_{ij} joins d_i to d_j. The complement of $b_{12} \cup b_{13} \cup b_{23}$ in S_0 consists of two components, an annulus A containing d_0 and a disk D. We assume that the bridges b_{12}, b_{13} and b_{23} are chosen so that D is on the right of b_{12} as we traverse b_{12} from d_1 to d_2. (This assumption is equivalent to the assumption that D is on the right of b_{23} as we traverse b_{23} from d_2 to d_3, as well as to the assumption that D is on the left of b_{13} as we traverse b_{13} from d_1 to d_3.)

Let P_{ij} be a regular neighborhood in S_0 of the union $b_{ij} \cup d_i \cup d_j$, so that P_{ij} is a pair of pants in S_0 with boundary components, d_i, d_j and d_{ij} for some circle d_{ij} on S_0. Then we have the following "lantern" relation ([4], [9]):

$$t_{d_0} t_{d_1} t_{d_2} t_{d_3} = t_{d_{12}} t_{d_{13}} t_{d_{23}}. \tag{2.5}$$

Consider a torus T with two holes embedded in S such that the boundary components of T are circles on S. Let q_1 and q_2 denote the two boundary components of T. Let a, b and c denote circles on T such that (i) a, c and q_1 cobound a pair of pants P_1 in T, (ii) a, c and q_2 cobound a pair of pants P_2 in T, (iii) a and b are dual circles on T, and (iv) b and c are dual circles on T. Then we have the following "two-holed torus" relation ([10]):

$$t_{q_1} t_{q_2} = (t_a t_c t_b)^4. \tag{2.6}$$

3. *Homological results for $\mathcal{M}_S$*

In this section, S denotes a surface of genus g with h holes. Using the results of Section 2, we shall compute $H_1(\mathcal{M}_S)$. Except for Theorem 3.13, the results of this section are not essential for the proof of the main theorem of this paper, Theorem A.

If the genus of S is 0, we have the following well-known presentation of $\mathcal{M}_S$ ([1], Theorem 4.5).

THEOREM 3.1. *Let S be a sphere with h holes. If $h = 0$ or 1, then $\mathcal{M}_S$ is trivial. If $h \geq 2$, then $\mathcal{M}_S$ has a presentation consisting of generators $w_1, .., w_{h-1}$ and defining relations:*

$$w_i w_j = w_j w_i \quad |i - j| \geq 2$$

$$w_i w_{i+1} w_i = w_{i+1} w_i w_{i+1}$$

$$w_1 ... w_{h-2} w_{h-1}^2 w_{h-2} ... w_1 = 1$$

$$(w_1 w_2 ... w_{h-1})^h = 1.$$

The generators $w_1, .., w_{h-1}$ in the above theorem may be constructed as follows. Index the boundary components of S, $\partial_1, ..., \partial_h$. Choose a family of disjoint bridges, $a_1, ..., a_{h-1}$ such that a_i joins ∂_i to ∂_{i+1}. Finally, let w_i be the right elementary braid σ_{a_i}.

The following result is an immediate corollary of Theorem 3.1.

THEOREM 3.2. *Let S be a sphere with h holes. If $h = 0$ or 1, then $H_1(\mathcal{M}_S)$ is trivial. If h is an odd integer greater than 1, then $H_1(\mathcal{M}_S) = \mathbb{Z}_{h-1}$. If h is an even integer greater than 1, then $H_1(\mathcal{M}_S) = \mathbb{Z}_{2(h-1)}$.*

If $g \geq 1$, let τ denote the class in $H_1(\mathcal{M}_S)$ of a right Dehn twist about a nonseparating circle a on S. If $h \geq 2$, let ψ denote the class in $H_1(\mathcal{M}_S)$ of a right elementary braid σ_a

on S, where a is a bridge on S. Since right Dehn twists about nonseparating circles on S (resp. right elementary braids on S) are all conjugate in $\mathcal{M}_S$, we see that the class τ (resp. ψ) is well-defined independently of the choice of nonseparating circle (resp. bridge) a on S. Hence, we have the following immediate corollary of Theorem 2.4.

THEOREM 3.3. *Let S be a surface of positive genus with h holes. If $h = 0$ or 1, then $H_1(\mathcal{M}_S)$ is generated by the class τ. If $h \geq 2$, then $H_1(\mathcal{M}_S)$ is generated by the two elements, τ and ψ.*

THEOREM 3.4. *Let S be a torus with h holes. Then $\tau^{12} = 1$.*

Proof: By assumption, $g = 1$. Let a and b be a pair of dual circles on S. A regular neighborhood R of the union $a \cup b$ is a torus with one hole c, where c is an h-separating circle on S. It is well-known that:

$$(t_a t_b)^6 = t_c.$$

If $h = 0$ or 1, then $t_c = 1$. Hence, if $h = 0$ or 1, $\tau^{12} = 1$.

Suppose, therefore, that $h \geq 2$. Let k be an integer, where $0 \leq k \leq h$. Let γ_k denote the class in $H_1(\mathcal{M}_S)$ of a right Dehn twist about any k-separating circle on S. Note that any two k-separating circles on S are topologically equivalent. Hence, γ_k is a well-defined class depending only on k.

Now suppose that k is less than h. We may embed the four-holed sphere S_0 of Section 2 in S so that d_0 is a 1-separating circle on S, d_1 and d_2 are nonseparating circles on S, and d_3 is a k-separating circle on S. It follows that d_{12} is $(k+1)$-separating, whereas d_{13} and d_{23} are nonseparating. Since d_0 is a 1-separating circle on S, t_{d_0} is trivial. Hence, by the lantern relation (2.5), it follows that:

$$1 \times \tau \times \tau \times \gamma_k = \gamma_{k+1} \times \tau \times \tau.$$

Hence, $\gamma_{k+1} = \gamma_k$. On the other hand, γ_1 is trivial. Hence, by induction, γ_h is trivial. Since a and b are dual circles, they are both nonseparating. Hence, t_a and t_b both map to τ in $H_1(\mathcal{M}_S)$. On the other hand, since c is h-separating, t_c maps to γ_h in $H_1(\mathcal{M}_S)$. Hence, since $(t_a t_b)^6 = t_c$, we conclude that $\tau^{12} = 1$. □

THEOREM 3.5. *Suppose that S is a surface of genus 2 with h holes. Then $\tau^{10} = 1$.*

Proof: Since $g = 2$, we may embed the torus T with two holes in S so that the boundary components q_1 and q_2 of T are nonseparating circles on S. (The complement of T in S is then a sphere with $h+2$ holes.) It follows that the curves q_1, q_2, a, b and c involved in the two-holed torus relation (2.6) are all nonseparating circles on S. It follows that the twists t_{q_1}, t_{q_2}, t_a, t_b and t_c all map to τ in $H_1(\mathcal{M}_S)$. Hence, the two-holed torus relation (2.6) implies that $\tau^2 = \tau^{12}$. In other words, $\tau^{10} = 1$. □

THEOREM 3.6. *Suppose that S is a surface of genus $g \geq 3$ with h holes. Then τ is trivial.*

Proof: Since $g \geq 3$, we may embed a sphere S_0 with 4 holes in S so that the circles d_i and d_{jk} associated with the lantern relation (2.5) are all nonseparating circles on S. Since

Dehn twists about nonseparating circles on S are all conjugate in $\mathcal{M}_S$, it follows that t_{d_i} and $t_{d_{jk}}$ map to τ. We conclude, by the lantern relation (2.5), that $\tau^4 = \tau^3$. Hence, τ is trivial. □

Theorems 3.4, 3.5 and 3.6 give an upper bound to the order of τ. Using the following well-known result, we may obtain a lower bound and, hence, determine the order of τ.

THEOREM 3.7. *Let T be a closed, connected orientable surface of genus g. If $g = 1$, then $H_1(\mathcal{M}_T) = \mathbb{Z}_{12}$. If $g = 2$, then $H_1(\mathcal{M}_T) = \mathbb{Z}_{10}$. If $g \geq 3$, then $H_1(\mathcal{M}_T)$ is trivial.*

Now, let T be the closed surface of genus g obtained from S by filling in each hole of S with a disc, (i.e. by "coning off" each boundary component of S). There is a natural homomorphism $\phi : \mathcal{M}_S \to \mathcal{M}_T$ obtained by extending homeomorphisms $F : S \to S$ over T by "coning off" their restrictions to the boundary of S. We shall refer to ϕ as the filling homomorphism.

THEOREM 3.8. *Let S be a torus with h holes. Then τ has order* 12.

Proof: By Theorem 3.4, $\tau^{12} = 1$. Hence, the order of τ divides 12. On the other hand, the filling homomorphism maps τ to a generator τ_0 of $H_1(\mathcal{M}_T)$, where T is the torus obtained by capping off each boundary component of S. Since, by Theorem 3.7, $H_1(\mathcal{M}_T)$ is a cyclic group of order 12, we conclude that the order of τ is divisible by 12. Hence, the order of τ is equal to 12. □

THEOREM 3.9. *Suppose that S is a surface of genus* 2 *with h holes. Then τ has order* 10.

Proof: This follows from Theorem 3.5 and Theorem 3.7 by the same argument as in the proof of Theorem 3.8. □

Now suppose that S is a surface of positive genus with $h \geq 2$.

THEOREM 3.10. *Let S be a surface of positive genus with h holes, where $h \geq 2$. Then $\psi^2 = 1$.*

Proof: Let k be an integer with $0 \leq k \leq h$. We recall that γ_k denotes the class in $H_1(\mathcal{M}_S)$ of a right Dehn twist about any k-separating circle on S. By the same argument as in the proof of Theorem 3.4, γ_k is trivial. In particular, γ_2 is trivial.

Now let P be a regular neighborhood of the union $a \cup b \cup c$, where b and c are the two boundary components of S joined by the bridge a. The boundary components of P are b, c and d, where d is a 2-separating circle on S. By the definition of a right elementary braid, $\sigma_a^2 = t_d$. Hence, t_d maps to ψ^2. On the other hand, by the definition of γ_2, t_d maps to γ_2. We conclude that $\psi^2 = 1$. □

As Theorems 3.4, 3.5 and 3.6 did for τ, this result gives an upper bound to the order of ψ. In the same manner in which we computed the order of τ in the proofs of Theorems 3.8 and 3.9, we shall compute the order of ψ by finding an appropriate homomorphism from $\mathcal{M}_S$ to $\mathbb{Z}_2$. In addition to the filling homomorphism employed in the proofs of Theorems 3.8 and 3.9, there is another natural homomorphism from $\mathcal{M}_S$ which corresponds to the action of homeomorphisms $F : S \to S$ on the set of boundary components of S. After indexing the boundary components of S, this action yields a natural homomorphism $\beta : \mathcal{M}_S \to \Sigma_h$, where Σ_h denotes the trivial group if $h = 0$ and the group of permutations of the set $\{1, ..., h\}$ if $h \geq 1$. We shall refer to β as the boundary permutation homomorphism.

THEOREM 3.11. *Let S be a surface of positive genus with h holes, where $h \geq 2$. Then ψ has order 2.*

Proof: By Theorem 3.10, $\psi^2 = 1$. Hence, the order of ψ divides 2. On the other hand, the boundary permutation homomorphism $\beta : \mathcal{M}_S \to \Sigma_h$ maps ψ to a 2-cycle ψ_0. Since $h \geq 2$, $H_1(\Sigma_h)$ is a cyclic group of order 2 generated by the class of any 2-cycle. We conclude that the order of ψ is divisible by 2. Hence, the order of ψ is equal to 2. □

We are now ready to compute $H_1(\mathcal{M}_S)$.

THEOREM 3.12. *Let S be a surface of positive genus with h holes. If $g = 1$ and $h = 0$ or 1, then $H_1(\mathcal{M}_S) = \mathbb{Z}_{12}$. If $g = 1$ and $h \geq 2$, then $H_1(\mathcal{M}_S) = \mathbb{Z}_{12} \bigoplus \mathbb{Z}_2$. If $g = 2$ and $h = 0$ or 1, then $H_1(\mathcal{M}_S) = \mathbb{Z}_{10}$. If $g = 2$ and $h \geq 2$, then $H_1(\mathcal{M}_S) = \mathbb{Z}_{10} \bigoplus \mathbb{Z}_2$. If $g \geq 3$ and $h = 0$ or 1, then $H_1(\mathcal{M}_S)$ is trivial. If $g \geq 3$ and $h \geq 2$, then $H_1(\mathcal{M}_S) = \mathbb{Z}_2$.*

Proof: The abelianizations of the filling homomorphism ϕ and the boundary permutation homomorphism β afford a homomorphism $(\phi_*, \beta_*) : H_1(\mathcal{M}_S) \to H_1(\mathcal{M}_T) \bigoplus H_1(\Sigma_h)$. It suffices to show that (ϕ_*, β_*) is an isomorphism.

By the previous remark, $H_1(\mathcal{M}_T)$ is generated by the class of any right Dehn twist about a nonseparating circle c on T. We may assume that c lies in S and is, hence, a nonseparating circle on S. If $h = 0$ or 1, then $H_1(\Sigma_h) = 0$. On the other hand, if $h \geq 2$, then $H_1(\Sigma_h)$ is equal to $\mathbb{Z}_2$ and is generated by the class of any 2-cycle in Σ_h.

Let c be a nonseparating circle on S. Since t_c preserves each boundary component of S, it follows that $(\phi_*, \beta_*)([t_c]) = ([t_c], 0)$. It follows that $H_1(\mathcal{M}_T) \bigoplus \{0\}$ is contained in the image of (ϕ_*, β_*). If $h = 0$ or 1, it follows that (ϕ_*, β_*) is onto.

Suppose now that $h \geq 2$. Let σ_b be a right elementary braid on S. Let $F : S \to S$ be a right elementary braid representing σ_b. By coning off F, we obtain a homeomorphism $G : T \to T$ which is supported on a disc D embedded in T. Indeed, F is supported on a pair of pants P where two of the boundary components of P are boundary components of S. D is the disc obtained from P by capping off these two boundary components of P. It follows that G is isotopic to the identity. Hence, $(\phi_*, \beta_*)([\sigma_b]) = (0, [\beta(\sigma_b)])$. Since σ_b interchanges exactly two boundary components of S, $\beta(\sigma_b)$ is a 2-cycle in Σ_h. Hence, $\{0\} \bigoplus H_1(\Sigma_h)$ is contained in the image of (ϕ_*, β_*). Since $H_1(\mathcal{M}_T) \bigoplus H_1(\Sigma_h)$ is generated by $H_1(\mathcal{M}_T) \bigoplus \{0\}$ and $\{0\} \bigoplus H_1(\Sigma_h)$, it follows that (ϕ_*, β_*) is onto.

Hence, (ϕ_*, β_*) is onto for all surfaces of positive genus.

On the other hand, by the previous results $H_1(\mathcal{M}_S)$ is generated by the class τ if $h = 0$ or 1, and by the two classes τ and ψ if $h \geq 2$. We have seen that the orders of τ and ψ are equal to the orders of the cyclic groups $H_1(\mathcal{M}_T)$ and $H_1(\Sigma_h)$ respectively. Hence, $H_1(\mathcal{M}_S)$ is a finite group whose order is at most that of the direct product $H_1(\mathcal{M}_T) \bigoplus H_1(\Sigma_h)$. It follows that the epimorphism (ϕ_*, β_*) is an isomorphism. □

Note, in particular, that $\mathcal{M}_S$ is perfect if $g \geq 3$ and $h = 0$ or 1. Since $\mathcal{M}_S$ is Hopfian, it follows that $\mathcal{M}_S$ is hyperhopfian and ultrahopfian if $g \geq 3$ and $h = 0$ or 1.

As a corollary of this section, we have the following result.

THEOREM 3.13. *For any surface S, the index in $\mathcal{M}_S$ of the commutator subgroup of $\mathcal{M}_S$ is finite.*

4. *Perfect derived subgroups of finite index in $\mathcal{M}_S$*

In this section, S denotes a surface of genus g with h holes. Using the results of Section 1, we shall prove that certain derived subgroups of $\mathcal{M}_S$ are perfect subgroups of finite index in $\mathcal{M}_S$, provided $g \neq 1$. In most cases, we shall in fact establish this property for the commutator subgroup of $\mathcal{M}_S$. The exceptional cases occur when $g \geq 2$ and $h = 2$, 3 or 4.

Suppose that S is a sphere with h holes, where $h \geq 5$. Using the braid relation (2.1) for dual bridges and its reformulation (2.3), we have the following result.

THEOREM 4.1. *Let S be a sphere with h holes, where $h \geq 5$. Then the commutator subgroup of $\mathcal{M}_S$ is perfect.*

Proof: Let Γ denote the commutator subgroup of $\mathcal{M}_S$. By Theorem 2.2, Γ is generated by the elements of the form $\sigma_a\sigma_b^{-1}$, where a and b are dual bridges on S. Let a and b be dual bridges on S. Since $h \geq 5$, we may choose a bridge c such that c and the boundary components of S joined by c are disjoint from a and b. Since right elementary braids on S are all conjugate in $\mathcal{M}_S$, the elements $\sigma_c\sigma_b^{-1}$ and $\sigma_c\sigma_a^{-1}$ lie in Γ. Hence, the commutator of these elements lies in $[\Gamma, \Gamma]$. Hence, by the reformulation (2.3) of the braid relation for dual bridges, $\sigma_a\sigma_b^{-1}$ lies in $[\Gamma, \Gamma]$. We conclude that Γ is contained in $[\Gamma, \Gamma]$. Hence, $\Gamma = [\Gamma, \Gamma]$. In other words, Γ is perfect. □

Suppose now that S is a torus with h holes. Let Γ denote the commutator subgroup of $\mathcal{M}_S$. Let T be the torus obtained from S by capping off the boundary components of S. It is well-known that $\mathcal{M}_T$ is isomorphic to $SL(2, \mathbb{Z})$. On the other hand, the commutator subgroup Γ_0 of $SL(2, \mathbb{Z})$ maps onto a free group of rank 2. The natural "filling" homomorphism $\phi : \mathcal{M}_S \to \mathcal{M}_T$ described in Section 3 is an epimorphism. We conclude that Γ maps onto Γ_0 and, hence, onto a free group of rank 2. It follows that the derived subgroups of $\mathcal{M}_S$ have infinite index commutator subgroups. Moreover, they all map onto free groups of positive rank. Hence, none of the derived subgroups of $\mathcal{M}_S$ are perfect.

Suppose now that $g \geq 2$.

THEOREM 4.2. *Let S be a surface of genus $g \geq 2$ with h holes. Then the commutator subgroup of $\mathcal{PM}_S$ is perfect.*

Proof: Let Γ denote the commutator subgroup of $\mathcal{PM}_S$. By Theorem 2.5, Γ is generated by the elements of the form $t_at_b^{-1}$, where a and b are dual circles on S. Let a and b be dual circles on S. Since $g \geq 2$, we may choose a nonseparating circle c on S such that c is disjoint from a and b. Since right Dehn twists about nonseparating circles on S are all conjugate in $\mathcal{PM}_S$, the elements $t_ct_b^{-1}$ and $t_ct_a^{-1}$ lie in Γ. Hence, the commutator of these elements lies in $[\Gamma, \Gamma]$. Hence, by the reformulated braid relation (2.4) for dual circles, $t_a t_b^{-1}$ lies in $[\Gamma, \Gamma]$. We conclude that Γ is contained in $[\Gamma, \Gamma]$. Hence, $\Gamma = [\Gamma, \Gamma]$. In other words, Γ is perfect. □

Since $\mathcal{P}M_S = \mathcal{M}_S$ when $h = 0$ or 1, we have the following immediate corollary of Theorem 4.2.

THEOREM 4.3. *Let S be a surface of genus $g \geq 2$ with h holes, where $h = 0$ or 1. Then the commutator subgroup of $\mathcal{M}_S$ is perfect.*

Suppose now that $g \geq 3$. Then we have the following additional results.

THEOREM 4.4. *Let S be a surface of genus $g \geq 3$ with h holes. Then $\mathcal{P}M_S$ is perfect.*

Proof: Since right Dehn twists about nonseparating circles on S are all conjugate in $\mathcal{P}M_S$, they map to the same element τ in $H_1(\mathcal{P}M_S)$. It follows, by Theorem 2.3, that $H_1(\mathcal{P}M_S)$ is generated by τ. Hence, by Theorem 3.6, $H_1(\mathcal{P}M_S)$ is trivial. □

Since $\mathcal{P}M_S = \mathcal{M}_S$ when $h = 0$ or 1, we have the following immediate corollary of Theorem 4.4.

THEOREM 4.5. *Let S be a surface of genus $g \geq 3$ with h holes, where $h = 0$ or 1. Then $\mathcal{M}_S$ is perfect.*

Suppose now that $g \geq 2$ and $h = 2$, 3 or 4.

THEOREM 4.6. *Let S be a surface of genus $g \geq 2$ with h holes, where $h = 2$, 3 or 4. Then the fourth derived subgroup $\mathcal{M}_S^{(4)}$ of $\mathcal{M}_S$ is equal to the commutator subgroup of $\mathcal{P}M_S$. In particular, $\mathcal{M}_S^{(4)}$ is a perfect derived subgroup of finite index in $\mathcal{M}_S$.*

Proof: Since $h \leq 4$, Σ_h is solvable. Indeed, the third derived subgroup of Σ_h is trivial. Hence, we have a sequence of subgroups of Σ_h:

$$G_3 \subset G_2 \subset G_1 \subset G_0$$

where $G_0 = \Sigma_h$, G_3 is the trivial group, G_{i+1} is normal in G_i, and the quotient of G_i by G_{i+1} is abelian.

Let Γ_i denote the preimage of G_i in $\mathcal{M}_S$ under the boundary permutation homomorphism $\beta : \mathcal{M}_S \to \Sigma_h$ of Section 3. Then, we obtain a sequence of subgroups of $\mathcal{M}_S$:

$$\Gamma_3 \subset \Gamma_2 \subset \Gamma_1 \subset \Gamma_0$$

where $\Gamma_0 = \mathcal{M}_S$, $\Gamma_3 = \mathcal{P}M_S$, Γ_{i+1} is normal in Γ_i, and the quotient of Γ_i by Γ_{i+1} is abelian.

Since the quotient of Γ_i by Γ_{i+1} is abelian, it follows by induction that the ith derived subgroup $\Gamma_0^{(i)}$ of $\mathcal{M}_S$ is contained in Γ_i. Hence, the commutator subgroup $\Gamma_0^{(4)}$ of $\Gamma_0^{(3)}$ is contained in the commutator subgroup $\Gamma_3^{(1)}$ of Γ_3.

Since Γ_3 is equal to $\mathcal{P}M_S$, $\Gamma_3^{(1)}$ is perfect, by Theorem 4.2. On the other hand, Γ_0 is equal to $\mathcal{M}_S$. Hence, $\Gamma_3^{(1)}$ is a perfect subgroup of Γ_0. It follows by induction that $\Gamma_3^{(1)}$ is contained in $\Gamma_0^{(n)}$ for all n. In particular, $\Gamma_3^{(1)}$ is contained in $\Gamma_0^{(4)}$.

We conclude that $\Gamma_0^{(4)}$ is equal to $\Gamma_3^{(1)}$, the commutator subgroup of $\mathcal{P}M_S$.

By Theorem 2.3 and the usual argument, $H_1(\mathcal{P}M_S)$ is generated by the class $\tilde{\tau}$ in $H_1(\mathcal{P}M_S)$ of a right Dehn twist about a nonseparating circle on S. By the arguments in the proofs of Theorem 3.5 and 3.6, $\tilde{\tau}$ has finite order. Hence, $H_1(\mathcal{P}M_S)$ is finite. In other words, the commutator subgroup $\Gamma_0^{(4)}$ of $\mathcal{P}M_S$ has finite index in $\mathcal{P}M_S$. Since $\mathcal{P}M_S$ has finite index in $\mathcal{M}_S$, we conclude that $\Gamma_0^{(4)}$ has finite index in $\mathcal{M}_S$. Hence, $\Gamma_0^{(4)}$ is a perfect derived subgroup of finite index in $\mathcal{M}_S$. □

Suppose now that $g \geq 2$ and $h \geq 5$.

Theorem 4.7. *Let S be a surface of genus $g \geq 2$ with h holes, where $h \geq 5$. Then the commutator subgroup of $\mathcal{M}_S$ is perfect.*

Proof: Let Γ denote the commutator subgroup of $\mathcal{M}_S$. By Theorem 2.8, Γ is generated by the collection consisting of all elements of the form $\sigma_a\sigma_b^{-1}$, where a and b are dual bridges on S, and all elements of the form $t_c t_d^{-1}$, where c and d are dual circles on S. Using the reformulations, (2.3) and (2.4), of the braid relations for dual bridges and dual circles in S, we deduce, as in the proofs of Theorems 4.1 and 4.2, that each of these elements lie in $[\Gamma, \Gamma]$. As before, we conclude that Γ is perfect. □

5. *The main result*

In this section, S denotes a surface of genus g with h holes. We shall prove the main theorem described in the introduction, Theorem A. Let us denote by $\mathcal{M}_S^*$ the extended mapping class group of S, the group of isotopy classes of all diffeomorphisms of S, including orientation-reversing ones.

We shall require the following results.

Theorem 5.1. ([7]) *Let S be a surface of genus $g \geq 2$ with h holes. Suppose that S is not a closed surface of genus 2. Let G_1 and G_2 be subgroups of finite index in $\mathcal{M}_S$. Then any isomorphism $G_1 \to G_2$ is induced by some inner automorphism of $\mathcal{M}_S^*$.*

Theorem 5.2.([10]) *Let S be a sphere with at least five holes or a torus with at least three holes. Let G_1 and G_2 be subgroups of finite index in $\mathcal{M}_S$. Then any isomorphism $G_1 \to G_2$ is induced by some inner automorphism of $\mathcal{M}_S^*$.*

The proof of Theorem 5.1 from [7] may be adapted to prove the following result.

Theorem 5.3. *Let S be a closed surface of genus 2. Let Γ be a perfect subgroup of finite index in $\mathcal{M}_S$ and let G be a subgroup of finite index in $\mathcal{M}_S$. Then any isomorphism $\psi : \Gamma \to G$ is induced by an inner automorphism of $\mathcal{M}_S^*$.*

Proof: Let a be a nontrivial circle on S. As in the proof of Theorem 5.1 from [7], there exist nonzero integers M and N and a circle b on S such that $t_a^M \in \Gamma$ and $\psi(t_a^M) = t_b^N$.

We recall that the complex of curves $C(S)$ of S is the simplicial complex whose vertices are isotopy classes α of nontrivial circles A on S and whose k-simplices are sets $\{\alpha_0, ..., \alpha_k\}$ of isotopy classes of $k+1$ pairwise nonisotopic and disjoint nontrivial circles $A_0, ..., A_k$ on S. As in the proofs in [7] and [10], it follows that the correspondence $a \mapsto b$ defines an automorphism of the complex of curves $C(S)$ of S.

It is shown in [7] that this automorphism is induced by a mapping class f in $\mathcal{M}_S^*$. That is, $\psi(t_a^M) = t_{f(a)}^N$ for each circle a on S.

Now, as in the proof in [7], we have the following equations, for any mapping class g in Γ and for appropriately chosen nonzero powers, J, K and L:

$$\psi(g)\psi(t_a^J)\psi(g)^{-1} = \psi(gt_a^J g^{-1}) = \psi(t_{g(a)}^J) = t_{f(g(a))}^K,$$

$$\psi(g)\psi(t_a^J)\psi(g)^{-1} = \psi(g)t_{f(a)}^L\psi(g)^{-1} = t_{\psi(g)f(a)}^L.$$

Hence, $t_{f(g(a))}^K = t_{\psi(g)f(a)}^L$. This implies that $f(g(a)) = \psi(g)f(a)$. Since this last identity holds for every circle a on S, we conclude that $fg = \psi(g)f\sigma_g$, where σ_g is either 1 or σ, depending upon g, and σ is the isotopy class of the hyperelliptic involution on S.

If g and h are two mapping classes in Γ, we have:

$$\psi(gh)f\sigma_{gh} = fgh.$$

On the other hand, since σ is of order two and since σ is in the center of $\mathcal{M}_S$:

$$\begin{aligned}\psi(gh)f\sigma_{gh} &= \psi(g)\psi(h)f\sigma_{gh} = \psi(g)fh\sigma_h\sigma_{gh} \\ &= fg\sigma_g h\sigma_h\sigma_{gh} = fgh\sigma_g\sigma_h\sigma_{gh}.\end{aligned}$$

It follows that $\sigma_{gh} = \sigma_g\sigma_h$. Hence, the correspondence $g \mapsto \sigma_g$ defines a homomorphism $\Gamma \to C(\mathcal{M}_S)$, where $C(\mathcal{M}_S)$ is the center $\{1, \sigma\}$ of $\mathcal{M}_S$, a cyclic group of order 2. Since Γ is perfect, the homomorphism $\Gamma \to C(\mathcal{M}_S)$ is trivial. Hence, $fg = \psi(g)f$ for each g in Γ. In other words, $\psi(g) = fgf^{-1}$ for each g in Γ. □

Since $\mathcal{M}_S$ is invariant under all inner automorphisms of $\mathcal{M}_S^*$, we have the following result from Theorems 5.1, 5.2 and 5.3.

Theorem 5.4. *Let S denote either a sphere with at least five holes or a torus with at least three holes or a connected orientable surface of genus at least two. Let G_1 and G_2 be subgroups of finite index in $\mathcal{M}_S$. If S is a closed surface of genus two, suppose that G_1 is a perfect subgroup of finite index in $\mathcal{M}_S$. Then any isomorphism $G_1 \to G_2$ is induced by an automorphism of $\mathcal{M}_S$.*

We are now ready to prove the main result of this paper.

Theorem A. *Let S denote a compact, connected orientable surface with genus g and h boundary components. Let $\mathcal{M}_S$ denote the mapping class group of S, the group of isotopy classes of orientation-preserving homeomorphisms $S \to S$. Suppose that S is not an annulus, a sphere with four holes or a torus with at least two holes. Then $\mathcal{M}_S$ is ultrahopfian.*

Proof: Let G denote $\mathcal{M}_S$. Suppose that $\psi : G \to G$ is a homomorphism with $\psi(G)$ normal in G and $G/\psi(G)$ abelian. It follows that $[G, G] \subset \psi(G) \subset G$.

If S is a sphere with at most one hole, then G is the trivial group and the result is clear.

Suppose that S is a sphere with three holes. By the presentation given in Theorem 3.1, we see that G is isomorphic to the symmetric group Σ_3 on 3 letters. (Indeed, this isomorphism is exhibited by the boundary permutation homomorphism $\beta : G \to \Sigma_3$.) The abelianization of G is a cyclic group of order 2. Hence, the commutator subgroup of G is a cyclic group of order 3. Suppose that $\psi : G \to G$ is not onto. Then, since $[G, G] \subset \psi(G) \subset G$, we conclude that ψ maps G onto the cyclic group $[G,G]$ of order 3. Since the abelianization of G is a cyclic group of order 2, this is impossible. Hence, the result holds in this case.

Suppose now that S is a torus with at most one hole. Then G is isomorphic to $SL(2,\mathbb{Z})$. In particular, G is an infinite group.

Let a and b be dual circles on S and let $X = t_a t_b t_a$, $Y = t_a t_b$ and $T = (t_a t_b t_a)^2 = (t_a t_b)^3$. G has a standard presentation with 3 generators, X, Y and T, and relations, $T = X^2 = Y^3$ and $T^2 = 1$. Let I denote the identity matrix in $SL(2,\mathbb{Z})$. We may obtain an isomorphism $G \to SL(2,\mathbb{Z})$ by $X \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $Y \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ and $T \mapsto -I$. The abelianization of G is a cyclic group of order 12. This implies that the commutator subgroup of G is of finite index in G. Since G is an infinite group, the commutator subgroup of G is infinite. Since $\psi(G)$ contains the commutator subgroup of G, $\psi(G)$ is an infinite group.

We may express the natural map from G to its abelianization as a homomorphism $\mu : G \to \mathbb{Z}_{12}$ such that $X \mapsto 3$, $Y \mapsto 2$ and $T \mapsto 6$. It is a well-known fact that if X' is an element of order 4 in $SL(2,\mathbb{Z})$, then either X' is conjugate to X or X' is conjugate to TX. Likewise, if Y' is an element of order 6 in $SL(2,\mathbb{Z})$, then either Y' is conjugate to Y or Y' is conjugate to TY.

Let $X' = \psi(X)$ and $Y' = \psi(Y)$. Since $T = X^2$, G is generated by X and Y. Hence, $\psi(G)$ is generated by X' and Y'. We shall show that X' and Y' have orders 4 and 6 respectively. Since $X^4 = T^2 = 1$, $(X')^4 = 1$. Likewise, $(Y')^6 = 1$. This implies that the order of X' divides 4 and the order of Y' divides 6. Suppose that X' does not have order 4. Then $(X')^2 = 1$. Since I and $-I$ are the only two elements in $SL(2,\mathbb{Z})$ whose order divides 2, we conclude that X' corresponds to either I or $-I$. Since I and $-I$ commute with every element in $SL(2,\mathbb{Z})$, we conclude that X' commutes with Y'. We conclude that $\psi(G)$ is an abelian group generated by two elements X' and Y' of finite order. This implies that $\psi(G)$ is a finite group. This is a contradiction. Hence, X' has order 4.

Now suppose that Y' does not have order 6. Then either $(Y')^2 = 1$ or $(Y')^3 = 1$. Since $X^2 = Y^3$, we conclude that $(X')^2 = (Y')^3$. Hence, either $(X')^2 = 1$ or $(X')^2 = Y'$. We have already ruled out the first possibility. Hence, $Y' = (X')^2$. Since $\psi(G)$ is generated by X' and Y', we conclude that $\psi(G)$ is generated by X'. Since X' has order 4, we conclude that $\psi(G)$ is a finite group. Again, this is a contradiction. Hence Y' has order 6.

Consider the "abelianization" homomorphism $\mu : G \to \mathbb{Z}_{12}$ described above. The kernel of this homomorphism is the commutator subgroup of G. Note that $\psi(G)$ contains this kernel and is contained in G. In order to show that $\psi(G) = G$, therefore, it remains only to prove that μ maps $\psi(G)$ onto $\mathbb{Z}_{12}$. Let $p = \mu(X')$ and $q = \mu(Y')$. Since $\psi(G)$ is generated by X' and Y', we need only show that p and q generate $\mathbb{Z}_{12}$. Since X' has order 4, X' is conjugate to one of the two elements X and TX in G. Hence, $p = 3$ or $p = 6 + 3 = 9$. On the other hand, since Y' has order 6, Y' is conjugate to one of the two elements, Y and TY in G. Hence, $q = 2$ or $q = 6 + 2 = 8$. It follows that p and q are relatively prime. Hence, p and q generate $\mathbb{Z}_{12}$.

This completes the argument for a torus with at most one hole.

It remains to consider the cases where S is a sphere with at least five holes or a surface of genus $g \geq 2$ with h holes.

Since $[G, G] \subset \psi(G)$, $[\psi^n(G), \psi^n(G)] = \psi^n([G, G]) \subset \psi^{n+1}(G)$ for any positive integer n.

We now show that $G^{(n)} \subset \psi^n(G)$ for each positive integer n. For $n = 1$, $G^{(1)} = [G, G] \subset \psi(G)$ by the hypothesis. Suppose that $G^{(n)} \subset \psi^n(G)$. Then $G^{(n+1)} = [G^{(n)}, G^{(n)}] \subset [\psi^n(G), \psi^n(G)] \subset \psi^{n+1}(G)$.

Since S is a sphere with at least five holes or a surface of genus $g \geq 2$ with h holes, Theorems 3.13, 4.1, 4.3, 4.6 and 4.7 imply that $G^{(4)}$ is a perfect subgroup of finite index in G. Since $G^{(4)}$ is perfect, $G^{(n)} = G^{(4)}$ for each integer $n \geq 4$. We conclude that $G^{(4)}$ is contained in $\psi^n(G)$ for each integer n. Since $\psi^{n+1}(G)$ is contained in $\psi^n(G)$ for each positive integer n, we conclude that:

$$G^{(4)} \subset ... \subset \psi^{n+1}(G) \subset \psi^n(G) \subset ... \subset \psi(G) \subset G$$

for each integer n. Since $G^{(4)}$ has finite index in G, we conclude that $\psi^{n+1}(G) = \psi^n(G)$ for some integer n. Hence, restricting ψ to $\psi^n(G)$ we obtain an epimorphism:

$$\psi| : \psi^n(G) \to \psi^{n+1}(G) = \psi^n(G).$$

Since $G^{(4)}$ has finite index in G and $G^{(4)} \subset \psi^n(G) \subset G$, $\psi^n(G)$ has finite index in G.

By Grossman's result ([5]), G is a residually finite group. Since every subgroup of a residually finite group is residually finite, $\psi^n(G)$ is residually finite. On the other hand, by Theorems 2.1 and 2.4, G is finitely generated. Since $\psi^n(G)$ has finite index in G, $\psi^n(G)$ is also finitely generated. Finally, since a finitely generated residually finite group is hopfian ([12]), $\psi^n(G)$ is hopfian. Hence, the restriction $\psi| : \psi^n(G) \to \psi^n(G)$ is an automorphism of the finite index subgroup $\psi^n(G)$ of G.

Suppose, first, that S is a closed surface of genus 2. By Theorem 4.3, $[G, G]$ is perfect. Since $[G, G] \subset \psi^n(G) \subset G$ and since $[G, G]$ is perfect, we conclude that $[G, G] = [[G, G], [G, G]] \subset [\psi^n(G), \psi^n(G)] \subset [G, G]$. Hence, $[\psi^n(G), \psi^n(G)] = [G, G]$. It follows that $[G, G]$ is a characteristic subgroup of $\psi^n(G)$. That is, it is invariant under all automorphisms of $\psi^n(G)$. In particular, the restriction of ψ to $[G, G]$ is an automorphism of $[G, G]$. By Theorem 5.4, the restriction of ψ to $[G, G]$ is the restriction of an automorphism of G. Composing ψ with the inverse of this automorphism, we can assume that ψ restricts to the identity of $[G, G]$. Since the index of $[G, G]$ in G is 10, $\psi(t_a^{10}) = t_a^{10}$ for each circle a. For each $f \in G$ and for each circle a, we have $t_{\psi(f)(a)}^{10} = \psi(f) t_a^{10} \psi(f)^{-1} = \psi(f t_a^{10} f^{-1}) = \psi(t_{f(a)}^{10}) = t_{f(a)}^{10}$. This implies that $\psi(f)(a) = f(a)$. Hence, $\psi(f) = f \sigma_f$, where σ_f is either the hyperelliptic involution σ or the identity, depending on f. Then:

$$fg\sigma_{fg} = \psi(fg) = \psi(f)\psi(g) = f\sigma_f g\sigma_g.$$

Since σ is in the center of G, $\sigma_{fg} = \sigma_f \sigma_g$. Hence, the correspondence $f \mapsto \sigma_f$ defines a homomorphism $G \to C(G)$, where $C(G)$ is the center $\{1, \sigma\}$ of G, a cyclic group of order 2. Since G is generated by right Dehn twists about nonseparating circles and since all such generators are conjugate, $\sigma_{t_a} = \sigma_{t_b}$ for any two nonseparating circles a, b. Let a be a nonseparating circle on S. If $\sigma_{t_a} = 1$, then $\psi(t_a) = t_a$. Since this is true for any nonseparating a, ψ is the identity. If $\sigma_{t_a} = \sigma$, then $\psi(t_a) = t_a\sigma$. Then ψ is the exceptional automorphism of G ([13]). In particular, ψ is an automorphism.

Suppose now that S is not a closed surface of genus 2.

By Theorem 5.4, $\psi|$ is induced by an automorphism of G. By composing ψ with the inverse of this automorphism of G, we may assume that $\psi| = id : \psi^n(G) \to \psi^n(G)$.

Since $\psi^n(G)$ has finite index in G, we may choose a positive integer N such that $t_a^N \in \psi^n(G)$ for every circle a on S. Since $\psi| = id$, we conclude that $\psi(t_a^N) = t_a^N$ for every circle a on S. Let g be a mapping class. For any nontrivial circle a on S:

$$\begin{aligned} t^N_{\psi(g)(a)} &= \psi(g) t_a^N \psi(g)^{-1} = \psi(g)\psi(t_a^N)\psi(g)^{-1} \\ &= \psi(g t_a^N g^{-1}) = \psi(t^N_{g(a)}) = t^N_{g(a)}. \end{aligned}$$

Hence, $\psi(g)(a) = g(a)$, for every nontrivial circle a on S. Since S is a sphere with at least five holes or a surface of genus $g \geq 2$ with h holes and S is not a closed surface of genus 2, this implies that $\psi(g) = g$. We conclude that $\psi = id : G \to G$. $\square$

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