THE SUBSTITUTION RULE FOR INDEFINITE INTEGRALS

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Abstract. In this note, we explain the meaning of the Substitution Rule for Indefinite Integrals

We recall the Substitution Rule for Indefinite Integrals.

Theorem 1. *If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then*

$$\int f(g(x))g'(x)dx = \int f(u)du. \tag{1}$$

To better understand this rule, let us make some basic observations about indefinite integrals. What does the following notation mean?

$$\int h(z)dz \tag{2}$$

In this notation, z is a real variable and $h(z)$ is a real valued function of the real variable z. The notation stands for the family of all antiderivatives $H(z)$ of $h(z)$; that is to say, the family of all functions $H(z)$ of z whose derivatives $H'(z)$ with respect to z are equal to $h(z)$.

Note that, in this notation, z, is not a function of some variable. That is to say, in classical language, z is an independent real variable.

Now, let $p(x) = f(g(x))g'(x)$. Then the left hand side of equation (1):

$$\int f(g(x))g'(x)dx = \int p(x)dx \tag{3}$$

is the family of all functions $Q(x)$ of the independent real variable x whose derivative $Q'(x)$ with respect to x is equal to $p(x) = f(g(x))g'(x)$.

Likewise, the right hand side of equation (1):

$$\int f(u)du \tag{4}$$

is the family of all functions $G(u)$ of the independent real variable u whose derivative $G'(u)$ with respect to u is equal to $f(u)$.

Note that the u on the right hand side of equation (1) is an independent real variable, not a function of some variable. Why is this? Because this is what the notation in equation (4) means.

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In particular, the u on the right hand side of (1) is not equal to the function $g(x)$ of the independent real variable x. If the u on the right hand side of (1) were equal to $g(x)$, then this u would not be an independent real variable; it would be a dependent real variable, dependent upon the independent real variable x. But the u on the right hand side of (1) is an independent real variable. Hence, it cannot be equal to $g(x)$.

Note what this means. The left hand side of equation (1) is a family of functions of the independent real variable x; while the right hand side of equation (1) is a family of functions of the independent real variable u.

Then how are we to understand equation (1), an equation that asserts that a certain family of functions of the independent variable x is equal to a certain family of functions of the independent variable u?

The answer to this question is in the following precise formal statement of the Substitution Rule for Indefinite Integrals:

Theorem 2. *Suppose that $g(x)$ is a differentiable function whose range is an interval I and f is continuous on I. If*

$$\int f(u)du = F(u) + C \tag{5}$$

for some antiderivative $F(u)$ of $f(u)$ as C varies over $\mathbb{R}$, then:

$$\int f(g(x))g'(x)dx = F(g(x)) + C \tag{6}$$

as C varies over $\mathbb{R}$.

Hence, equation (1) is a shorthand way of expressing the conditional statement in Theorem 2. Rather than writing out this conditional statement, we express it by the following shorthand:

$$\int f(g(x))g'(x)dx = \int f(u)du = F(u) + C = F(g(x)) + C. \tag{7}$$

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