ABELIAN SUBGROUPS AND AUTOMORPHISMS OF THE TORELLI GROUP

By

William R. Vautaw

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

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Let $\mathbf{S}$ be a closed, connected, oriented surface of genus $g \geq 3$. The *mapping class group* $\mathcal{M}$ of $\mathbf{S}$ is defined to be the group of isotopy classes of self-homeomorphisms of $\mathbf{S}$, while the *Torelli group* $\mathcal{T}$ of $\mathbf{S}$ is the subgroup of $\mathcal{M}$ consisting of the isotopy classes of those self-homeomorphisms of $\mathbf{S}$ that induce the identity permutation of the first homology group of $\mathbf{S}$.

This work considers two aspects of the Torelli group. The first is the Abelian subgroups of $\mathcal{T}$. This portion of the work, where graph theory is the principal tool, contains two primary theorems. One gives a complete description of the multitwist subgroups of $\mathcal{T}$, and the other states that any Abelian subgroup of $\mathcal{T}$ has rank at most $2g-3$.

The second subject of investigation is automorphisms of the Torelli group; specifically, we ask whether any automorphism $\Psi : \mathcal{T} \to \mathcal{T}$ is induced by a homeomorphism of $\mathbf{S}$. In several formal and informal announcements made between October 2001 and March 2002, Benson Farb stated that he was able to prove that this is indeed the case for $g \geq 4$. In this work, we lay the foundation for proving that it is also true for $g = 3$. This involves three basic steps. The first is to characterize

algebraically certain elements of the Torelli group, namely powers of Dehn twists about separating curves and powers of bounding pair maps. The characterization given by Farb is valid for $g \geq 4$, while our's is valid for $g \geq 3$. The second step is to show that Ψ induces an automorphism Ψ_* of C, the complex of curves of $\mathbf{S}$. This difficult step remains incomplete at the time of this writing. In the last step we use a theorem of Ivanov which states that Ψ_* is induced by a homeomorphism h of the surface $\mathbf{S}$, and conclude, under the assumption that it is possible to complete the second step, that the automorphism of the Torelli group induced by the homeomorphism h agrees with our automorphism Ψ.

I will sing unto the Lord, because he hath dealt bountifully with me.

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INTRODUCTION

The subject of this work is surface mapping class groups, generally classified as within the realm of geometric topology along with other areas of mathematics such as knot theory, cobordism theory, the theory of retracts, and so on. This in spite of the fact that, although surface topology plays an essential role, our true object of interest is an algebraic one - a group. For the key technique is to consider configurations of curves on surfaces. This strategy will lead us to graph theory in chapters 1 and 2, and in chapter 3 to a well-known abstract simplicial complex first introduced by William Harvey [8], the "complex of curves."

In this introductory chapter we give the notation, definitions, and essential foundational theorems that we will rely on in the later chapters. We also present the specific problem we will be addressing and some historical background.

Throughout this work, $\mathbf{S}$ represents a closed, connected, oriented surface of genus $g \geq 3$. We use the symbols $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$, etc., to denote (the isotopy classes of) simple closed curves on $\mathbf{S}$. In general, we confuse a curve and its isotopy class, and thus to say that two curves are distinct means that they are not isotopic. We use the symbol $\mathfrak{S}$ to denote the set of all nonoriented (isotopy classes of) homotopically nontrivial simple closed curves on $\mathbf{S}$. The *geometric intersection number* of $\mathfrak{a}$ and

$\mathfrak{b}$ is given by

$i(\mathfrak{a}, \mathfrak{b}) = \min\{\mathrm{card}\{\mathfrak{a}' \cap \mathfrak{b}'\} : \ \mathfrak{a}' \text{ is isotopic to } \mathfrak{a} \text{ and } \mathfrak{b}' \text{ is isotopic to } \mathfrak{b}\}.$

If $\mathfrak{a} \in \mathfrak{S}$ is separating, then $\mathbf{S} \setminus \mathfrak{a}$ has two components $\mathbf{S}_1$ and $\mathbf{S}_2$, and we define the *genus* of $\mathfrak{a}$ to be the smaller of the two numbers genus$(\mathbf{S}_1)$ and genus$(\mathbf{S}_2)$. A *reduction system* $\mathfrak{C}$ on $\mathbf{S}$ is a set of distinct, mutually disjoint, homotopically nontrivial simple closed curves on $\mathbf{S}$, and we use $\mathbf{S}_{\mathfrak{C}}$ to denote the natural compactification of $\mathbf{S} \setminus \bigcup_{\mathfrak{a} \in \mathfrak{C}} \mathfrak{a}$; that is, "$\mathbf{S}$ cut along $\mathfrak{C}$." If $\mathfrak{C} = \{\mathfrak{a}\}$, then we write $\mathbf{S}_{\mathfrak{a}}$ instead of $\mathbf{S}_{\{\mathfrak{a}\}}$. Note that a reduction system consists of at most $3g - 3$ elements, and a reduction system containing $3g - 3$ elements gives a pants decomposition of $\mathbf{S}$.

The *mapping class group* of $\mathbf{S}$, $\mathcal{M}(\mathbf{S})$, is the group of isotopy classes, or *mapping classes*, of orientation-preserving self-homeomorphisms of $\mathbf{S}$. In general, our notation will not distinguish between a homeomorphism $f : \mathbf{S} \to \mathbf{S}$ and its mapping class. Given $\mathfrak{a} \in \mathfrak{S}$ and $f \in \mathcal{M}(\mathbf{S})$, we let $f(\mathfrak{a})$ denote the isotopy class of the image of any curve representing the class of $\mathfrak{a}$ under any map representing the class of f.

The symbol $D_{\mathfrak{a}}$ will denote the right Dehn twist about $\mathfrak{a} \in \mathfrak{S}$. Two standard results are that Dehn twists $D_{\mathfrak{a}}$ and $D_{\mathfrak{b}}$ commute if and only if $i(\mathfrak{a}, \mathfrak{b}) = 0$, and that if $f \in \mathcal{M}(\mathbf{S})$ then $f D_{\mathfrak{a}} f^{-1} = D_{f(\mathfrak{a})}$. Given a reduction system $\mathfrak{C}$ on $\mathbf{S}$, we define a *multitwist* on $\mathfrak{C}$ to be a composition of left and right Dehn twists about the curves in $\mathfrak{C}$, and denote the group of all multitwists on $\mathfrak{C}$ by $\mathcal{D}_{\mathfrak{C}}$. Clearly $\mathcal{D}_{\mathfrak{C}}$ is a free Abelian group with basis $\{D_{\mathfrak{a}} : \mathfrak{a} \in \mathfrak{C}\}$, and so $\mathrm{rank}(\mathcal{D}_{\mathfrak{C}}) = \mathrm{card}(\mathfrak{C})$.

The mapping class group acts in an obvious way on the first homology group of $\mathbf{S}$: given an *oriented* simple closed curve $\mathfrak{b}$ on $\mathbf{S}$, f sends the homology class of $\mathfrak{b}$ to the homology class of $f(\mathfrak{b})$. In the case of a right Dehn twist we have a specific formula for the image class. If $\mathfrak{a}$ and $\mathfrak{b}$ are oriented simple closed curves on $\mathbf{S}$, then in $H_1(\mathbf{S})$ we have

$$D_{\mathfrak{a}}(\mathfrak{b}) = \mathfrak{b} + \langle \mathfrak{a}, \mathfrak{b} \rangle \mathfrak{a},$$

where $\langle \mathfrak{a}, \mathfrak{b} \rangle$ denotes the algebraic intersection number of $\mathfrak{a}$ with $\mathfrak{b}$.

The *Torelli group* of $\mathbf{S}$, which we denote by $\mathcal{T}(\mathbf{S})$ or simply $\mathcal{T}$, is defined to be the kernel of this action of $\mathcal{M}(\mathbf{S})$ on $H_1(\mathbf{S})$. The Torelli group is torsion-free, and is trivial in the case of the sphere or torus. Also, the center of the Torelli group of any closed surface is trivial.

We define the mapping class group and Torelli group of a surface with boundary analogously; we simply require that homeomorphisms and admissible isotopies fix each component of the boundary *setwise*. Note that the Torelli group of a surface with boundary (with the exceptions of a disc and an annulus) is still torsion-free, and that the Torelli groups of a pair of pants and a one-holed torus are trivial.

Given a reduction system $\mathfrak{C}$ on $\mathbf{S}$, we use the symbol $\mathcal{M}_{\mathfrak{C}}(\mathbf{S})$ to denoted the stabilizer of $\mathfrak{C}$ in $\mathcal{M}(\mathbf{S})$, and define the *reduction homomorphism* $\Lambda : \mathcal{M}_{\mathfrak{C}}(\mathbf{S}) \to \mathcal{M}(\mathbf{S}_{\mathfrak{C}})$ as follows: For $f \in \mathcal{M}_{\mathfrak{C}}(\mathbf{S})$, there exist a representative $\bar{f}$ of f and a set $\bar{\mathfrak{C}}$ of disjoint representatives of $\mathfrak{C}$ such that $\bar{f}(\bar{\mathfrak{C}}) = \bar{\mathfrak{C}}$. We let $\Lambda(f)$ be the unique extension of $\bar{f}|_{\mathbf{S} \setminus \bar{\mathfrak{C}}}$ to $\mathbf{S}_{\mathfrak{C}}$. The kernel of Λ is $\mathcal{D}_{\mathfrak{C}}$.

We say that $f \in \mathcal{M}(\mathbf{S})$ is *reducible* if $f \in \mathcal{M}_{\mathfrak{C}}(\mathbf{S})$ for some nonempty reduction system $\mathfrak{C}$. In this case we call $\mathfrak{C}$ a *reduction system for f* and each $\mathfrak{a} \in \mathfrak{C}$ a *reduction class* for f. Nikolai Ivanov has proved a theorem ([9], Theorem 1.2) which implies the following useful result concerning the Torelli group, which we refer to as Ivanov's Theorem:

Theorem 0.1 (Ivanov's Theorem) *Let $f \in \mathcal{T}$ and suppose $\mathfrak{C}$ is a reduction system for f. Then f leaves each curve in $\mathfrak{C}$ invariant and $\Lambda(f)$ leaves each component and each boundary component of $\mathbf{S}_{\mathfrak{C}}$ invariant.*

Hence if $f \in \mathcal{T}$ reduces along $\mathfrak{C}$, then $\Lambda(f)$ restricts to each component of $\mathbf{S}_{\mathfrak{C}}$. We call each component of $\mathbf{S}_{\mathfrak{C}}$ as well as each restriction of $\Lambda(f)$ a *component of f (determined by $\mathfrak{C}$).* In particular, if $\mathbf{S}'$ is a component of $\mathbf{S}_{\mathfrak{C}}$, and the restriction of $\Lambda(f)$ to $\mathbf{S}'$ is a pseudo-Anosov element of $\mathcal{M}(\mathbf{S}')$, then we say that $\mathbf{S}'$ is a pseudo-Anosov component of f.

The notion of an adequate reduction system, which we describe presently, was first introduced by Birman, Lubotzky, and McCarthy in [2]. However, their paper deals with a more general situation (the mapping class group of a disconnected surface) than we will need to consider (the Torelli group of a connected surface), and so the definition we give is specialized to suit our needs. We say that a reduction system $\mathfrak{C}$ for an element $f \in \mathcal{T}(\mathbf{S})$ is an *adequate reduction system for f* if each component of f is either trivial or is pseudo-Anosov, and in this case, we say that $\Lambda(f)$, the reduction of f along $\mathfrak{C}$, is *adequately reduced.* Using this concept and fact that the Torelli group is torsion-free, Thurston's classification of mapping classes (cf. [7]) implies that every element f of the Torelli group is either reducible or pseudo-Anosov, and if f is reducible, then it has an adequate reduction system.

We say that a reduction class $\mathfrak{a}$ for $f \in \mathcal{M}(\mathbf{S})$ is an *essential reduction class* for f if and only if for each $\mathfrak{b} \in \mathfrak{S}$ such that $i(\mathfrak{a}, \mathfrak{b}) \neq 0$, the isotopy classes $f^m(\mathfrak{b})$ and $\mathfrak{b}$ are distinct for all $m \neq 0$. Given $f \in \mathcal{M}(\mathbf{S})$, we define the *essential reduction system for* f by

$$\mathfrak{E}_f = \{\mathfrak{a} \in \mathfrak{S} : \mathfrak{a} \text{ is an essential reduction class for } f\}.$$

The following facts are proved in [2]:

Theorem 0.2 (Birman, Lubotzky, and McCarthy)

i) Let $\mathfrak{a}$ and $\mathfrak{b}$ be reduction classes for $f \in \mathcal{M}(\mathbf{S})$. If $\mathfrak{a}$ is essential, then $i(\mathfrak{a}, \mathfrak{b}) = 0$.

ii) Let $\mathfrak{E}$ be an adequate reduction system for f and let $\mathfrak{a} \in \mathfrak{E}$. Then $\mathfrak{a}$ is essential if and only if $\mathfrak{E} \setminus \{\mathfrak{a}\}$ is not an adequate reduction system for f^m, for any $m \neq 0$.

iii) For all $g \in \mathcal{M}(\mathbf{S})$, $g(\mathfrak{E}_f) = \mathfrak{E}_{gfg^{-1}}$.

iv) $\mathfrak{E}_{f^m} = \mathfrak{E}_f$ for all $m \neq 0$.

v) $\mathfrak{E}_f$ is an adequate reduction system for f.

vi) $\mathfrak{E}_f \subseteq \mathfrak{E}$ for each adequate reduction system $\mathfrak{E}$ for f.

Note that *(ii)* above implies that the essential reduction system of an infinite order reducible mapping class is nonempty, and *(iii)* implies that if f and g commute, then the essential reduction system for f is a reduction system for g. A standard result is that if $f = D_{\mathfrak{a}_1}^{n_1} D_{\mathfrak{a}_2}^{n_2} \cdots D_{\mathfrak{a}_k}^{n_k}$

is a multitwist about the reduction system $\mathfrak{E} = \{\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_k\}$, where each exponent is nonzero, then $\mathfrak{E}_f = \mathfrak{E}$. In particular, $\mathfrak{E}_{D_\mathfrak{a}} = \{\mathfrak{a}\}$.

John McCarthy has proved the following, which we refer to as McCarthy's Theorem:

Theorem 0.3 (McCarthy's Theorem)([11], Theorem 1 and Corollary 3) *Let f be a pseudo-Anosov element in the mapping class group $\mathcal{M}$ of a compact, connected, orientable surface with possibly nonempty boundary. Then the centralizer $\mathcal{C}_{\mathcal{M}}(f)$ of the cyclic subgroup of $\mathcal{M}$ generated by f is a finite extension of an infinite cyclic group, and consequently every torsion-free subgroup of $\mathcal{C}_{\mathcal{M}}(f)$ is infinite cyclic.*

This theorem implies the following useful corollary concerning the Torelli group:

Corollary 0.4 *Let $\mathcal{A}$ be an Abelian subgroup of the Torelli group of a compact, connected, orientable surface with possibly nonempty boundary. If $\mathcal{A}$ contains a pseudo-Anosov mapping class, then $\mathcal{A}$ is infinite cyclic, and any generator is pseudo-Anosov.*

The specific issue addressed in this work concerns automorphisms of the Torelli group of $\mathbf{S}$. Namely, are all automorphisms of $\mathcal{T}$ topological (i.e., induced by homeomorphisms of the surface)?

In the 1980's Nikolai Ivanov answered affirmatively the analogous question for the mapping class group (genus ≥ 3). One of the inspirations for Ivanov's work was the results concerning Abelian subgroups of the mapping class group published in 1983 by Birman, Lubotzky, and

McCarthy [2]. The idea of the essential reduction system, introduced in their paper, played a central role in Ivanov's arguments as he characterizes certain elements of the mapping class group in terms of their centralizers and centers of centralizers. He then proceeded by using algebraic methods involving relations in the mapping class group.

Using Harvey's complex of curves, Ivanov later proved a stronger theorem [9], which includes as a special case the result about automorphisms of the mapping class group. Specifically, he first showed that an automorphism Ψ of the mapping class group induces an automorphism of the complex of curves, and then that any automorphism of the complex of curves is induced by a homeomorphism h of the surface. The theorem is proved when he shows that the automorphism of the mapping class group induced by h is the same as the automorphism Ψ.

We follow the same basic steps in the case of the Torelli group. First we investigate the Abelian subgroups, proving two primary theorems. The first gives a complete description of "Torelli multitwists," and the second states that any Abelian subgroup of $\mathcal{T}$ has rank at most $2g-3$. In this portion of the work, in chapters 1 and 2, graph theory is the principal tool.

We then deal with the automorphism question directly. In several formal and informal announcements made between October 2001 and March 2002, Benson Farb announced that he was able to prove that every automorphism of the Torelli group is induced by a homeomorphism of $\mathbf{S}$, for genus $g \geq 4$, [4], [6]. He outlined his proof in [5].

In chapter 3 we lay the foundation for proving that this is also true for $g = 3$. Following Ivanov, we use centralizers and centers of centralizers to give algebraic charactertizations of certain elements of the Torelli group, namely powers of Dehn twists about separating curves and powers of bounding pair maps. We note that Farb's characterization is valid only for $g \geq 4$, whereas the characterization we give is valid for $g \geq 3$. The second step, showing that any automorphism Ψ of the Torelli group induces an automorphism Ψ_* of the complex of curves, remains incomplete at the time of this writing. In the last step we use Ivanov's theorem stating that Ψ_* is induced by a homeomorphism h of the surface $\mathbf{S}$ to conclude, under the assumption that it is possible to complete the second step, that the automorphism of the Torelli group induced by the homeomorphism h agrees with our automorphism Ψ. The apparent difficulty in proving this result purely algebraically (i.e., without using the complex of curves) suggests that the automorphism theorem for the Torelli group is a deeper result than for the mapping class group.

CHAPTER 1

Reduction Systems and Reduction System Graphs

1.1 Graph Terminology

We use graph-theoretic terminology consistent with its use in [3]. We remind the reader of the less familiar terms, and give the graph-theoretic definitions of those terms that may be used in different ways in ordinary topology.

We let G denote a connected, finite linear graph. We include the possibility that G may contain loops or parallel edges. $E = E(G)$ will denote the edge set of G, and we use the symbols a, b, c, etc. to denote edges of G. For $E' \subset E(G)$, $G - E'$ denotes the subgraph obtained from G by deleting the edges in E', while $G + E''$ is the graph obtained from G by adding a set of edges E''. If $E' = \{e\}$, then we write $G - e$ and $G + e$ instead of $G - \{e\}$ and $G + \{e\}$. A *bond* E' in G is a minimal subset of $E(G)$ such that $G - E'$ is disconnected. Note that $G - E'$ consists of precisely two components. We say that the edge e is a *cut edge* if $G - e$ is disconnected. We use the symbols u, v, x, y to denote vertices of G. The *degree* of a vertex v is the number of edges incident with v, each loop counting as two edges.

A (v_0, v_n)–*walk* W of *length* n is a finite nonempty alternating sequence, $W = v_0 e_1 v_1 e_2 v_2 \ldots e_n v_n$, of vertices and edges such that the ends of the edge e_i are the vertices v_{i-1} and v_i for $1 \leq i \leq n$. If the edges of W are distinct, W is called a *trail.* A *cycle* in G is a closed trail of positive length whose origin and internal vertices are distinct. Thus a cycle is an embedded circle in G. For our purposes, to denote a trail or cycle, it will be enough to give its sequence of edges, and we do not distinguish between a closed trail W and another closed trail whose sequence of edges is a cyclic permutation of W's.

A *spanning tree* T is a subgraph of G with the same vertex set as G such that T contains no cycles. The number of edges in any spanning tree is equal to one less than the number of vertices of G. Note that if T is a spanning tree, and e is an edge of G not in T, then $T + e$ contains a unique cycle C, and e is an edge of C, so the rank of $\pi_1(G)$ is equal to the number of edges of G outside any spanning tree. Every connected graph contains a spanning tree.

Given a subgraph H of G, we let $G \bullet H$ denote the graph obtained by deleting every edge e of H and identifying the ends of e. Equivalently, thinking of G as a CW–complex and H as a subcomplex, $G \bullet H$ is the complex obtained from G by crushing each component of H to a point. Thus, we have a quotient ("contraction") map $p : G \to G \bullet H$. Next, by a *cut vertex* of G, we mean a vertex v of G such that when v, *and only* v, is removed from the topological space G, the resulting space is disconnected. (This is not the definition used by graph theorists, but is an equivalent topological one.) A *block* is a connected graph without cut vertices, and a *block of a graph* is a subgraph that is a block and is maximal with repsect to that property. Any graph is the union of its blocks.

1.2 Reduction Systems and Reduction System Graphs

Let $\mathfrak{E}$ be a reduction system on $\mathbf{S}$, as defined in the introduction.

We partition the set $\mathfrak{E} = \{\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_n\}$ according to the equivalence relation $\sim$ generated by the rule

$$\mathfrak{e}_i \sim \mathfrak{e}_j \text{ if } \begin{cases} \mathfrak{e}_i = \mathfrak{e}_j \\ \text{or} \\ \{\mathfrak{e}_i, \mathfrak{e}_j\} \text{ is a minimal separating set in } \mathfrak{E} \end{cases}$$

Here, "$\{\mathfrak{e}_i, \mathfrak{e}_j\}$ is a minimal separating set" means that $\mathbf{S}_{\{\mathfrak{e}_i,\mathfrak{e}_j\}}$ is disconnected, but both $\mathbf{S}_{\mathfrak{e}_i}$ and $\mathbf{S}_{\mathfrak{e}_j}$ are connected. There are three types of $\sim$–equivalence classes:

i) Singleton classes $\{\mathfrak{a}_1\}, \{\mathfrak{a}_2\}, \ldots, \{\mathfrak{a}_p\}$ consisting of the separating curves $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_p$ in $\mathfrak{E}$. Such a curve wil be called an *a–type curve.*

ii) Classes $\{\mathfrak{b}_{11}, \ldots, \mathfrak{b}_{1q_1}\}, \{\mathfrak{b}_{21}, \ldots, \mathfrak{b}_{2q_2}\}, \ldots, \{\mathfrak{b}_{r1}, \ldots, \mathfrak{b}_{rq_r}\}$ of cardinality at least 2. Each such class $\{\mathfrak{b}_{i1}, \ldots, \mathfrak{b}_{in_i}\}$ is characterized by the following three properties:

a) No curve $\mathfrak{b}_{ij}$ is separating.

b) $\mathfrak{b}_{ij}$ is homologous to $\mathfrak{b}_{ij'}$ for every pair $\mathfrak{b}_{ij}$, $\mathfrak{b}_{ij'}$.

c) Maximal with respect to (a) and (b).

A curve in such a class will be called a *b–type curve.*

iii) Singleton classes $\{\mathfrak{c}_1\}, \{\mathfrak{c}_2\}, \ldots, \{\mathfrak{c}_s\}$ where each $\mathfrak{c}_i$ is non-separating and is homologous to no other curve in $\mathfrak{E}$. Such a curve will be called a *c–type curve.*

According to (i), (ii), and (iii) above, we write

$$\mathfrak{E} = \{\mathfrak{a}_1, \ldots, \mathfrak{a}_p, \mathfrak{b}_{11}, \ldots, \mathfrak{b}_{1q_1}, \mathfrak{b}_{21}, \ldots, \mathfrak{b}_{2q_2}, \ldots, \mathfrak{b}_{r1}, \ldots, \mathfrak{b}_{rq_r}, \mathfrak{c}_1, \ldots, \mathfrak{c}_s\}.$$

We use $\mathfrak{E}$ to define a graph $G_{\mathfrak{E}}$, which we call the *reduction system graph* of $\mathfrak{E}$, as follows:

Vertices of $G_{\mathfrak{E}}$ correspond to the components of $\mathbf{S}_{\mathfrak{E}}$.

Edges of $G_{\mathfrak{E}}$ correspond to the curves in the reduction system $\mathfrak{E}$, with

(Links) Two distinct vertices are connected by the edge e_i if and only if the curve $\mathfrak{e}_i$ in $\mathfrak{E}$ is a common boundary curve of the two components of $\mathbf{S}_{\mathfrak{E}}$ which correspond to the vertices in question.

(Loops) A vertex has a loop e_i if and only if the curve $\mathfrak{e}_i$ in $\mathfrak{E}$ represents two boundary curves of the component of $\mathbf{S}_{\mathfrak{E}}$ which corresponds to the vertex in question.

Note that $G_{\mathfrak{E}}$ is connected, and that any connected graph G is $G_{\mathfrak{E}}$ for some surface $\mathbf{S}$ and some reduction system $\mathfrak{E}$ on $\mathbf{S}$. However, the genus of $\mathbf{S}$ is not determined by G, any two possible $\mathbf{S}$'s differing by the genera of their complementary components. But, unless G is the graph consisting of a single vertex and either no edges or a single loop, then genus($\mathbf{S}$) $\geq$ rank($\pi_1(G)$) + (number of vertices of degree ≤ 2).

Since $\mathbf{S}$ and $\mathfrak{C}$ will be fixed, we will denote $G_{\mathfrak{C}}$ simply by G.

The equivalence relation $\sim$ on the curves in $\mathfrak{C}$ induces a equivalence relation $\sim$ on the edge set $E(G) = \{e_1, e_2, ..., e_n\}$ of G. It is generated by

$$e_i \sim e_j \text{ if } \begin{cases} e_i = e_j \\ \text{or} \\ \{e_i, e_j\} \text{ is a bond} \end{cases}$$

(Again, it should be noted that this equivalence relation may be defined for *any* graph G.) The three types of equivalence classes described above become, for G,

i) Singleton classes $\{a_1\}, \ldots, \{a_p\}$ consisting of the cut edges $a_1, \ldots, a_p$ of G. Such an edge will be called an *a-type edge.*

ii) Classes $\{b_{11}, \ldots, b_{1q_1}\}, \{b_{21}, \ldots, b_{2q_2}\}, \ldots, \{b_{r1}, \ldots, b_{rq_r}\}$ of cardinality at least 2. Each such class is characterized by the following three properties:

  a) No edge b_{ij} is a cut edge.

  b) $\{b_{ij}, b_{ij'}\}$ is a bond for every pair b_{ij}, $b_{ij'}$.

  c) Maximal with respect to (a) and (b).

  An edge in such a class will be called a *b-type edge.*

iii) Singleton classes $\{c_1\}, \ldots, \{c_s\}$ where each c_i is not a cut edge, and forms a 2–edge bond with no other edge of G. Such an edge will be called a *c-type edge.*

According to (i), (ii), and (iii) above, we write

$$E(G) = \{a_1, \ldots, a_p, b_{11}, \ldots, b_{1q_1}, b_{21}, \ldots, b_{2q_2}, \ldots, b_{r1}, \ldots, b_{rq_r}, c_1, \ldots, c_s\}$$

Figure 1.1 shows a typical example of a reduction system and its graph.

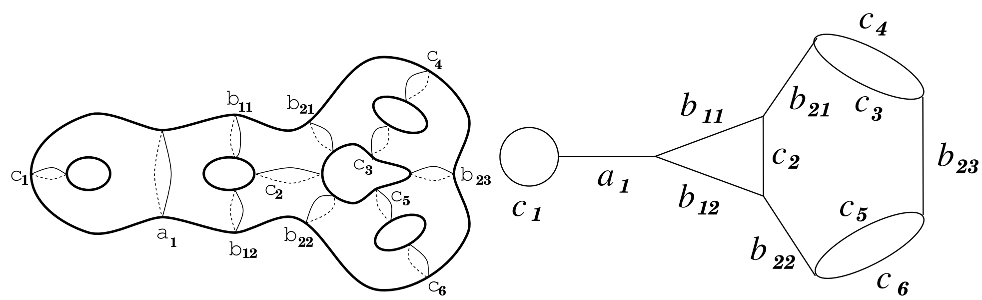

Figure 1.1: On left: A surface with a reduction system. On right: The reduction system graph.

Now let $\mathfrak{h}$ be a simple closed curve on $\mathbf{S}$ that intersects each element of $\mathfrak{C}$ transversely at most once. Starting at any point on $\mathfrak{h}$ and travelling in either direction gives a cyclic ordering of the reduction curves which $\mathfrak{h}$ intersects, thus defining a closed trail H in G. Note that H is a cycle in G if and only if $\mathfrak{h} \cap \mathbf{S}_i$ is either empty or is a single (that is, connected) arc, for every component $\mathbf{S}_i$ of $\mathbf{S}_{\mathfrak{C}}$. Likewise, given a closed trail H in G, there is such a curve $\mathfrak{h}$ on $\mathbf{S}$ defining H. The fact that the isotopy class of $\mathfrak{h}$ is never unique is not important for our purposes.

Figure 1.2 shows a typical example. Note that $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are nonisotopic curves which both define the cycle $H = b_{11}c_2b_{12}$.

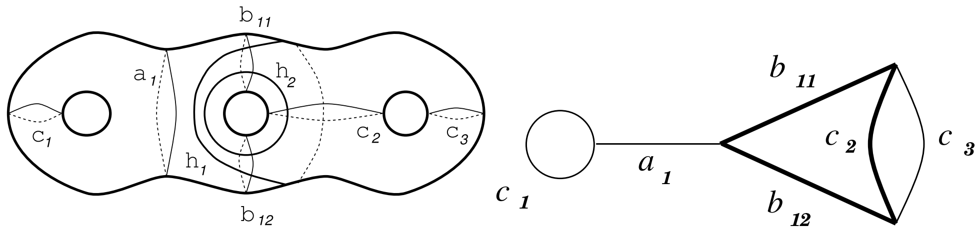

Figure 1.2: On left: Curves $\mathfrak{h}_1$ and $\mathfrak{h}_2$. On right: The cycle H "produced" by $\mathfrak{h}_1$ and $\mathfrak{h}_2$.

1.3 Results on Graphs

This section presents some purely graph-theoretic results, concluding with Theorem 1.4, which is used to prove the main theorem of section 2.1. Here, G denotes an arbitrary connected, finite linear graph.

Lemma 1.1 *If G has no cut edges, then any two vertices of G are connected by two edge-disjoint paths.*

Proof:

We prove by induction on $d(u, v)$ that any two vertices u and v in G are connected by two edge-disjoint paths.

$d(u, v) = 1$:

In this case, uv is an edge, and so by hypothesis uv is not a cut edge. By [3], page 27, uv is contained within a cycle C. Then uv and $C - uv$ are two edge-disjoint (u, v)-paths.

Now suppose the theorem is true for any two vertices at distance less than k, and let $d(u, v) = k \geq 2$. Consider a (u, v)-path of length k, and let w be the vertex preceding v on this path. Then $d(u, w) = k - 1$, so by the induction hypothesis, there are two edge-dijoint (u, w)-paths P and Q in G. Since G has no cut edges, $G - wv$ is connected, so there is a

(u, v)-path P' in $G - wv$. Lew x be the last vertex of P' lying in $P \cup Q$, and suppose that $x \in P$. Then G has two edge-disjoint (u, v)-paths: one composed of the portion of P from u to x, the other is $Q + wv$. $\square$

Lemma 1.2 *Let b_1 and b_2 be edges of G such that $\{b_1, b_2\}$ is a bond. If C is a cycle in G, and b_1 is an edge of C, then so is b_2.*

Proof:

Let b_1 have ends u_1 and v_1, and let b_2 have ends u_2 and v_2. Since $\{b_1, b_2\}$ is a bond, $G - b_1$ and $G - b_2$ are connected, but $G - \{b_1, b_2\}$ is not, having two components which, without loss of generality, separate u_2 and v_2.

Let $C = b_1 e_1 e_2 \cdots e_n$, and suppose b_2 is not an edge of C. Then $e_1 e_2 \cdots e_n$ is a path in $G = \{b_1, b_2\}$ connecting u_1 and v_1, so u_1 and v_1 lie in the same component of $G - \{b_1, b_2\}$. Since there is no (u_2, v_2)-path in $G - \{b_1, b_2\}$, but there *is* a (u_2, v_2)-path P in $G - b_2$, P must contain b_1. Let P_1 be the portion of P from u_2 to, say, u_1, and let P_2 be the portion of P from v_1 to v_2. Then $P_1 + e_1 e_2 \cdots e_n + P_2$ is a (u_2, v_2)-path in $G - \{b_1, b_2\}$. This is a contradiction.

Hence b_2 must also be an edge of C. $\square$

Lemma 1.3 *Let c be a c–type edge in G that is not a loop. Then c is contained within two cycles of G, the intersection of whose edge sets is precisely c.*

Proof:

Assume that G is a block. If G has exactly two vertices, then each edge of G is a link, and G must have at least three edges, since c is a c–type edge. The result is clear in this case. Otherwise, G has at least three vertices and no cut edges. Consider the graph $G - c$. If $G - c$ has a cut edge e, then $G - \{c, e\}$ is not connected, so $\{c, e\}$ is a bond of G. This contradicts the fact that c is a c–type edge. So $G - c$ has no cut edges. By Lemma 1.1, there are two edge-disjoint paths P and P' in $G - c$ connecting the ends of c. Then the cycles $C = P + c$ and $C' = P' + c$ have exactly the edge c in common. In the case that G is not a block, we let B be the block of G containing c. It is easy to see that c is a c–type edge of B, so we apply the first case to B and find two such cycles within B. □

Theorem 1.4 *Let G have edge set*

$$E(G) = \{a_1, \dots, a_p, b_{11}, \dots, b_{1q_1}, \dots, b_{r1}, \dots, b_{rq_r}, c_1, \dots, c_s\},$$

notated according to a– , b–, *and* c–*type equivalence classes. Let $w : E(G) \to \mathbb{Z}$ be a weighting of G. Then $w(H) = 0$ for every cycle H in G if and only if*

i) $w(c_i) = 0, \;\; 1 \leq i \leq s,$ *and*

ii) $w(b_{j1}) + w(b_{j2}) + \cdots + w(b_{jq_j}) = 0, \;\; 1 \leq j \leq r.$

Proof:

$\Rightarrow$) Assume that $w(H) = 0$ for every cycle H in G.

i) Let c be a c–type edge with ends u and v. If c is a loop, then $w(c) = 0$, by hypothesis. Otherwise, there are two edge-disjoint (u, v)–paths, P and $\overline{P'}$, in $G - c$. We have three cycles: $P + c$, $P' + c$, and $P + P'$. Thus,

$$\left.\begin{aligned} w(P) + w(c) = w(P + c) = 0 \\ w(P') + w(c) = w(P' + c) = 0 \\ w(P) + w(P') = w(P + P') = 0 \end{aligned}\right\} \Longrightarrow w(c) = 0$$

ii) Let B be the equivalence class of the b–type edge b, and $\overline{B} = B \diagdown \{b\}$. Let $p : G \to G \bullet \overline{B}$ be the contraction map. Suppose that b is a cut edge of $G \bullet \overline{B}$, separating it into two components G_1 and G_2. Then the restriction of p to $G - b$ maps onto the disconnected space $G_1 \cup G_2$, and so $G - b$ is disconnected. This is a contradiction to the hypothesis that b is a b–type edge of G. We obtain a similar contradiction if we suppose $\{b, e\}$ is a bond in $G \bullet \overline{B}$. Thus b is a c–type edge in $G \bullet \overline{B}$. If b is a loop in $G \bullet \overline{B}$, then $p^{-1}(b) = B$, which therefore forms a cycle in G. So equation (ii) holds for the equivalence class of b.

If b is not a loop in $G \bullet \overline{B}$, then by Lemma 1.3 there are two cycles $\overline{H}$ and $\overline{H'}$ in $G \bullet \overline{B}$, the intersection of whose edge sets is $\{b\}$. Lemma 1.2 implies that $p^{-1}(\overline{H})$ and $p^{-1}(\overline{H'})$ are cycles H and H', respectively, the intersection of whose edge sets is precisely B. Thus we have

$$\left.\begin{aligned} 0 = w(H) = w(B) + w(H - B) \\ 0 = w(H') = w(B) + w(H' - B) \end{aligned}\right\} \Rightarrow \; 0 = 2w(B) + w(H \Delta H') = 2w(B)$$

And so, $w(B) = 0$. Here we have used the fact that the symmetric difference $H \Delta H'$ of the cycles H and H' is a disjoint union of cycles (regarded as sets of edges).

$\Leftarrow$) Assume that

i) $w(c_i) = 0, \;\; 1 \leq i \leq s,$ and

ii) $w(b_{j1}) + w(b_{j2}) + \cdots + w(b_{jq_j}) = 0, \;\; 1 \leq j \leq r.$

Let H be a cycle in G. H contains no a–type edges, since they are cut edges, and by Lemma 1.2, if H contains one edge of a b–type class, then it contains the whole class. So the assumptions imply that $w(H) = 0$.

$\square$

CHAPTER 2

Abelian Subgroups of the Torelli Group

2.1 Multitwists in the Torelli Group

We at first consider a specific type of Abelian subgroup of the Torelli group $\mathcal{T}(\mathbf{S})$, namely one consisting of *multitwists* — that is, compositions of left and right Dehn twists about a fixed reduction system $\mathfrak{C}$ on $\mathbf{S}$.

Theorem 2.1 *Let* $\mathbf{S}$ *be a closed, connected, oriented surface, and let*

$$\mathfrak{C} = \{\mathfrak{a}_1, \ldots, \mathfrak{a}_p, \mathfrak{b}_{11}, \ldots, \mathfrak{b}_{1q_1}, \mathfrak{b}_{21}, \ldots, \mathfrak{b}_{2q_2}, \ldots, \mathfrak{b}_{r1}, \ldots, \mathfrak{b}_{rq_r}, \mathfrak{c}_1, \ldots, \mathfrak{c}_s\}$$

be a reduction system on $\mathbf{S}$*, notated by* a–*,* b–*, and* c*–type* $\sim$*–equivalence classes as in section 1.2. Let* $\mathcal{D}_{\mathfrak{C}}$ *be the multitwist group on* $\mathfrak{C}$*, and let*

$$f = D_{\mathfrak{a}_1}^{\alpha_1} \cdots D_{\mathfrak{a}_p}^{\alpha_p} D_{\mathfrak{b}_{11}}^{\beta_{11}} \cdots D_{\mathfrak{b}_{1q_1}}^{\beta_{1q_1}} D_{\mathfrak{b}_{21}}^{\beta_{21}} \cdots D_{\mathfrak{b}_{2q_2}}^{\beta_{2q_2}} \cdots D_{\mathfrak{b}_{rqr}}^{\beta_{rqr}} D_{\mathfrak{c}_1}^{\gamma_1} \cdots D_{\mathfrak{c}_s}^{\gamma_s}$$

be an element of $\mathcal{D}_{\mathfrak{C}}$*. Then* f *is an element of* $\mathcal{D}_{\mathfrak{C}} \cap \mathcal{T} \equiv \mathcal{T}_{\mathfrak{C}}$*, which we call the* Torelli multitwist group of $\mathfrak{C}$*, if and only if*

i) $\gamma_i = 0,\ 1 \leq i \leq s,$ *and*

ii) $\beta_{j1} + \beta_{j2} + \cdots + \beta_{jq_j} = 0,\ 1 \leq j \leq r.$

Consequently, $\mathcal{T}_{\mathfrak{E}}$ is a free Abelian group of rank

$$p + (q_1 - 1) + (q_2 - 1) + \cdots + (q_r - 1) = p + q_1 + q_2 + \cdots + q_r - r.$$

Proof:

$\Rightarrow$) Assume that $f \in \mathcal{T}_{\mathfrak{E}}$.

Let G be the reduction system graph of $\mathfrak{E}$ with edge set $E(G)$. We weight each edge of G according to the exponent in f of the twist about its corresponding curve in $\mathfrak{E}$, giving $w : E(G) \to \mathbb{Z}$.

Let $H = e_1 e_2, \ldots, e_n$ be a cycle in G. Then, as in section 1.2, H is defined by any simple closed curve $\mathfrak{h}$ on $\mathbf{S}$ that intersects each of the corresponding curves $\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_n$ of $\mathfrak{E}$ exactly once, and does not intersect any of the other curves of $\mathfrak{E}$. Orient $\mathfrak{h}$. Then orient the curves $\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_n$ so that $\langle \mathfrak{h}, \mathfrak{e}_i \rangle = 1$. So we have

$$0 = \langle \mathfrak{h}, \mathfrak{h} \rangle = \langle \mathfrak{h}, f(\mathfrak{h}) \rangle = \langle \mathfrak{h}, \mathfrak{h} + \epsilon_1 \mathfrak{e}_1 + \epsilon_2 \mathfrak{e}_2 + \cdots + \epsilon_n \mathfrak{e}_n \rangle = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n,$$

where $\epsilon_i = w(e_i)$. Hence the weight of every cycle in G is zero. The conclusion follows from Theorem 1.4.

$\Leftarrow$) Assume that

i) $\gamma_i = 0$, $1 \leq i \leq s$, and

ii) $\beta_{j1} + \beta_{j2} + \cdots + \beta_{jq_j} = 0$, $1 \leq j \leq r$.

Since $H_1(\mathbf{S})$ has a basis consisting of simple closed curves, in order to prove that $f \in \mathcal{T}$, it suffices to show that in $H_1(\mathbf{S})$, we have $f(\mathfrak{h}) = \mathfrak{h}$ for any simple closed curve $\mathfrak{h}$ on $\mathbf{S}$. Note that for any such $\mathfrak{h}$, we have $\langle \mathfrak{a}_i, \mathfrak{h} \rangle = 0$, $1 \leq i \leq p$, and after orienting $\mathfrak{h}$ and then each $\mathfrak{b}_{ij}$ so that

$\langle \mathfrak{b}_{ij}, \mathfrak{h} \rangle = \langle \mathfrak{b}_{i1}, \mathfrak{h} \rangle$, we have $\mathfrak{b}_{ij} = \mathfrak{b}_{i1}$, $2 \leq j \leq q_i$, $1 \leq i \leq r$. Let $\delta_i = \langle \mathfrak{b}_{i1}, \mathfrak{h} \rangle$. Then in $H_1(\mathbf{S})$ we have

$$\begin{aligned}
f(\mathfrak{h}) &= D_{\mathfrak{a}_1}^{\alpha_1} \cdots D_{\mathfrak{a}_p}^{\alpha_p} D_{\mathfrak{b}_{11}}^{\beta_{11}} \cdots D_{\mathfrak{b}_{1q_1}}^{\beta_{1q_1}} D_{\mathfrak{b}_{21}}^{\beta_{21}} \cdots D_{\mathfrak{b}_{2q_2}}^{\beta_{2q_2}} \cdots D_{\mathfrak{b}_{rq_r}}^{\beta_{rq_r}} D_{\mathfrak{c}_1}^{\gamma_1} \cdots D_{\mathfrak{c}_s}^{\gamma_s}(\mathfrak{h}) \\
&= \mathfrak{h} + \beta_{11} \langle \mathfrak{b}_{11}, \mathfrak{h} \rangle \mathfrak{b}_{11} + \cdots + \beta_{1q_1} \langle \mathfrak{b}_{1q_1}, \mathfrak{h} \rangle \mathfrak{b}_{1q_1} + \cdots \\
&\quad + \beta_{r1} \langle \mathfrak{b}_{r1}, \mathfrak{h} \rangle \mathfrak{b}_{r1} + \cdots + \beta_{rq_r} \langle \mathfrak{b}_{rq_r}, \mathfrak{h} \rangle \mathfrak{b}_{rq_r} \\
&= \mathfrak{h} + \delta_1(\beta_{11} + \cdots + \beta_{1q_1})\mathfrak{b}_{11} + \cdots + \delta_r(\beta_{r1} + \cdots + \beta_{rq_r})\mathfrak{b}_{r1} \\
&= \mathfrak{h}
\end{aligned}$$

□

Theorem 2.2 *Let* $\mathbf{S}$ *be a closed connected oriented surface, and let* $\mathfrak{E} = \{\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_n\}$ *be a reduction system on* $\mathbf{S}$. *Let* $f = D_{\mathfrak{e}_1}^{\epsilon_1} D_{\mathfrak{e}_2}^{\epsilon_2} \cdots D_{\mathfrak{e}_n}^{\epsilon_n}$ *be a multitwist on* $\mathfrak{E}$. *Let* G *be the reduction system graph of* $\mathfrak{E}$, *and define a weighting* $w : E(G) \to \mathbb{Z}$ *of* G *by* $w(e_i) = \epsilon_i$. *Then* f *is in the Torelli multitwist group* $\mathcal{T}_{\mathfrak{E}}$ *if and only if the weight of every cycle in* G *is zero.*

Proof:

Partition $\mathfrak{E}$ into $\sim$–equivalence classes and write

$$\mathfrak{E} = \{\mathfrak{a}_1, \ldots, \mathfrak{a}_p, \mathfrak{b}_{11}, \ldots, \mathfrak{b}_{1q_1}, \mathfrak{b}_{21}, \ldots, \mathfrak{b}_{2q_2}, \ldots, \mathfrak{b}_{r1}, \ldots, \mathfrak{b}_{rq_r}, \mathfrak{c}_1, \ldots, \mathfrak{c}_s\}.$$

Theorems 1.4 and 2.1 show the conditions to be equivalent. □

Given a pair, $\mathfrak{e}_1$ and $\mathfrak{e}_2$, of disjoint, non-separating, but homologous simple closed curves on $\mathbf{S}$, we call $D_{\mathfrak{e}_1} D_{\mathfrak{e}_2}^{-1}$ a *bounding-pair map*. Powell

[12] has shown that the Torelli group $\mathcal{T}$ is generated by bounding pair maps and Dehn twists about separating simple closed curves.

Corollary 2.3 *Let* $\mathbf{S}$*,* $\mathfrak{E}$*,* $\mathcal{D}_{\mathfrak{E}}$*, and* $\mathcal{T}_{\mathfrak{E}}$ *be as in Theorem 2.1. Let* $\mathcal{D}'$ *be the subgroup of* $\mathcal{M}(\mathbf{S})$ *generated by*

i) bounding pair maps about bounding pairs in $\mathfrak{E}$*, and*

ii) Dehn twists about separating curves in $\mathfrak{E}$*.*

Then $\mathcal{D}' = \mathcal{D}_{\mathfrak{E}} \cap \mathcal{T} = \mathcal{T}_{\mathfrak{E}}$*.*

Proof:

By the definition of $\mathcal{D}_{\mathfrak{E}}$, it is clear that every generator of $\mathcal{D}'$ is in $\mathcal{D}_{\mathfrak{E}}$. By Powell's result noted above, every generator of $\mathcal{D}'$ is in $\mathcal{T}$. Thus $\mathcal{D}' \subseteq \mathcal{D}_{\mathfrak{E}} \cap \mathcal{T}$. We must show that $\mathcal{D}_{\mathfrak{E}} \cap \mathcal{T} \subseteq \mathcal{D}'$.

Let $f \in \mathcal{D}_{\mathfrak{E}} \cap \mathcal{T}$. By Theorem 3.1, we know that

$$f = D_{\mathfrak{a}_1}^{\alpha_1} \cdots D_{\mathfrak{a}_p}^{\alpha_p} D_{\mathfrak{b}_{11}}^{\beta_{11}} \cdots D_{\mathfrak{b}_{1q_1}}^{\beta_{1q_1}} D_{\mathfrak{b}_{21}}^{\beta_{21}} \cdots D_{\mathfrak{b}_{2q_2}}^{\beta_{2q_2}} \cdots D_{\mathfrak{b}_{r1}}^{\beta_{r1}} \cdots D_{\mathfrak{b}_{rq_r}}^{\beta_{rq_r}},$$

where $\beta_{i1}+\beta_{i2}+\cdots+\beta_{iq_i} = 0,\ 1 \leq i \leq r$. Since each $D_{\mathfrak{a}_i}^{\alpha_i}$ is a product of type–(ii) generators of $\mathcal{D}'$, we will be done if we write $D_{\mathfrak{b}_{i1}}^{\beta_{i1}} D_{\mathfrak{b}_{i2}}^{\beta_{i2}} \cdots D_{\mathfrak{b}_{iq_i}}^{\beta_{iq_i}}$ as a product of bounding pair maps. We do this:

$$D_{\mathfrak{b}_{i1}}^{\beta_{i1}} D_{\mathfrak{b}_{i2}}^{\beta_{i2}} \cdots D_{\mathfrak{b}_{iq_i}}^{\beta_{iq_i}} = (D_{\mathfrak{b}_{i2}} D_{\mathfrak{b}_{i1}}^{-1})^{\beta_{i2}} (D_{\mathfrak{b}_{i3}} D_{\mathfrak{b}_{i1}}^{-1})^{\beta_{i3}} \cdots (D_{\mathfrak{b}_{iq_1}} D_{\mathfrak{b}_{i1}}^{-1})^{\beta_{iq_i}},$$

where we note that $-\beta_{i2} - \beta_{i3} - \cdots - \beta_{iq_i} = \beta_{i1}$. $\square$

Corollary 2.4 *Let* $\mathbf{S}$ *be a closed, connected, oriented surface, and let*

$$\mathfrak{E} = \{\mathfrak{a}_1, \ldots, \mathfrak{a}_p, \mathfrak{b}_{11}, \ldots, \mathfrak{b}_{1q_1}, \mathfrak{b}_{21}, \ldots, \mathfrak{b}_{2q_2}, \ldots, \mathfrak{b}_{r1}, \ldots, \mathfrak{b}_{rq_r}, \mathfrak{c}_1, \ldots, \mathfrak{c}_s\}$$

be a reduction system on $\mathbf{S}$*, notated by* a–, b–, *and* c*–type* $\sim$*–equivalence classes as in section 2. Let* $\mathcal{D}_{\mathfrak{C}}$ *be the multitwist group on* $\mathfrak{C}$*, and let*

$$f = D_{\mathfrak{a}_1}^{\alpha_1} \cdots D_{\mathfrak{a}_p}^{\alpha_p} D_{\mathfrak{b}_{11}}^{\beta_{11}} \cdots D_{\mathfrak{b}_{1q_1}}^{\beta_{1q_1}} D_{\mathfrak{b}_{21}}^{\beta_{21}} \cdots D_{\mathfrak{b}_{2q_2}}^{\beta_{2q_2}} \cdots D_{\mathfrak{b}_{rq_r}}^{\beta_{rq_r}} D_{\mathfrak{c}_1}^{\gamma_1} \cdots D_{\mathfrak{c}_s}^{\gamma_s}$$

be an element of $\mathcal{D}_{\mathfrak{C}}$*. Let* $m \geq 2$ *be an integer.*
Then $f \in \Gamma_{\mathbf{S}}(m) \equiv \{g \in \mathcal{M}(\mathbf{S}) : g \text{ acts trivially on } H_1(\mathbf{S};\mathbb{Z}_m)\}$ *if and only if*

i) $\gamma_i \equiv 0 \pmod{m}$, $1 \leq i \leq s$, *and*

ii) $\beta_{j1} + \beta_{j2} + \cdots + \beta_{jq_j} \equiv 0 \pmod{m}$, $1 \leq j \leq r$.

Let $\mathbf{S}$ be the surface of genus $g \geq 2$ and $\mathfrak{C}$ the reduction system on $\mathbf{S}$ shown in Figure 2.1. Since $\mathfrak{C}$ consists of $2g-3$ a–type curves, $\text{rank}(\mathcal{T}_{\mathfrak{C}}) = 2g-3$. This example, along with Theorem 2.7 below, shows that the maximal rank of an Abelian subgroup of the Torelli group is attained by a multitwist group.

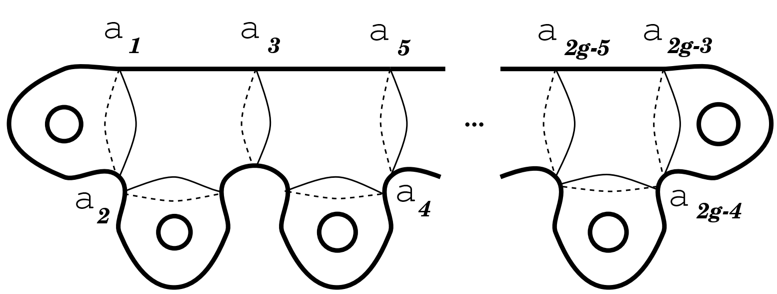

Figure 2.1: The rank of the Torelli multitwist group on this reduction system is $2g-3$.

2.2 The Rank of Abelian Subgoups of the Torelli Group

We prove that for any closed oriented surface of genus $g \geq 2$, the general Abelian subgoup of its Torelli group has rank $\leq 2g-3$. We first give two lemmas.

Lemma 2.5 *Let* $\mathbf{S}$ *be a closed, connected, oriented surface, and* $\mathfrak{C}$ *a reduction system on* $\mathbf{S}$ *with reduction system graph* G. *Let* $\mathcal{T}_{\mathfrak{C}}$ *be the Torelli multitwist group on* $\mathfrak{C}$, *as in Theorem 2.1. Then* $\operatorname{rank}(\mathcal{T}_{\mathfrak{C}}) \leq \nu - 1$, *where* ν *is the number of vertices of* G, *or, equivalently, the number of components of* $\mathbf{S}_{\mathfrak{C}}$.

Proof:

Let G have edge set

$E(G) = \{a_1, \ldots, a_p, b_{11}, \ldots, b_{1q_1}, b_{21}, \ldots, b_{2q_2}, \ldots, b_{r1}, \ldots, b_{rq_r}, c_1, \ldots, c_s\}$

notated according to a- b-, and c-type equivalence classes. Let

$$E' = \{b_{11}, \ldots, b_{1(q_1-1)}, b_{21}, \ldots, b_{2(q_2-1)}, \ldots, b_{r1}, \ldots, b_{r(q_r-1)}\} \subseteq E(G)$$

and let $G' = G[E']$. Then G' contains no cycles, since any cycle containing one edge of a b–type class contains the whole class. Therefore, G' is contained in a spanning tree T. T contains each cut edge a_i, $1 \leq i \leq p$, so T contains the set of edges $E' \cup \{a_1, a_2, \ldots, a_p\}$, and by Theorem 2.1, the cardinality of this set equals the rank of $\mathcal{T}_{\mathfrak{C}}$. Hence

$$\nu - 1 = \operatorname{card}(E(T)) \geq p + (q_1 - 1) + \cdots + (q_r - 1) = \operatorname{rank}(\mathcal{T}_{\mathfrak{C}})$$

$\square$

Lemma 2.6 *Let* $\mathbf{S}$ *be a closed, connected, oriented surface of genus* $g \geq 2$*, and let* $\mathfrak{C}$ *be a reduction system on* $\mathbf{S}$*. Let* Ω *denote the number of components of* $\mathbf{S}_{\mathfrak{C}}$ *not homeomorphic to a pair of pants or a one-holed torus. Let* $\mathcal{T}_{\mathfrak{C}}$ *be the Torelli multitwist group on* $\mathfrak{C}$*. Then* $\operatorname{rank}(\mathcal{T}_{\mathfrak{C}})+\Omega \leq 2g-3$*.*

Proof:

Let G be the reduction system graph of $\mathfrak{C}$. We use the following notation:

- Γ is the maximum genus of any component of $\mathbf{S}_{\mathfrak{C}}$.
- Δ is the maximum degree of any vertex of G, or, equivalently, the maximum number of boundary curves of any component of $\mathbf{S}_{\mathfrak{C}}$.
- ν_b is the number of vertices of G of degree b, or, equivalently, the number of components of $\mathbf{S}_{\mathfrak{C}}$ with b boundary curves.
- ν_b^γ ($\nu_b^{\geq\gamma}$) is the number of components of $\mathbf{S}_{\mathfrak{C}}$ of genus γ ($\geq \gamma$) having b boundary curves, or, equivalently, the number of vertices of G of degree b corresponding to a component of $\mathbf{S}_{\mathfrak{C}}$ of genus γ ($\geq \gamma$).

So we have

$$\nu_b = \sum_{\gamma=0}^{\Gamma} \nu_b^\gamma \qquad \text{and} \qquad \nu = \sum_{b=1}^{\Delta} \nu_b$$

But the assumption that each element of $\mathfrak{C}$ is homotopically nontrivial means $\nu_1^0 = 0$, and the assumption that the elements of $\mathfrak{C}$ are pairwise nonisotopic means $\nu_2^0 = 0$. So, in fact, $\nu = \nu_1^{\geq 1} + \nu_2^{\geq 1} + \nu_3 + \nu_4 + \cdots + \nu_\Delta$. Now, ν_1^1 is the number of one-holed tori, and ν_3^0 is the number of pairs of pants, so by the definition of Ω, we have

$$\Omega = \nu_1^{\geq 2} + \nu_2^{\geq 1} + \nu_3^{\geq 1} + \nu_4 + \cdots + \nu_\Delta.$$

$$\begin{aligned}
\text{Hence } 2g-2 &= -\chi(\mathbf{S}) \\
&= \sum_{\text{components } \mathbf{V} \text{ of } \mathbf{S}_{\mathfrak{E}}} -\chi(\mathbf{V}) \\
&= \sum_{\gamma=1}^{\Gamma} (2\gamma-1)\nu_1^\gamma + \sum_{\gamma=1}^{\Gamma} (2\gamma)\nu_2^\gamma + \sum_{b=3}^{\Delta} \sum_{\gamma=0}^{\Gamma} (2\gamma+b-2)\nu_b^\gamma.
\end{aligned}$$

By Lemma 3.1, $\text{rank}(\mathcal{T}_{\mathfrak{E}}) \leq \nu - 1$, so we have

$$\begin{aligned}
\text{rank}(\mathcal{T}_{\mathfrak{E}}) + \Omega &\leq \nu + \Omega - 1 \qquad (2.1) \\
&= (\nu_1^{\geq 1} + \nu_1^{\geq 2} + \cdots + \nu_\Delta) + (\nu_1^{\geq 2} + \nu_2^{\geq 1} + \nu_3^{\geq 1} + \nu_4 + \cdots + \nu_\Delta) - 1 \\
&= [(\nu_1^1 + 2\nu_1^{\geq 2}) + 2\nu_2^{\geq 1} + (\nu_3^0 + 2\nu_3^{\geq 1}) + 2\nu_4 + 2\nu_5 + \cdots + 2\nu_\Delta] - 1 \\
&\leq \left[\sum_{\gamma=1}^{\Gamma} (2\gamma-1)\nu_1^\gamma + \sum_{\gamma=1}^{\Gamma} (2\gamma)\nu_2^\gamma + \sum_{b=3}^{\Delta} \sum_{\gamma=0}^{\Gamma} (2\gamma+b-2)\nu_b^\gamma \right] - 1 \\
&= -\chi(\mathbf{S}) - 1 \\
&= 2g - 3
\end{aligned}$$

□

Theorem 2.7 *Let* $\mathbf{S}$ *be a closed, connected, oriented surface of genus* $g \geq 2$*, and let* $\mathcal{A}$ *be an Abelian subgroup of* $\mathcal{T}$*, the Torelli group of* $\mathbf{S}$*. Then* $\text{rank}(\mathcal{A}) \leq 2g - 3$*.*

Proof:

Let $f \in \mathcal{A}$, $f \neq 0$. As mentioned in the introduction, by Thurston's classification, f is either reducible or pseudo-Anosov.

Case 1: f is pseudo-Anosov.

By the corollary to McCarthy's Theorem given in the introduction, $\mathcal{A}$ is infinite cyclic.

Case 2: f is reducible.

Let $\mathfrak{C} = \bigcup_{h\in\mathcal{A}} \mathfrak{C}_h$. Then $\mathfrak{C}$ is an adequate reduction system for each $h \in \mathcal{A}$ ([2], Lemma 3.1(1)), and f reducible implies $\mathfrak{C} \neq \emptyset$, so every element of $\mathcal{A}$ is reducible.

Let $\Lambda : \mathcal{M}_{\mathfrak{C}}(\mathbf{S}) \to \mathcal{M}(\mathbf{S}_{\mathfrak{C}})$ be the reduction homomorphism. Then $\ker(\Lambda) = \mathcal{D}_{\mathfrak{C}}$, the multitwist group on $\mathfrak{C}$, and thus

$$\ker(\Lambda|_{\mathcal{A}}) = \ker(\Lambda) \cap \mathcal{A} = \mathcal{D}_{\mathfrak{C}} \cap \mathcal{A} = \mathcal{D}_{\mathfrak{C}} \cap \mathcal{T} \cap \mathcal{A} = \mathcal{T}_{\mathfrak{C}} \cap \mathcal{A}.$$

We now have a short exact sequence

$$0 \longrightarrow \mathcal{T}_{\mathfrak{C}} \cap \mathcal{A} \longrightarrow \mathcal{A} \xrightarrow{\Lambda|_{\mathcal{A}}} \Lambda(\mathcal{A}) \longrightarrow 0$$

of free Abelian groups, which shows that

$$\mathrm{rank}(\mathcal{A}) = \mathrm{rank}(\mathcal{T}_{\mathfrak{C}} \cap \mathcal{A}) + \mathrm{rank}(\Lambda(\mathcal{A})) \leq \mathrm{rank}(\mathcal{T}_{\mathfrak{C}}) + \mathrm{rank}(\Lambda(\mathcal{A}))$$

By applying Lemma 2.6, we will be done if we show that $\mathrm{rank}(\Lambda(\mathcal{A})) \leq \Omega$, the number of components of $\mathbf{S}_{\mathfrak{C}}$ not homeomorphic to a pair of pants or a one-holed torus.

Ivanov's theorem, given in the introduction, implies that $\Lambda(f)$ restricts to each component $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_\nu$ of $\mathbf{S}_{\mathfrak{C}}$, giving "projections" $p_i : \Lambda(\mathcal{A}) \longrightarrow \mathcal{M}(\mathbf{S}_i)$ induced by restricting representatives. Set $\mathcal{A}_i = p_i(\Lambda(\mathcal{A})) \subseteq \mathcal{M}(\mathbf{S}_i)$. Then $\Lambda(\mathcal{A}) \subseteq \bigoplus \mathcal{A}_i$, so $\mathrm{rank}(\Lambda(\mathcal{A})) \leq \sum \mathrm{rank}(\mathcal{A}_i)$. We make the following observations:

i) If $\mathbf{S}_i$ is a pair of pants, then $\mathcal{M}(\mathbf{S}_i)$ is finite, so $\mathrm{rank}(\mathcal{A}_i) = 0$.

ii) If $\mathbf{S}_i$ is a one-holed torus, then the homomorphism $H_1(\mathbf{S}_i) \to H_1(\mathbf{S})$ induced by inclusion is injective. Any homeomorphism f representing an element of $\mathcal{A}$ maps a circle $\mathfrak{c}$ in $\mathbf{S}_i$ to a circle $\mathfrak{c}'$ in $\mathbf{S}_i$, so $\mathcal{A}_i$ lies within the Torelli group of $\mathbf{S}_i$, which is trivial in this case.

iii) If $\mathbf{S}_i$ is neither a pair of pants nor a one-holed torus, then $\mathcal{A}_i$ is either trivial or is an adequately reduced torsion-free Abelian subgroup of $\mathcal{M}(\mathbf{S}_i)$. So again by McCarthy's theorem, $\mathrm{rank}(\mathcal{A}_i) \leq 1$.

These observations tell us that

$$\mathrm{rank}(\Lambda(\mathcal{A})) \leq \sum_{i=1}^{\nu} \mathrm{rank}(\mathcal{A}_i) \leq \Omega. \tag{2.2}$$

$\square$

CHAPTER 3

Automorphisms of the Torelli Group

Our reason for investigating Abelian subgroups of the Torelli group is to provide the necessary algebraic information about the Torelli group so that we may prove the following, the principal problem of this work.

Conjecture 1 *Let* $\mathbf{S}$ *be a closed, connected, oriented surface of genus* $g \geq 3$*, and let* $\Psi : \mathcal{T} \to \mathcal{T}$ *be an automorphism of the Torelli group* $\mathcal{T}$ *of* $\mathbf{S}$*. Then* Ψ *is induced by an homeomorphism* $h : \mathbf{S} \to \mathbf{S}$*. That is, for any* $f \in \mathcal{T}$*, we have* $\Psi(f) = hfh^{-1}$*.*

As stated in the introduction, our strategy for proving this conjecture is comprised of three basic steps. The first, in section 3.1, is to give algebraic characterizations of power of Dehn twists about separating curves and powers of bounding pair maps. From this it will follow that Ψ must permute the set of left and right Dehn twists about separating curves and that Ψ must permute the set of bounding pair maps. The second step, in section 3.2, is to show that Ψ therefore induces an automorphism of $C(\mathbf{S})$, the complex of curves of $\mathbf{S}$. At the time of this writing, this step is incomplete. Finally, a theorem of Ivanov states that any automorphism of $C(\mathbf{S})$ (in particular, Ψ) is induced by a homeomorphism h of $\mathbf{S}$. In section 3.3 we apply this theorem and show, assuming that it is possible to complete step two, that the two automorphisms of $\mathcal{T}$, Ψ and the one induced by the homeomorphism h, are the same. This method of proof was first employed by Ivanov in [9] to prove the analogous theorem about the mapping class group.

In several formal and informal announcements made between October 2001 and March 2002, Benson Farb stated that he is able to prove this conjecture in the case $g \geq 4$, [4], [5], [6]. His strategy is the same, but we note that his characterization of powers of Dehn twists about separating curves and powers of bounding pair maps is only valid for genus at least 4, whereas our characterization is valid for genus 3 and above. Also, his proof that an automorphism of the Torelli group induces an automorphism of the complex of curves appears to rely on the ability to produce three curves on the surface, each pair of which is a bounding pair. This is possible only for genus 4 and above.

3.1 The Algebraic Characterization of Elementary $\mathcal{T}$ Classes

In this section assume that the genus of $\mathbf{S}$ is at least 3.

Recall that if $f \in \mathcal{T}(\mathbf{S})$ is reducible along $\mathfrak{C}$, and $\Lambda : \mathcal{M}_{\mathfrak{C}}(\mathbf{S}) \to \mathcal{M}(\mathbf{S}_{\mathfrak{C}})$ is the reduction homomorphism, then, by Ivanov's Theorem given on page 4, $\Lambda(f)$ leaves each component of $\mathbf{S}_{\mathfrak{C}}$ invariant, and so $\Lambda(f)$ restricts to the mapping class group of each component. Also, $f(\mathfrak{a}) = \mathfrak{a}$ for each $\mathfrak{a} \in \mathfrak{C}$.

Lemma 3.1 *i) Let $f \in \mathcal{T}(\mathbf{S})$ be reducible along the separating simple* *closed curve $\mathfrak{a}$, and Λ the reduction along $\mathfrak{a}$. Let $\mathbf{S}'$ be a component of $\mathbf{S}_{\mathfrak{a}}$ and f' the restriction of $\Lambda(f)$ to $\mathcal{M}(\mathbf{S}')$. Then $f' \in \mathcal{T}(\mathbf{S}')$.*

ii) Let $f \in \mathcal{T}(\mathbf{S})$ be reducible along the bounding pair $\{\mathfrak{a}, \mathfrak{b}\}$, and Λ the reduction along $\{\mathfrak{a}, \mathfrak{b}\}$. Let $\mathbf{S}'$ be a component of $\mathbf{S}_{\{\mathfrak{a},\mathfrak{b}\}}$ and f' the restriction of $\Lambda(f)$ to $\mathcal{M}(\mathbf{S}')$. Then $f' \in \mathcal{T}(\mathbf{S}')$.

Proof:

i) Since $\mathfrak{a}$ is separating, we may consider $\mathbf{S}'$ to be a subsurface of $\mathbf{S}$. Then a symplectic basis for $H_1(\mathbf{S}')$ may be extended to a symplectic basis for $H_1(\mathbf{S})$ (see figure 3.1). Thus $H_1(\mathbf{S}')$ is a direct summand of $H_1(\mathbf{S})$, and since f acts trivially on $H_1(\mathbf{S})$, its restriction to $\mathbf{S}'$, which is f', acts trivially on $H_1(\mathbf{S}')$. That is, $f' \in \mathcal{T}(\mathbf{S}')$.

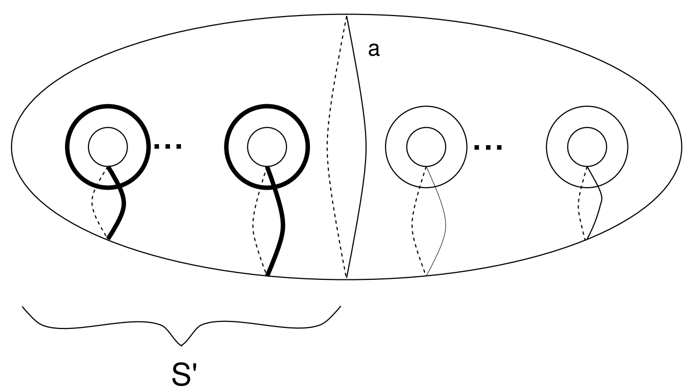

Figure 3.1: A symplectic basis for $H_1(\mathbf{S}')$ completed to a symplectic basis for $H_1(\mathbf{S})$.

ii) Since the two boundary components of $\mathbf{S}'$ correspond to the distinct curves $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathbf{S}$, we may again consider $\mathbf{S}'$ to be a subsurface of $\mathbf{S}$. Again, a basis for $H_1(\mathbf{S}')$ may be extended to a basis for $H_1(\mathbf{S})$ (see figure 3.2) so $H_1(\mathbf{S}')$ is a direct summand of $H_1(\mathbf{S})$. As in (i), $f' \in \mathcal{T}(\mathbf{S}')$.

$\square$

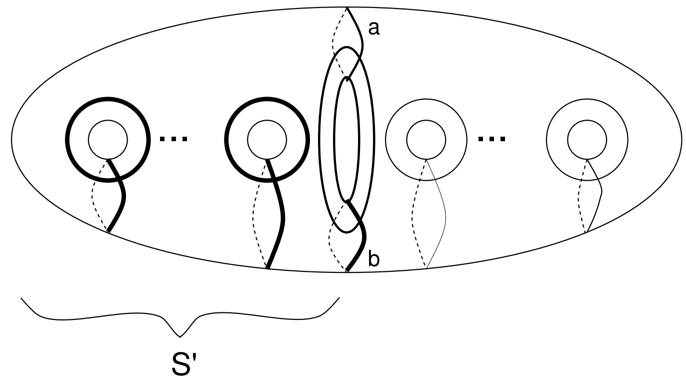

Figure 3.2: A basis for $H_1(\mathbf{S}')$ containing $\mathfrak{b}$ completed to a symplectic basis for $H_1(\mathbf{S})$.

Lemma 3.2 *Let $\mathfrak{C}$ consist of either a single separating curve or a single bounding pair, and let $\Lambda : \mathcal{M}_{\mathfrak{C}}(\mathbf{S}) \to \mathcal{M}(\mathbf{S}_{\mathfrak{C}})$ denote the reduction along $\mathfrak{C}$. Let $\mathbf{S}'$ and $\mathbf{S}''$ be the components of $\mathbf{S}_{\mathfrak{C}}$. If $f \in \mathcal{T}(\mathbf{S})$ is reducible along $\mathfrak{C}$, and if f' is the restriction of $\Lambda(f)$ to $\mathcal{M}(\mathbf{S}')$, then there exists $h \in \mathcal{T}(\mathbf{S})$, reducible along $\mathfrak{C}$, such that the restriction of $\Lambda(h)$ to $\mathcal{M}(\mathbf{S}')$ is equal to f' and the restriction of $\Lambda(h)$ to $\mathcal{M}(\mathbf{S}'')$ is trivial.*

Note: By Lemma 3.1, all mapping classes mentioned in this lemma actually lie in the Torelli group of their respective surfaces.

Proof:

As in the preceding lemma, we may consider $\mathbf{S}'$ and $\mathbf{S}''$ to be subsurfaces of $\mathbf{S}$. Now, take a homeomorphism $F' : \mathbf{S}' \to \mathbf{S}'$ representing f' and extend F' trivially to a map $F'' : \mathbf{S} \to \mathbf{S}$. In the case that $\mathfrak{C}$ is a separating curve, we may let h be the mapping class of F''. In the case that $\mathfrak{C}$ is a bounding pair, we may compose F'' with a multitwist about $\mathfrak{C}$ so that the resulting mapping class h lies in the Torelli group of $\mathbf{S}$.

In either case then, $h \in \mathcal{T}(\mathbf{S})$ is reducible along $\mathfrak{E}$, and the restriction of $\Lambda(h)$ to $\mathcal{T}(\mathbf{S}')$ is equal to f', while $\mathbf{S}''$ is a trivial component of h. $\Box$

We use the symbol $\mathcal{C}_{\mathcal{G}}(x)$ to denote the centralizer of x in the group $\mathcal{G}$, and $\mathcal{Z}(\mathcal{G})$ to denote the center of $\mathcal{G}$.

Lemma 3.3 *Let $\mathfrak{a}$ be a separating simple closed curve on $\mathbf{S}$ and $0 \neq n \in \mathbb{Z}$. Then $\mathcal{C}_{\mathcal{T}}(D_{\mathfrak{a}}^n)$ consists of all elements of $\mathcal{T}$ that are reducible along $\mathfrak{a}$, and $\mathcal{Z}(\mathcal{C}_{\mathcal{T}}(D_{\mathfrak{a}}^n)) = \mathcal{Z}(\mathcal{C}_{\mathcal{T}}(D_{\mathfrak{a}})) = \langle D_{\mathfrak{a}} \rangle$.*

Proof:

Let $f \in \mathcal{T}$ be reducible along $\mathfrak{a}$. Then $fD_{\mathfrak{a}}^n f^{-1} = D_{f(\mathfrak{a})}^n = D_{\mathfrak{a}}^n \Rightarrow fD_{\mathfrak{a}}^n = D_{\mathfrak{a}}^n f$, so $f \in \mathcal{C}_{\mathcal{T}}(D_{\mathfrak{a}}^n)$. Now let $f \in \mathcal{C}_{\mathcal{T}}(D_{\mathfrak{a}}^n)$. That is, $f \in \mathcal{T}$ and $fD_{\mathfrak{a}}^n = D_{\mathfrak{a}}^n f$. Then $f(\mathfrak{a}) = f(\mathfrak{E}_{D_{\mathfrak{a}}^n}) = \mathfrak{E}_{fD_{\mathfrak{a}}^n f^{-1}} = \mathfrak{E}_{D_{f(\mathfrak{a})}^n} = \mathfrak{E}_{D_{\mathfrak{a}}^n} = \mathfrak{a}$, so f is reducible along $\mathfrak{a}$. This proves the first assertion.

Now we show that $\mathcal{Z}(\mathcal{C}_{\mathcal{T}}(D_{\mathfrak{a}})) = \langle D_{\mathfrak{a}} \rangle$.

First, $D_{\mathfrak{a}} \in \mathcal{Z}(\mathcal{C}(D_{\mathfrak{a}}))$, so $\langle D_{\mathfrak{a}} \rangle \subseteq \mathcal{Z}(\mathcal{C}(D_{\mathfrak{a}}))$. Now, $\mathbf{S}_{\mathfrak{a}}$ consists of two components, $\mathbf{S}_1$ and $\mathbf{S}_2$, at most one of which is a one-holed torus. If $\mathbf{S}_i$, $i \in \{1, 2\}$, is not a one-holed torus, then it is a surface of genus at least 2 with one boundary component. In each such component of $\mathbf{S}_{\mathfrak{a}}$, perform the following construction: Choose a separating curve $\mathfrak{c}$, and a family $\{\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_m\}$ of curves which together with $\mathfrak{c}$ fill $\mathbf{S}_i$. Let $\mathfrak{d} = D_{\mathfrak{e}_m} D_{\mathfrak{e}_{m-1}} \cdots D_{\mathfrak{e}_1}(\mathfrak{c})$, where $\mathfrak{e}_j \cap D_{\mathfrak{e}_{j-1}} D_{\mathfrak{e}_{j-2}} \cdots D_{\mathfrak{e}_1}(\mathfrak{c}) \neq \emptyset$. Then $\mathfrak{d}$ is a separating curve in $\mathbf{S}_i$ and $\mathfrak{c}$ and $\mathfrak{d}$ fill $\mathbf{S}_i$, since the complementary components of $\mathfrak{c} \cup \mathfrak{d}$ are the same as the complementary components of $\mathfrak{c} \cup \mathfrak{e}_1 \cup \mathfrak{e}_2 \cup \cdots \cup \mathfrak{e}_m$. Let $f_i = D_{\mathfrak{c}} D_{\mathfrak{d}}^{-1}$ and $g_i = D_{\mathfrak{c}}^2 D_{\mathfrak{d}}^{-1}$. Then f_i and

g_i do not commute, and both f_i and g_i are pseudo-Anosov elements of the Torelli group of $\mathbf{S}_i$ (cf. [7]).

It follows from the two preceding lemmas that there exist elements f and g of $\mathcal{T}(\mathbf{S})$ that are reducible along $\mathfrak{a}$ such that the reduction of f to each component $\mathbf{S}_i$ which is not a one-holed torus is f_i and the reduction of g to $\mathbf{S}_i$ is g_i. (The reduction to any one-holed torus component of $\mathbf{S}_\mathfrak{a}$ must be trivial.)

Let h be a nontrivial element of $\mathcal{Z}(\mathcal{C}_\mathcal{T}(D_\mathfrak{a}))$. Then $h \in \mathcal{T}$, so h has infinite order, and $h \in \mathcal{C}_\mathcal{T}(D_\mathfrak{a})$, so h is reducible along $\mathfrak{a}$. As remarked in the introduction, then, $\mathfrak{E}_h \neq \emptyset$. We claim that $\mathfrak{E}_h = \{\mathfrak{a}\}$.

Suppose $\mathfrak{a} \neq \mathfrak{b} \in \mathfrak{E}_h$. Since $\mathfrak{b}$ is essential and $\mathfrak{a}$ is a reduction class for h, $\mathfrak{a}$ and $\mathfrak{b}$ are disjoint (Theorem 0.2 *(i)*), so $\mathfrak{b}$ lies in either $\mathbf{S}_1$ or $\mathbf{S}_2$. If $\mathbf{S}_i$ is a one-holed torus, then h is trivial on $\mathbf{S}_i$, and in this case, $\mathfrak{b} \subset \mathbf{S}_i$ implies that $\mathfrak{b}$ is not essential, since the removal of $\mathfrak{b}$ from any adequate reduction system for h leaves an adequate reduction system contradicting Theorem 0.2 (ii). So $\mathfrak{b}$ lies within $\mathbf{S}_i$, where $\mathbf{S}_i$ is not a one-holed torus component of $\mathbf{S}_\mathfrak{a}$. Since each of f and g constucted above are reducible along $\mathfrak{a}$, we know that f and g are elements of $\mathcal{C}_\mathcal{T}(D_\mathfrak{a})$. Thus, both f and g commute with h. In particular, the facts that f and h commute and $\mathfrak{b}$ is an essential reduction class for h imply that $\mathfrak{b}$ is a reduction class for f. But f has no reduction class within $\mathbf{S}_i$, since by construction the restriction of f to $\mathbf{S}_i$ is pseudo-Anosov. We conclude that $\mathfrak{E}_h = \{\mathfrak{a}\}$.

So $\mathcal{Z}(\mathcal{C}_\mathcal{T}(D_\mathfrak{a}))$ is a finitely generated, torsion free Abelian subgroup of $\mathcal{T}$, and $\displaystyle\bigcup_{h\in\mathcal{Z}(\mathcal{C}_{\mathcal{T}}(D_{\mathfrak{a}}))} \mathfrak{C}_h = \{\mathfrak{a}\}$.

Let $\Lambda : \mathcal{M}_{\mathfrak{a}}(\mathbf{S}) \to \mathcal{M}(\mathbf{S}_{\mathfrak{a}})$ be the reduction homomorphism, and denote $\mathcal{Z}(\mathcal{C}_{\mathcal{T}}(D_{\mathfrak{a}}))$ by simply $\mathcal{Z}$. Now, $\ker(\Lambda) = \langle D_{\mathfrak{a}}\rangle$, so $\ker(\Lambda|_{\mathcal{Z}}) = \langle D_{\mathfrak{a}}\rangle \cap \mathcal{Z} = \langle D_{\mathfrak{a}}\rangle$. We have the short exact sequence of Abelian groups:

$$1 \longrightarrow \langle D_{\mathfrak{a}}\rangle \longrightarrow \mathcal{Z} \xrightarrow{\Lambda|_{\mathcal{Z}}} \Lambda(\mathcal{Z}) \longrightarrow 1$$

Since h commutes with both f and g, $\Lambda(h)$ commutes with both $\Lambda(f)$ and $\Lambda(g)$. Let $\mathbf{S}_i$ be a component of $\mathbf{S}_{\mathfrak{a}}$ that is not a one-holed torus, and let h_i be the restriction of $\Lambda(h)$ to $\mathbf{S}_i$. Then h_i commutes with both f_i and g_i. By McCarthy's theorem given in the introduction $\mathcal{C}_{\mathcal{T}(\mathbf{S}_i)}(f_i)$, the centralizer of f_i in $\mathcal{T}(\mathbf{S}_i)$, is infinite cyclic since $\mathcal{T}(\mathbf{S}_i)$, and thus also $\mathcal{C}_{\mathcal{T}(\mathbf{S}_i)}(f_i)$, is torsion-free. Similarly, $\mathcal{C}_{\mathcal{T}(\mathbf{S}_i)}(g_i)$ is infinite cyclic. Since f_i and g_i do not commute, the cyclic subgroups $\mathcal{C}_{\mathcal{T}(\mathbf{S}_i)}(f_i)$ and $\mathcal{C}_{\mathcal{T}(\mathbf{S}_i)}(g_i)$ intersect trivially. But h_i is in this intersection, and hence we see that each component of h is trivial. Therefore $\Lambda(\mathcal{Z}) = \{1\}$, so that $\langle D_{\mathfrak{a}}\rangle \longrightarrow \mathcal{Z}(\mathcal{C}_{\mathcal{T}}(D_{\mathfrak{a}}))$ is an isomorphism.

Finally, it is clear that $\mathcal{Z}(\mathcal{C}_{\mathcal{T}}(D_{\mathfrak{a}}^n)) = \mathcal{Z}(\mathcal{C}_{\mathcal{T}}(D_{\mathfrak{a}}))$. $\square$

Lemma 3.4 *Let $\{\mathfrak{a}, \mathfrak{b}\}$ be a bounding pair on $\mathbf{S}$, and let $0 \neq n \in \mathbb{Z}$. Then $\mathcal{C}_{\mathcal{T}}((D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1})^n)$ consists of all elements of $\mathcal{T}$ that are reducible along both $\mathfrak{a}$ and $\mathfrak{b}$, and*

$$\mathcal{Z}(\mathcal{C}_{\mathcal{T}}((D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1})^n)) = \mathcal{Z}(\mathcal{C}_{\mathcal{T}}(D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1})) = \langle D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1}\rangle.$$

Proof:

Let $f \in \mathcal{T}$ be reducible along both $\mathfrak{a}$ and $\mathfrak{b}$. Then $fD_{\mathfrak{a}}D_{\mathfrak{b}}^{-1}f^{-1} =$

$fD_{\mathfrak{a}}f^{-1}fD_{\mathfrak{b}}^{-1}f^{-1} = D_{f(\mathfrak{a})}D_{f(\mathfrak{b})}^{-1} = D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1}$, so $f \in \mathcal{C}_{\mathcal{T}}(D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1})$.

Now let $f \in \mathcal{C}_{\mathcal{T}}(D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1})$. That is, $f \in \mathcal{T}$, and $fD_{\mathfrak{a}}D_{\mathfrak{b}}^{-1} = D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1}f$. Then

$$f(\{\mathfrak{a},\mathfrak{b}\}) = f(\mathfrak{E}_{D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1}}) = \mathfrak{E}_{fD_{\mathfrak{a}}D_{\mathfrak{b}}^{-1}f^{-1}} = \mathfrak{E}_{D_{f(\mathfrak{a})}D_{f(\mathfrak{b})}^{-1}} = \mathfrak{E}_{D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1}} = \{\mathfrak{a},\mathfrak{b}\}.$$

Therefore f is reducible along both $\mathfrak{a}$ and $\mathfrak{b}$. This proves the first assertion.

The proof of the second assertion follows the proof of Lemma 3.3. $\square$

Definition Let $1 \neq f \in \mathcal{T}$. We say that f is an **elementary $\mathcal{T}$ class** if and only if f is either a power of a Dehn twist about a separating curve or a power of a bounding pair map.

Theorem 3.5 *Let $f \in \mathcal{T}$. Then f is an elementary $\mathcal{T}$ class if and only if each of the following are true:*

i) $\mathcal{C}_{\mathcal{T}}(f)$ is not infinite cyclic,

ii) $\mathcal{Z}(\mathcal{C}_{\mathcal{T}}(f))$ is infinite cyclic, and

iii) f is contained in an Abelian subgroup of $\mathcal{T}$ having rank $2g-3$.

Proof:

$\Rightarrow$) Assume f is an elementary $\mathcal{T}$ class.

Case 1: f is a power of a Dehn twist about a separating curve $\mathfrak{a}$, say, $f = D_{\mathfrak{a}}^n$, where $n \neq 0$. In this case, $\mathfrak{a}$ can be completed to a pants decomposition $\mathfrak{E}$ on $\mathbf{S}$ whose reduction system graph is shown in Figure 3.3.

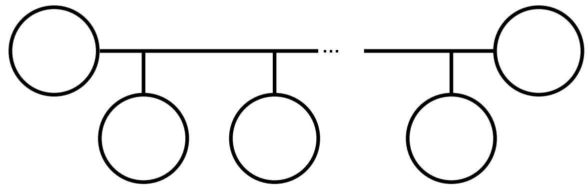

Figure 3.3: The Torelli multitwist group on a reduction system with this graph has rank $2g-3$.

The Torelli multitwist group on $\mathfrak{C}$, $\mathcal{T}_{\mathfrak{C}}$, is an Abelian subgroup of $\mathcal{T}$ with rank $2g-3$, and $f \in \mathcal{T}_{\mathfrak{C}}$, so $f \in \mathcal{T}_{\mathfrak{C}} \subset \mathcal{C}_{\mathcal{T}}(f)$. Since $g \geq 3$, it follows that $2g-3 \geq 3$, so $\mathcal{C}_{\mathcal{T}}(f)$ is not infinite cyclic. So both *(i)* and *(iii)* hold. Finally, by Lemma 3.3, $\mathcal{Z}(\mathcal{C}_{\mathcal{T}}(f))$ is infinite cyclic.

Case 2: f is a power of a bounding pair map, say, $f = (D_{\mathfrak{a}} D_{\mathfrak{b}}^{-1})^n$, where $n \neq 0$. In this case, $\{\mathfrak{a}, \mathfrak{b}\}$ can be completed to a pants decomposition $\mathfrak{C}$ whose reduction system graph is shown in Figure 3.4.

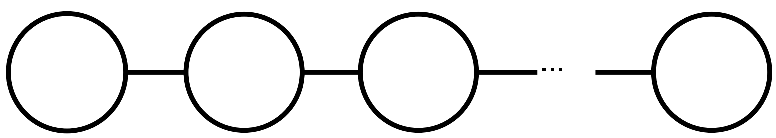

Figure 3.4: The Torelli multitwist group on a reduction system with this graph has rank $2g-3$.

The Torelli multitwist group on $\mathfrak{C}$, $\mathcal{T}_{\mathfrak{C}}$, is an Abelian subgroup of $\mathcal{T}$ with rank $2g-3$, and $f \in \mathcal{T}_{\mathfrak{C}}$, so $f \in \mathcal{T}_{\mathfrak{C}} \subset \mathcal{C}_{\mathcal{T}}(f)$. Since $g \geq 3$, it follows that $2g-3 \geq 3$, so that $\mathcal{C}_{\mathcal{T}}(f)$ is not infinite cyclic. So both *(i)* and *(iii)* hold. Finally, by lemma 3.3, $\mathcal{Z}(\mathcal{C}_{\mathcal{T}}(f))$ is infinite cyclic.

$\Leftarrow$) Assume *(i)*, *(ii)*, and *(iii)* hold.

Since $\mathcal{C}_{\mathcal{T}}(1) = \mathcal{T}$ and $\mathcal{Z}(\mathcal{T}) = \{1\}$, we have by *(ii)* that $f \neq 1$. Then $f \in \mathcal{T}$ implies that f is either pseudo-Anosov or infinite-order reducible. But $\mathcal{C}_{\mathcal{T}}(f)$ is not infinte cyclic by *(i)*, so f is not pseudo-Anosov. Hence f is infinite-order reducible, and therefore $\mathfrak{C}_f$, the essential reduction system for f, is nonempty.

Assume that f is not an elementary $\mathcal{T}$ class. We obtain a contradiction by considering $\mathfrak{E}_f$.

First, suppose $\mathfrak{E}_f$ contains a separating curve $\mathfrak{a}$.

Let $g \in \mathcal{C}_{\mathcal{T}}(f)$. Then $g(\mathfrak{E}_f) = \mathfrak{E}_{gfg^{-1}} = \mathfrak{E}_f$, so $\mathfrak{a}$ is a reduction class for g. Ivanov's theorem implies that $g(\mathfrak{a}) = \mathfrak{a}$, and so

$$gD_{\mathfrak{a}}g^{-1} = D_{g(\mathfrak{a})} = D_{\mathfrak{a}} \Rightarrow gD_{\mathfrak{a}} = D_{\mathfrak{a}}g.$$

Since $f \in \mathcal{C}_{\mathcal{T}}(f)$, the paragraph above shows that $D_{\mathfrak{a}} \in \mathcal{C}_{\mathcal{T}}(f)$, as well as that $D_{\mathfrak{a}} \in \mathcal{Z}(\mathcal{C}_{\mathcal{T}}(f))$. Consider the short exact sequence of Abelian groups

$$0 \longrightarrow \langle D_{\mathfrak{a}} \rangle \longrightarrow \mathcal{Z}(\mathcal{C}_{\mathcal{T}}(f)) \longrightarrow \mathcal{Z}(\mathcal{C}_{\mathcal{T}}(f))/\langle D_{\mathfrak{a}} \rangle \longrightarrow 0$$

Claim 1: $\mathcal{Z}(\mathcal{C}_{\mathcal{T}}(f))/\langle D_{\mathfrak{a}} \rangle$ is not a torsion group.

Proof: Now, $f \in \mathcal{Z}(\mathcal{C}_{\mathcal{T}}(f))$ and $f \notin \langle D_{\mathfrak{a}} \rangle$ since f is not an elementary $\mathcal{T}$ class, but suppose that $f^m = D_{\mathfrak{a}}^n$ for some integers m and n. Then $\{\mathfrak{a}\} = \mathfrak{E}_{D_{\mathfrak{a}}} = \mathfrak{E}_{D_{\mathfrak{a}}^n} = \mathfrak{E}_{f^m} = \mathfrak{E}_f$, so the reduction of f along $\mathfrak{a}$ is adequately reduced. But $f \notin \langle D_{\mathfrak{a}} \rangle$ implies that the reduction of f along $\mathfrak{a}$ is not trivial, so that f has a pseudo-Anosov component. But then $f^m = D_{\mathfrak{a}}^n$ is impossible unless $m = n = 0$. Hence f generates an infinite cyclic subgroup in $\mathcal{Z}(\mathcal{C}_{\mathcal{T}}(f))/\langle D_{\mathfrak{a}} \rangle$, proving Claim 1.

Therefore, the rank of $\mathcal{Z}(\mathcal{C}_{\mathcal{T}}(f))/\langle D_{\mathfrak{a}} \rangle$ is at least one, so the rank of $\mathcal{Z}(\mathcal{C}_{\mathcal{T}}(f))$ is at least 2, contradicting condition *(ii)*. We conclude that $\mathfrak{E}_f$ can contain no separating curves.

Second, suppose that $\mathfrak{E}_f$ contains a bounding pair $\{\mathfrak{a}, \mathfrak{b}\}$.

Let $g \in \mathcal{Z}(\mathcal{C}_{\mathcal{T}}(f))$. Then $g(\mathfrak{E}_f) = \mathfrak{E}_{gfg^{-1}} = \mathfrak{E}_f$, so both $\mathfrak{a}$ and $\mathfrak{b}$ are reduction classes for g. Ivanov's theorem implies that $g(\mathfrak{a}) = \mathfrak{a}$ and $g(\mathfrak{b}) = \mathfrak{b}$, so

$$gD_{\mathfrak{a}}D_{\mathfrak{b}}^{-1}g^{-1} = D_{g(\mathfrak{a})}D_{g(\mathfrak{b})}^{-1} = D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1} \Rightarrow gD_{\mathfrak{a}}D_{\mathfrak{b}}^{-1} = D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1}g.$$

Since $f \in \mathcal{C}_{\mathcal{T}}(f)$, the paragraph above shows that $D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1} \in \mathcal{C}_{\mathcal{T}}(f)$, as well as that $D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1} \in \mathcal{Z}(\mathcal{C}_{\mathcal{T}}(f))$. Consider the short exact sequence of Abelian groups

$$0 \longrightarrow \langle D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1}\rangle \longrightarrow \mathcal{Z}(\mathcal{C}_{\mathcal{T}}(f)) \longrightarrow \mathcal{Z}(\mathcal{C}_{\mathcal{T}}(f))/\langle D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1}\rangle \longrightarrow 0$$

Claim 2: $\mathcal{Z}(\mathcal{C}_{\mathcal{T}}(f))/\langle D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1}\rangle$ is not a torsion group.

Proof: Now, $f \in \mathcal{Z}(\mathcal{C}_{\mathcal{T}}(f))$ and $f \notin \langle D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1}\rangle$, since f is not an elementary $\mathcal{T}$ class, but suppose that $f^m = (D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1})^n$ for some integers m and n. Then $\{\mathfrak{a}, \mathfrak{b}\} = \mathfrak{E}_{D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1}} = \mathfrak{E}_{(D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1})^n} = \mathfrak{E}_{f^m} = \mathfrak{E}_f$, so the reduction of f along $\{\mathfrak{a}, \mathfrak{b}\}$ is adequately reduced. If the reduction of f along $\{\mathfrak{a}, \mathfrak{b}\}$ were trivial, it would follow that

$$f \in \langle D_{\mathfrak{a}}, D_{\mathfrak{b}}\rangle \cap \mathcal{T} = \mathcal{T}_{\{\mathfrak{a},\mathfrak{b}\}} = \langle D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1}\rangle$$

which is a contradiction. So we conclude that f has a pseudo-Anosov component. But then $f^m = (D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1})^n$ is impossible unless $m = n = 0$. Hence f generates an infinite cyclic supgroup in $\mathcal{Z}(\mathcal{C}_{\mathcal{T}}(f))/\langle D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1}\rangle$, proving Claim 2.

Therefore, the rank of $\mathcal{Z}(\mathcal{C}_{\mathcal{T}}(f))/\langle D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1}\rangle$ is at least one, so the rank of $\mathcal{Z}(\mathcal{C}_{\mathcal{T}}(f))$ is at least two, contradicting condition *(ii)*. We conclude that $\mathfrak{E}_f$ contains no bounding pairs.

Therefore, $\mathfrak{E}_f$ consists of a collection of c-type curves.

Let $\mathcal{A}$ be an Abelian subgroup of $\mathcal{T}$ containing f, and let $\mathfrak{C} = \bigcup_{g \in \mathcal{A}} \mathfrak{C}_g$. Then $\mathfrak{C}_f \subset \mathfrak{C}$.

By equations 2.1 and 2.2, we know that

$$\operatorname{rank}(\mathcal{A}) \leq \operatorname{rank}(\mathcal{T}_{\mathfrak{C}}) + \Omega \leq \nu + \Omega - 1 \leq 2g - 3.$$

By considering equation 2.1, we see that the last inequality above will be strict unless each of the following are true:

- Each component of $\mathbf{S}_{\mathfrak{C}}$ having one or two boundary components has genus 1.
- Each component of $\mathbf{S}_{\mathfrak{C}}$ having three or four boundary components has genus 0.
- No component of $\mathbf{S}_{\mathfrak{C}}$ has more than four boundary components.

Suppose each of these conditions hold. Consider the pseudo-Anosov components of f; each of them must be a pseudo-Anosov component of each $1 \neq h \in \mathcal{A}$. None can have exactly one or two boundary curves, since such boundary curves are elements of $\mathfrak{C}$, and we have already seen that $\mathfrak{C}$ contains no separating curves or bounding pairs. None can be a pair of pants, since pants do not support pseudo-Anosov mapping classes. So every pseudo-Anosov component of f must be a four-holed sphere, and we note that in the reduction system graph of $\mathfrak{C}$, such a component would correspond to a non-cut vertex of degree 4. But by the following lemma, then, $\operatorname{rank}(\mathcal{A}) < 2g - 3$. That is, f cannot lie in an Abelian subgroup of $\mathcal{T}$ having rank $2g - 3$, contradicting condition *(iii)*.

We conclude that f must be an elementary $\mathcal{T}$ class. $\square$

Lemma 3.6 *Let $\mathcal{A}$ be a reducible Abelian subgroup of $\mathcal{T}$, and let $\mathfrak{E} = \bigcup_{g \in \mathcal{A}} \mathfrak{E}_g$ be the essential reduction system for $\mathcal{A}$. If the reduction system graph for $\mathfrak{E}$ has a non-cut vertex of degree ≥ 3, then* $\operatorname{rank}(\mathcal{A}) < 2g - 3$.

Proof:

In this proof, we use the notation and results found in Chapters 1 and 2.

Let v be a non-cut vertex of degree ≥ 3. Then no two of the at-least three edges incident with v are $\sim$-equivalent b-type edges and none are cut edges (a-type edges). So there exists a spanning tree T in the graph containing none of the b-type edges incident with v. Hence T contains a c-type edge, incident with v, so

$$\nu - 1 = \operatorname{card}(E(T)) > p + (q_1 - 1) + (q_2 - 1) + \cdots + (q_r - 1) = \operatorname{rank}(\mathcal{T}_{\mathfrak{E}})$$

But then $\operatorname{rank}(\mathcal{A}) \leq \operatorname{rank}(\mathcal{T}_{\mathfrak{E}}) + \Omega < \nu - 1 + \Omega \leq 2g - 3$. $\square$

Theorem 3.7 *Let f be an elementary $\mathcal{T}$ class. Then f is a power of a Dehn twist about a separating curve of genus 1 if and only if f is contained in an Abelian subgroup of $\mathcal{T}$ of rank 2 which is not contained within any Abelian subgroup of $\mathcal{T}$ having higher rank.*

Proof:

$\Rightarrow$) Assume the f is a power of a Dehn twist about a separating curve of genus 1, say $f = D_{\mathfrak{a}}^p$, where $0 \neq p \in \mathbb{Z}$.

Let $\mathbf{S}'$ be the component of $\mathbf{S}_{\mathfrak{a}}$ having genus > 1, and let $\mathbf{S}''$ be the genus 1 component of $\mathbf{S}_{\mathfrak{a}}$. Let $h \in \mathcal{T}$ be reducible along $\mathfrak{a}$ such that $\mathbf{S}'$ is a pseudo-Anosov component of h. Since h reduces along $\mathfrak{a}$, f and h commute, so $\langle f, h\rangle$ is Abelian. We have a short exact sequence of Abelian groups

$$0 \longrightarrow \langle f\rangle \longrightarrow \langle f, h\rangle \longrightarrow \langle f, h\rangle/\langle f\rangle \longrightarrow 0$$

Suppose that $h^m = f^n$ for some integers m and n. Since h^m has a pseudo-Anosov component for all nonzero integers m, whereas f^n has no pseudo-Anosov component for any integer n, we must have $n = m = 0$, so that the coset of h has infinite order in $\langle f, h\rangle/\langle f\rangle$. Thus $\langle f, h\rangle/\langle f\rangle$ is not a torsion group, and hence has rank 1. By our sequence, then, we see that that $\langle f, h\rangle$ has rank 2.

We must show now that if $\mathcal{A}$ is an Abelian group and $\langle f, h\rangle \in \mathcal{A} \subset \mathcal{T}$, then the rank of $\mathcal{A}$ is at most 2. We have a short exact sequence of Abelian groups

$$0 \longrightarrow \langle f, h\rangle \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}/\langle f, h\rangle \longrightarrow 0$$

so $\operatorname{rank}(\mathcal{A}) = 2 + \operatorname{rank}(\mathcal{A}/\langle f, h\rangle)$. We show that $\mathcal{A}/\langle f, h\rangle$ is a torsion group, and hence has rank 0.

Let $1 \neq a \in \mathcal{A}$. We prove that $a^m \in \langle f, h\rangle$ for some nonzero integer m, which will establish that $\mathcal{A}/\langle f, h\rangle$ is a torsion group. Now, $f, a \in \mathcal{A}$ imply that a and f commute, so a is reducible along $\mathfrak{a}$, and $a \in \mathcal{T}$ implies that a has infinite order. Therefore, the essential reduction system for a, $\mathfrak{E}_a$, is nonempty. Suppose there exists $\mathfrak{a} \neq \mathfrak{b} \in \mathfrak{E}_a$. Then

$\mathfrak{a}$ and $\mathfrak{b}$ must be disjoint, so either $\mathfrak{b}$ lies within $\mathbf{S}'$, or $\mathfrak{b}$ lies within $\mathbf{S}''$. But in the second case, since a is trivial on this component, $\mathfrak{b}$ would not be essential. So $\mathfrak{b} \subset \mathbf{S}'$. But the fact that a commutes with h implies that h is reducible along $\mathfrak{b}$. This is a contradiction, since $\mathbf{S}'$ is a pseudo-Anosov component of h. Hence $\mathfrak{C}_a = \{\mathfrak{a}\}$.

Let $\Lambda : \mathcal{M}_{\mathfrak{a}}(\mathbf{S}) \to \mathcal{M}(\mathbf{S}_{\mathfrak{a}})$ be the reduction homomorphism. So $f \in \ker(\Lambda) = \langle D_{\mathfrak{a}} \rangle$. Now, for any $d \in \mathcal{A}$, the restriction of $\Lambda(d)$ to $\mathbf{S}''$ is trivial. Thus $\Lambda(\mathcal{A})$ is isomorphic to a torsion-free Abelian subgroup of $\mathcal{T}(\mathbf{S}')$ (Lemma 3.1) containing the pseudo-Anosov class $\Lambda(h)$. By the corollary to McCarthy's Theorem, $\Lambda(\mathcal{A})$ is cyclic. Hence if $a \notin \ker(\Lambda)$, then there are nonzero integers m and n such that $\Lambda(a)^m = \Lambda(h)^n$. But then $a^m h^{-n} \in \ker(\Lambda)$, so have $a^m = D_{\mathfrak{a}}^k h^n$ for some integer k, and therefore $a^{pm} = D_{\mathfrak{a}}^{pk} h^{pn} = f^k h^{np}$. That is, $a^{pm} \in \langle f, h \rangle$. In the case that $a \in ker(\Lambda)$, so that $a = D_{\mathfrak{a}}^k$, it follows that $a^p = f^k \in \langle f, h \rangle$.

$\Leftarrow$) Assume that f is not a power of a Dehn twist about a separating curve of genus 1.

Then f is a power of a Dehn twist about a separating curve of genus at least 2, say $f = k^n$ where $k = D_{\mathfrak{a}}$, or else f is a power of a bounding pair map, say $f = k^n$ where $k = D_{\mathfrak{a}} D_{\mathfrak{b}}^{-1}$. In either case, we want to show that it is not true that f is contained in Abelian subgroup of $\mathcal{T}$ of rank 2 which is not contained in an Abelian subgroup of $\mathcal{T}$ with higher rank. In other words, if $f \in \mathcal{A} \subset \mathcal{T}$, where $\mathcal{A}$ is Abelian and of rank 2, then there exists an Abelian subgroup $\mathcal{B} \subset \mathcal{T}$ such that $\mathcal{A} \subset \mathcal{B}$ and $\text{rank}(\mathcal{B}) > 2$. We consider the two possibilites for k simultaneously.

Let p be the smallest positive integer such that $k^p \in \mathcal{A}$.

Claim: $\mathcal{A}/\langle k^p \rangle$ is torsion-free.

Proof: Suppose that $h \in \mathcal{A} \setminus \langle k^p \rangle$, but $h^q \in \langle k^p \rangle$ for some $0 \neq q \in \mathbb{Z}$. That is, $h^q = k^{pr}$ for some $0 \neq r \in \mathbb{Z}$. Since $\mathfrak{C}_h = \mathfrak{C}_{h^q} = \mathfrak{C}_{k^{pr}} = \mathfrak{C}_k$ by Theorem 0.2 *(iv)*, we see that h reduces along $\mathfrak{C}_k$, which is either the separating curve $\mathfrak{a}$ or the bounding pair $\{\mathfrak{a}, \mathfrak{b}\}$, whichever the case may be. Let Λ denote the reduction along $\mathfrak{C}_k$. So $\Lambda(h)^q = \Lambda(h^q) = \Lambda(k^{pr}) = 1$. By Ivanov's Theorem, $\Lambda(h)$ leaves both of the two components of $\mathbf{S}_{\mathfrak{C}_k}$ invariant, and since h is in the Torelli group of $\mathbf{S}$ and we are reducing along either a separating curve of a bounding pair, by Lemma 3.1 each restriction of $\Lambda(h)$ lies in the Torelli group of its component, which is torsion-free. These facts imply that $\Lambda(h) = 1$, so that h is a Torelli multitwist about $\mathfrak{C}_k$, which then implies that h is a power of k. But then it is easy to see that this will contradict either the minimality of p or the assumption that $h \notin \langle k^p \rangle$, proving the claim.

Since $\mathcal{A}$ has rank two, it follows that there exists $t \in \mathcal{A}$ such that $\mathcal{A} = \langle k^p \rangle \oplus \langle t \rangle$, and since t and k commute, t reduces along $\mathfrak{C}_k$. Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be the two components of $\mathbf{S}_{\mathfrak{C}_k}$ (neither of which is a one-holed torus), and Λ the reduction along $\mathfrak{C}_k$. Note that $\Lambda(t)$ is not trivial; we may assume without loss of generality that $\mathbf{S}_1$ is a nontrivial component of t. By Lemma 3.2, there exists $t_1 \in \mathcal{T}(\mathbf{S})$ reducible along $\mathfrak{C}_k$ such that the restrictions of $\Lambda(t)$ and $\Lambda(t_1)$ to $\mathcal{T}(\mathbf{S}_1)$ are the same, and that $\mathbf{S}_2$ is a trivial component of t_1. If $\mathbf{S}_2$ is also a nontrivial component of t, then we have $t_2 \in \mathcal{T}(\mathbf{S})$ defined analogously to t_1. For the case that

$\mathbf{S}_2$ is a trivial component of t, note that $\mathbf{S}_2$ contains a separating curve $\mathfrak{c}$, so let $t_2 = D_{\mathfrak{c}}$. In either case, methods as employed in the preceding lemmas show that $\mathcal{B} = \langle k^p, t_1, t_2 \rangle$ is an Abelian subgroup of the Torelli group with rank 3. Then $t = t_1 t_2 k^r$ or $t = t_1 k^r$ for some integer r, so $t \in \mathcal{B}$. Thus $f \in \mathcal{A} \subset \mathcal{B} \subset \mathcal{T}$, and we are done. $\square$

Theorem 3.8 *Let f be an elementary $\mathcal{T}$ class. Then f is a power of a Dehn twist about a separating curve of genus ≥ 2 if and only if there exist g Dehn twists $D_{\mathfrak{a}_1}, D_{\mathfrak{a}_2}, \ldots D_{\mathfrak{a}_g}$ about separating curves of genus 1 such that $\langle f, D_{\mathfrak{a}_1}, D_{\mathfrak{a}_2}, \ldots D_{\mathfrak{a}_g} \rangle$ is an Abelian subgoup of $\mathcal{T}$ with rank $g+1$.*

Proof:

$\Rightarrow$) Assume f is a power about a separating curve of genus ≥ 2, say $f = D_{\mathfrak{a}}^p$.

Then there are g separating curves $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_g$ of genus 1 that are pairwise disjoint and each disjoint from $\mathfrak{a}$. See figure 3.5. It follows that $\langle f, D_{\mathfrak{a}_1}, D_{\mathfrak{a}_2}, \ldots D_{\mathfrak{a}_g} \rangle$ is an Abelian subgroup of $\mathcal{T}$ with rank $g+1$.

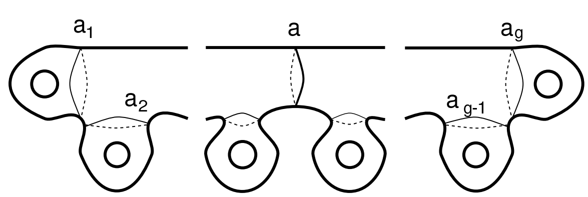

Figure 3.5: g pairwise disjoint separating curves of genus 1, each disjoint from $\mathfrak{a}$.

$\Leftarrow$) Assume that f is not a power of a Dehn twist about a separating curve of genus ≥ 2.

First, suppose that f is a power of a Dehn twist about a separating curve $\mathfrak{a}$ of genus 1. In this case, there do not exist g separating curves that are pairwise disjoint and disjoint from $\mathfrak{a}$.

Second, if f is a power of a bounding pair map, say $f = (D_{\mathfrak{a}} D_{\mathfrak{b}}^{-1})^p$, then there do not exist g separating curves of genus 1 that are pairwise disjoint and each disjoint from both $\mathfrak{a}$ and $\mathfrak{b}$.

So in either case, no such group $\langle f, D_{\mathfrak{a}_1}, D_{\mathfrak{a}_2}, \ldots D_{\mathfrak{a}_g} \rangle$ can exist. $\square$

Theorem 3.9 *Let $\Psi : \mathcal{T} \to \mathcal{T}$ be an automorphism of the Torelli group of* $\mathbf{S}$*. Then Ψ permutes that set of left and right Dehn twists about separating curves, and Ψ permutes the set of bounding pair maps.*

Proof:

Let $f \in \mathcal{T}$.

If $\mathcal{C}_{\mathcal{T}}(f)$ is not infinite cyclic, then $\mathcal{C}_{\mathcal{T}}(\Psi(f))$ is not infinite cyclic. If $\mathcal{Z}(\mathcal{C}_{\mathcal{T}}(f))$ is infinite cyclic, then $\mathcal{Z}(\mathcal{C}_{\mathcal{T}}(\Psi(f)))$ is infinite cyclic, and if f is contained within an Abelian subgroup $\mathcal{A} \subset \mathcal{T}$ with rank $2g-3$, then $\Psi(f)$ is contained within the Abelian subgroup $\Psi(\mathcal{A})$, which has rank $2g-3$. Thus, Ψ sends elementary $\mathcal{T}$ classes to elementary $\mathcal{T}$ classes. Now suppose that $\Psi(f)$ is an elementary $\mathcal{T}$ class. Since $\Psi^{-1} : \mathcal{T} \to \mathcal{T}$ is also an automorphism, and thus also sends elementary $\mathcal{T}$ classes to elementary $\mathcal{T}$ classes, we know that $\Psi^{-1}(\Psi(f)) = f$ is an elementary $\mathcal{T}$ class. Since Ψ is a bijection of $\mathcal{T}$, it follows that Ψ permutes the set of elementary $\mathcal{T}$ classes.

Similarly, we see that Ψ permutes the set of powers of Dehn twists about separating curves of genus 1, permutes the set of powers of Dehn twists about separating curves of genus ≥ 2, and permutes the set of powers of bounding pair maps.

If $f = D_{\mathfrak{a}}$ is a right Dehn twist about a separating curve, we know that $\Psi(f)$ is a power of a Dehn twist about a separating curve, say $\Psi(f) = D_{\mathfrak{b}}^p$, $0 \neq p \in \mathbb{Z}$. Then $\Psi^{-1}(D_{\mathfrak{b}})$ is also a power of a Dehn twist about a separating curve, say $\Psi^{-1}(D_{\mathfrak{b}}) = D_{\mathfrak{c}}^q$, $0 \neq q \in \mathbb{Z}$. Hence

$$D_{\mathfrak{c}}^{qp} = \Psi^{-1}(D_{\mathfrak{b}})^p = \Psi^{-1}\Psi(f) = f = D_{\mathfrak{a}}.$$

Since $\mathfrak{C}_{D_{\mathfrak{a}}} = \{\mathfrak{a}\}$ and $\mathfrak{C}_{D_{\mathfrak{c}}^{qp}} = \{\mathfrak{c}\}$, we see that $\mathfrak{a} = \mathfrak{c}$. Hence $D_{\mathfrak{a}} = D_{\mathfrak{a}}^{qp}$. But since $D_{\mathfrak{a}}$ has infinite order, we must have $qp = 1$, so that $p = \pm 1$. So Ψ permutes the set of left and right Dehn twists about separating curves.

If $f = D_{\mathfrak{a}} D_{\mathfrak{b}}^{-1}$ is a bounding pair map, we know that $\Psi(f)$ is a power of a bounding pair map, say $\Psi(f) = (D_{\mathfrak{c}} D_{\mathfrak{d}}^{-1})^p$, $0 \neq p \in \mathbb{Z}$. Then $\Psi^{-1}(D_{\mathfrak{c}} D_{\mathfrak{d}}^{-1})$ is also a power of a bounding pair map, say $\Psi^{-1}(D_{\mathfrak{c}} D_{\mathfrak{d}}^{-1}) = (D_{\mathfrak{e}} D_{\mathfrak{f}}^{-1})^q$, $0 \neq q \in \mathbb{Z}$. Hence

$$(D_{\mathfrak{e}} D_{\mathfrak{f}}^{-1})^{qp} = \Psi^{-1}(D_{\mathfrak{c}} D_{\mathfrak{d}}^{-1})^p = \Psi^{-1}\Psi(f) = f = D_{\mathfrak{a}} D_{\mathfrak{b}}^{-1}.$$

Since $\mathfrak{C}_{D_{\mathfrak{a}} D_{\mathfrak{b}}^{-1}} = \{\mathfrak{a}, \mathfrak{b}\}$ and $\mathfrak{C}_{(D_{\mathfrak{e}} D_{\mathfrak{f}}^{-1})^{qp}} = \{\mathfrak{e}, \mathfrak{f}\}$, we see that $\{\mathfrak{a}, \mathfrak{b}\} = \{\mathfrak{e}, \mathfrak{f}\}$. Hence $D_{\mathfrak{a}} D_{\mathfrak{b}}^{-1} = (D_{\mathfrak{a}} D_{\mathfrak{b}}^{-1})^{\pm qp}$. But since $D_{\mathfrak{a}} D_{\mathfrak{b}}^{-1}$ has infinite order, we must have $qp = \pm 1$, so that $p = \pm 1$. So Ψ also permutes the set of bounding pair maps. $\square$

3.2 The Complex of Curves

First introduced by William Harvey [8], the *complex of curves*, $C(\mathbf{S})$, of $\mathbf{S}$ is an abstract simplicial complex whose vertex set is $\mathfrak{S}$, the set of unoriented isotopy classes of homotopically nontrivial simple closed curves on $\mathbf{S}$, and a collection of vertices forms a simplex if and only if the vertices have a set of mutually disjoint representative curves. In other words, a simplex is a reduction system as defined in the introduction. By an *automorphism* $C(\mathbf{S}) \to C(\mathbf{S})$ we mean a bijective map $\mathfrak{S} \to \mathfrak{S}$ of the vertex set of $C(\mathbf{S})$ such that the image of any simplex is a simplex (i.e., the map is *simplicial.*)

Notation: Let $\mathfrak{a}$ be a nonseparating simple closed curve on $\mathbf{S}$. We write $\mathfrak{S}_{\mathfrak{a}} = \{\mathfrak{b} \in \mathfrak{S} : \{\mathfrak{a}, \mathfrak{b}\}$ is a bounding pair $\}$.

We want to show that our automorphism $\Psi : \mathcal{T} \to \mathcal{T}$ of the Torelli group induces an automorphism $\Psi_* : C(\mathbf{S}) \to C(\mathbf{S})$, where we make the following definition:

An automorphism $\Psi_* : C(\mathbf{S}) \to C(\mathbf{S})$ is *induced* by Ψ if and only if

i) for all $\mathfrak{a}$ separating, $\Psi_*(\mathfrak{a}) = \mathfrak{a}' \Leftrightarrow \Psi(D_{\mathfrak{a}}) = (D_{\mathfrak{a}'})^{\epsilon(\mathfrak{a})}$, where $\epsilon(\mathfrak{a}) = \pm 1$, and

ii) for all $\mathfrak{a}$ nonseparating, $\Psi_*(\mathfrak{a}) = \mathfrak{a}' \Leftrightarrow$ there exist functions

$$\sigma_{\mathfrak{a},\mathfrak{a}'} : \mathfrak{S}_{\mathfrak{a}} \to \mathfrak{S}_{\mathfrak{a}'} \quad \text{and} \quad \delta_{\mathfrak{a}} : \mathfrak{S}_{\mathfrak{a}} \to \{\pm 1\}$$

such that for all $\mathfrak{b} \in \mathfrak{S}_{\mathfrak{a}}$, we have

$$\Psi(D_{\mathfrak{a}} D_{\mathfrak{b}}^{-1}) = (D_{\mathfrak{a}'} D_{\sigma_{\mathfrak{a},\mathfrak{a}'}(\mathfrak{b})}^{-1})^{\delta_{\mathfrak{a}}(\mathfrak{b})}.$$

Our definition above makes it clear where Ψ_* should send the vertex corresponding to a separating curve. Conjecture 2 below says that there exists a unique well-defined image vertex for each vertex corresponding to a nonseparating curve. The proof of uniqueness is straight-forward and is stated and proved as the following lemma. The proof of existence, on the other hand, seems quite difficult, and is incomplete at the time of this writing. We must also show that the bijection $\mathfrak{S} \to \mathfrak{S}$ we get is simplicial.

Lemma 3.10 *Let $\mathfrak{a}$ be a nonseparating simple closed curve on $\mathbf{S}$. Then there exists at most one nonseparating simple closed curve $\mathfrak{a}'$ on $\mathbf{S}$ with the property that there exist functions $\sigma_{\mathfrak{a},\mathfrak{a}'} : \mathfrak{S}_{\mathfrak{a}} \to \mathfrak{S}_{\mathfrak{a}'}$ and $\delta_{\mathfrak{a}} : \mathfrak{S}_{\mathfrak{a}} \to \{\pm 1\}$ such that for all $\mathfrak{b} \in \mathfrak{S}_{\mathfrak{a}}$, we have $\Psi(D_{\mathfrak{a}} D_{\mathfrak{b}}^{-1}) = (D_{\mathfrak{a}'} D_{\sigma_{\mathfrak{a},\mathfrak{a}'}(\mathfrak{b})}^{-1})^{\delta_{\mathfrak{a}}(\mathfrak{b})}$.*

Proof:

Suppose there also exists $\mathfrak{a}''$ with $\sigma_{\mathfrak{a},\mathfrak{a}''} : \mathfrak{S}_{\mathfrak{a}} \to \mathfrak{S}_{\mathfrak{a}''}$ and $\gamma_{\mathfrak{a}} : \mathfrak{S}_{\mathfrak{a}} \to \{\pm 1\}$ such that for all $\mathfrak{b} \in \mathfrak{S}_{\mathfrak{a}}$, we have $\Psi(D_{\mathfrak{a}} D_{\mathfrak{b}}^{-1}) = (D_{\mathfrak{a}''} D_{\sigma_{\mathfrak{a},\mathfrak{a}''}(\mathfrak{b})}^{-1})^{\gamma_{\mathfrak{a}}(\mathfrak{b})}$.

Then for each $\mathfrak{b} \in \mathfrak{S}_{\mathfrak{a}}$, we have $(D_{\mathfrak{a}'} D_{\sigma_{\mathfrak{a},\mathfrak{a}'}(\mathfrak{b})}^{-1})^{\delta_{\mathfrak{a}}(\mathfrak{b})} = (D_{\mathfrak{a}''} D_{\sigma_{\mathfrak{a},\mathfrak{a}''}(\mathfrak{b})}^{-1})^{\gamma_{\mathfrak{a}}(\mathfrak{b})}$ and hence

$$\{\mathfrak{a}', \sigma_{\mathfrak{a},\mathfrak{a}'}(\mathfrak{b})\} = \{\mathfrak{a}'', \sigma_{\mathfrak{a},\mathfrak{a}''}(\mathfrak{b})\}. \tag{3.1}$$

Claim: $\displaystyle\bigcap_{\mathfrak{b} \in \mathfrak{S}_{\mathfrak{a}}} \{\mathfrak{a}', \sigma_{\mathfrak{a},\mathfrak{a}'}(\mathfrak{b})\} = \{\mathfrak{a}'\}$

Proof: Suppose there exists $\mathfrak{a}' \neq \mathfrak{c} \in \{\mathfrak{a}', \sigma_{\mathfrak{a},\mathfrak{a}'}(\mathfrak{b})\}$ for all $\mathfrak{b} \in \mathfrak{S}_{\mathfrak{a}}$. That is, $\mathfrak{c} = \sigma_{\mathfrak{a},\mathfrak{a}'}(\mathfrak{b})$ for all $\mathfrak{b} \in \mathfrak{S}_{\mathfrak{a}}$. This implies that

$$\operatorname{card}\{(D_{\mathfrak{a}'} D^{-1}_{\sigma_{\mathfrak{a},\mathfrak{a}'}(\mathfrak{b})})^{\delta_{\mathfrak{a}}(\mathfrak{b})} : \mathfrak{b} \in \mathfrak{S}_{\mathfrak{a}}\} \leq 2.$$

But this is a contradiction since $\mathfrak{S}_{\mathfrak{a}}$, and thus also $\{D_{\mathfrak{a}} D_{\mathfrak{b}}^{-1} : \mathfrak{b} \in \mathfrak{S}_{\mathfrak{a}}\}$, is infinite, and $\{(D_{\mathfrak{a}'} D^{-1}_{\sigma_{\mathfrak{a},\mathfrak{a}'}(\mathfrak{b})})^{\delta_{\mathfrak{a}}(\mathfrak{b})} : \mathfrak{b} \in \mathfrak{S}_{\mathfrak{a}}\}$ is the image of the set $\{D_{\mathfrak{a}} D_{\mathfrak{b}}^{-1} : \mathfrak{b} \in \mathfrak{S}_{\mathfrak{a}}\}$ under the injective map Ψ. This proves the claim.

Likewise, $\bigcap_{\mathfrak{b} \in \mathfrak{S}_{\mathfrak{a}}} \{\mathfrak{a}'', \sigma_{\mathfrak{a},\mathfrak{a}''}(\mathfrak{b}\} = \{\mathfrak{a}''\}$. But equation (3.1) implies that

$$\cap\{\mathfrak{a}', \sigma_{\mathfrak{a},\mathfrak{a}'}(\mathfrak{b})\} = \cap\{\mathfrak{a}'', \sigma\mathfrak{a}, \mathfrak{a}''(\mathfrak{b})\}.$$

That is, $\mathfrak{a}' = \mathfrak{a}''$.

$\square$

Conjecture 2 *Let $\mathfrak{a}$ be a nonseparating simple closed curve on $\mathbf{S}$. Then there exists a unique nonseparating simple closed curve $\mathfrak{a}'$ on $\mathbf{S}$ with the property that there exist functions $\sigma_{\mathfrak{a},\mathfrak{a}'} : \mathfrak{S}_{\mathfrak{a}} \to \mathfrak{S}_{\mathfrak{a}'}$ and $\delta_{\mathfrak{a}} : \mathfrak{S}_{\mathfrak{a}} \to \{\pm 1\}$ such that for all $\mathfrak{b} \in \mathfrak{S}_{\mathfrak{a}}$, we have $\Psi(D_{\mathfrak{a}} D_{\mathfrak{b}}^{-1}) = (D_{\mathfrak{a}'} D^{-1}_{\sigma_{\mathfrak{a},\mathfrak{a}'}(\mathfrak{b})})^{\delta_{\mathfrak{a}}(\mathfrak{b})}$.*

3.3 Proof that Conjecture 2 implies Conjecture 1

We now prove Conjecture 1, stated on page 30, under the assumption that Conjecture 2 is true.

Conjecture 2 implies that our automorphism $\Psi : \mathcal{T} \to \mathcal{T}$ induces an automorphism $\Psi_* : C(\mathbf{S}) \to C(\mathbf{S})$, given by

- for $\mathfrak{a} \in \mathfrak{S}$ separating, $\Psi_*(\mathfrak{a}) = \mathfrak{a}' \Leftrightarrow \Psi(D_{\mathfrak{a}}) = (D_{\mathfrak{a}'})^{\epsilon(\mathfrak{a})}$, and

- for $\mathfrak{a} \in \mathfrak{S}$ nonseparating, $\Psi_*(\mathfrak{a}) = \mathfrak{a}' \Leftrightarrow$ there exist unique functions

$$\sigma_{\mathfrak{a}} : \mathfrak{S}_{\mathfrak{a}} \to \mathfrak{S}_{\mathfrak{a}'} \quad \text{and} \quad \delta_{\mathfrak{a}} : \mathfrak{S}_{\mathfrak{a}} \to \{\pm 1\}$$

  such that for all $\mathfrak{b} \in \mathfrak{S}_{\mathfrak{a}}$ we have $\Psi(D_{\mathfrak{a}} D_{\mathfrak{b}}^{-1}) = (D_{\mathfrak{a}'} D_{\sigma_{\mathfrak{a}}(\mathfrak{b})}^{-1})^{\delta_{\mathfrak{a}}(\mathfrak{b})}$.

Now, if $h : \mathbf{S} \to \mathbf{S}$ is a homeomorphism, then h induces two functions:

- an automorphism $h_* : C(\mathbf{S}) \to C(\mathbf{S})$ defined for $\mathfrak{a} \in \mathfrak{S}$ by $\mathfrak{a} \mapsto h(\mathfrak{a})$
- an automorphism $h_\sharp : \mathcal{T} \to \mathcal{T}$ defined by $f \mapsto hfh^{-1}$.

By Ivanov ([9], Theorem 1), $\Psi_* = h_*$ for some homeomorphism $h : \mathbf{S} \to \mathbf{S}$. Our goal is to show that $\Psi = h_\sharp$.

Here, $\Psi_* = h_*$ means that if $\mathfrak{a} \in \mathfrak{S}$, then $\Psi_*(\mathfrak{a}) = h_*(\mathfrak{a})$. Specifically, if $\mathfrak{a}$ is separating, then $\Psi_*(\mathfrak{a}) = h_*(\mathfrak{a})$ means that $\Psi(D_{\mathfrak{a}}) = (D_{h(\mathfrak{a})})^{\epsilon(\mathfrak{a})}$, while if $\mathfrak{a}$ is nonseparating, $\Psi_*(\mathfrak{a}) = h_*(\mathfrak{a})$ means that for all $\mathfrak{b} \in \mathfrak{S}_{\mathfrak{a}}$, $\Psi(D_{\mathfrak{a}} D_{\mathfrak{b}}^{-1}) = (D_{h(\mathfrak{a})} D_{\sigma_{\mathfrak{a}}(\mathfrak{b})}^{-1})^{\delta_{\mathfrak{a}}(\mathfrak{b})}$.

Let $f \in \mathcal{T}$, and let $\mathfrak{a} \in \mathfrak{S}$. We have two cases.

Case 1: $\mathfrak{a}$ is separating. In this case we have

$$\Psi(fD_{\mathfrak{a}}f^{-1}) = \Psi(D_{f(\mathfrak{a})}) = D_{hf(\mathfrak{a})}^{\epsilon(\mathfrak{a})}.$$

On the other hand, we also have

$$\Psi(fD_{\mathfrak{a}}f^{-1}) = \Psi(f)\Psi(D_{\mathfrak{a}})\Psi(f)^{-1} = \Psi(f)D_{h(\mathfrak{a})}^{\epsilon(\mathfrak{a})}\Psi(f)^{-1} = D_{\Psi(f)(h(\mathfrak{a}))}^{\epsilon(\mathfrak{a})},$$

and so $\Psi(f)(h(\mathfrak{a})) = hf(\mathfrak{a})$. If we write $\mathfrak{a} = h^{-1}(\mathfrak{b})$, then this shows that $\Psi(f)(\mathfrak{b}) = hfh^{-1}(\mathfrak{b})$ for all $\mathfrak{b} \in \mathfrak{S}$ separating.

Case 2: $\mathfrak{a}$ is nonseparating. Let $\mathfrak{b} \in \mathfrak{S}_{\mathfrak{a}}$. Then we have

$$\Psi(fD_{\mathfrak{a}}D_{\mathfrak{b}}^{-1}f^{-1}) = \Psi(D_{f(\mathfrak{a})}D_{f(\mathfrak{b})}^{-1}) = (D_{hf(\mathfrak{a})}D_{\sigma_{f(\mathfrak{a})}(f(\mathfrak{b}))}^{-1})^{\delta_{f(\mathfrak{a})}(f(\mathfrak{b}))}.$$

On the other hand, we also have

$$
\begin{aligned}
\Psi(fD_{\mathfrak{a}}D_{\mathfrak{b}}^{-1}f^{-1}) &= \Psi(f)\Psi(D_{\mathfrak{a}}D_{\mathfrak{b}}^{-1})\Psi(f)^{-1} \\
&= \Psi(f)(D_{h(\mathfrak{a})}D_{\sigma_{\mathfrak{a}}(\mathfrak{b})}^{-1})^{\delta_{\mathfrak{a}}(\mathfrak{b})}\Psi(f)^{-1} \\
&= \Psi(f)D_{h(\mathfrak{a})}^{\delta_{\mathfrak{a}}(\mathfrak{b})}D_{\sigma_{\mathfrak{a}}(\mathfrak{b})}^{-\delta_{\mathfrak{a}}(\mathfrak{b})}\Psi(f)^{-1} \\
&= \Psi(f)D_{h(\mathfrak{a})}^{\delta_{\mathfrak{a}}(\mathfrak{b})}\Psi(f)^{-1}\Psi(f)D_{\sigma_{\mathfrak{a}}(\mathfrak{b})}^{-\delta_{\mathfrak{a}}(\mathfrak{b})}\Psi(f)^{-1} \\
&= D_{\Psi(f)(h(\mathfrak{a}))}^{\delta_{\mathfrak{a}}(\mathfrak{b})}D_{\Psi(f)(\sigma_{\mathfrak{a}}(\mathfrak{b}))}^{-\delta_{\mathfrak{a}}(\mathfrak{b})} \\
&= (D_{\Psi(f)(h(\mathfrak{a}))}D_{\Psi(f)(\sigma_{\mathfrak{a}}(\mathfrak{b}))}^{-1})^{\delta_{\mathfrak{a}}(\mathfrak{b})}.
\end{aligned}
$$

Hence, for all $\mathfrak{b} \in \mathfrak{S}_{\mathfrak{a}}$ we have

$$\{hf(\mathfrak{a}), \sigma_{f(\mathfrak{a})}(f(\mathfrak{b}))\} = \{\Psi(f)(h(\mathfrak{a})), \Psi(f)(\sigma_{\mathfrak{a}}(\mathfrak{b}))\}. \tag{3.2}$$

Claim: $\bigcap_{\mathfrak{b}\in\mathfrak{S}_{\mathfrak{a}}} \{hf(\mathfrak{a}), \sigma_{f(\mathfrak{a})}(f(\mathfrak{b}))\} = \{hf(\mathfrak{a})\}$

Proof: Suppose there exists $hf(\mathfrak{a}) \neq \mathfrak{c} \in \{hf(\mathfrak{a}), \sigma_{f(\mathfrak{a})}(f(\mathfrak{b}))\}$ for all $\mathfrak{b} \in \mathfrak{S}_{\mathfrak{a}}$. That is, $\mathfrak{c} = \sigma_{f(\mathfrak{a})}(f(\mathfrak{b}))$ for all $\mathfrak{b} \in \mathfrak{S}_{\mathfrak{a}}$. This implies that

$$\mathrm{card}\{(D_{hf(\mathfrak{a})}D_{\sigma_{f(\mathfrak{a})}(f(\mathfrak{b}))}^{-1})^{\delta_{f(\mathfrak{a})}(f(\mathfrak{b}))} : \mathfrak{b} \in \mathfrak{S}_{\mathfrak{a}}\} \leq 2.$$

But the set $\mathfrak{S}_{\mathfrak{a}}$ is infinite, and since f is injective, so is $\{D_{f(\mathfrak{a})}D_{f(\mathfrak{b})}^{-1} : \mathfrak{b} \in \mathfrak{S}_{\mathfrak{a}}\}$. And since $\{(D_{hf(\mathfrak{a})}D_{\sigma_{f(\mathfrak{a})}(f(\mathfrak{b}))}^{-1})^{\delta_{f(\mathfrak{a})}(f(\mathfrak{b}))} : \mathfrak{b} \in \mathfrak{S}_{\mathfrak{a}}\}$ is the image of the infinite set $\{D_{f(\mathfrak{a})}D_{f(\mathfrak{b})}^{-1} : \mathfrak{b} \in \mathfrak{S}_{\mathfrak{a}}\}$ under the injective map Ψ, we obtain a contradiction. This proves the claim.

Likewise, $\bigcap_{\mathfrak{b}\in\mathfrak{S}_{\mathfrak{a}}} \{\Psi(f)(h(\mathfrak{a})), \Psi(f)(\sigma_{\mathfrak{a}}(\mathfrak{b}))\} = \{\Psi(f)(h(\mathfrak{a}))\}$. So by (3.2) we have $hf(\mathfrak{a}) = \Psi(f)(h(\mathfrak{a}))$. If we write $\mathfrak{a} = h^{-1}(\mathfrak{d})$, then this shows that $hfh^{-1}(\mathfrak{d}) = \Psi(f)(\mathfrak{d})$ for all $\mathfrak{d} \in \mathfrak{S}$ nonseparating.

Cases 1 and 2 together show that $\Psi(f)(\mathfrak{a}) = hfh^{-1}(\mathfrak{a})$ for *all* $\mathfrak{a} \in \mathfrak{S}$, and since the genus g of $\mathbf{S}$ is at least 3, this implies that $\Psi(f) = hfh^{-1}$

for all $f \in \mathcal{T}$. That is, $\Psi = h_\sharp : \mathcal{T} \to \mathcal{T}$.

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