THE FUNDAMENTAL THEOREM OF CALCULUS

JOHN D. MCCARTHY

Abstract. In this note, we give a different proof of the Fundamental Theorem of Calculus Part 2 than that given in Thomas' Calculus, 11th Edition, Thomas, Weir, Hass, Giordano, ISBN-10: 0321185587, Addison-Wesley, ©2005. We also discuss the extent to which the Fundamental Theorem of Calculus Part 2 implies the Fundamental Theorem of Calculus Part 1.

1. The Fundamental Theorem of Calculus Part 1

We recall the Fundamental Theorem of Calculus Part 1, hereafter referred to as Part 1, with a slight revision from the formulation in Thomas' Calculus, 11th Edition, Thomas, Weir, Hass, Giordano, ISBN-10: 0321185587, Addison-Wesley, ©2005, hereafter referred to as Thomas' Calculus.

Theorem 1. ***The Fundamental Theorem of Calculus Part 1*** *If f is continuous on $[a, b]$ and F is the function on $[a, b]$ defined by the rule:*

$$F(x) = \int_a^x f(t)dt, \tag{1}$$

then F is continuous on $[a, b]$ and an antiderivative for f on (a, b); that is to say:

$$F'(x) = \frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x). \tag{2}$$

for all $x \in (a, b)$.

Part 1 has the following important corollary concerning the existence of antiderivatives.

Corollary 2. ***Existence of Antiderivatives*** *If f is continuous on $[a, b]$, then there exists a continuous function F on $[a, b]$ such that F is an antiderivative for f on (a, b).*

Proof. This is an immediate consequence of Part 1 which gives one such function F; namely the function F on $[a, b]$ defined by the rule $F(x) = \int_a^x f(t)dt$. □

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2. The Fundamental Theorem of Calculus Part 2

We recall the Fundamental Theorem of Calculus Part 2, hereafter referred to as Part 2, with a slight revision from the formulation in Thomas' Calculus.

Theorem 3. ***The Fundamental Theorem of Calculus Part 2*** *If f is continuous on $[a, b]$ and F is a continuous function on $[a, b]$ such that F is an antiderivative for f on (a, b), then:*

$$\int_a^b f(x)dx = F(x)\Big|_a^b \doteq F(b) - F(a). \tag{3}$$

In Thomas Calculus', Part 2 is proved by first proving Part 1 without using Part 2 and then deducing Part 2 from Part 1. In this section, we shall prove Part 2 without using Part 1.

Proof. **An Alternative Proof of Part 2 without using Part 1**

Since f is continuous on $[a, b]$, it follows from Theorem 1 of Section 5.3 of Thomas' Calculus that f is integrable on $[a, b]$; that is to say, $\int_a^b f(x)dx$ exists.

Let ϵ be a positive real number. By the definition of the definite integral as a limit of Riemann sums, it follows that there exists a positive real number δ such that if $P : a = x_0 < .... < x_n = b$ is any partition of $[a, b]$ with $||P|| < \delta$ and $c_i \in [x_{i-1}, x_i], 1 \leq i \leq n$, and:

$$R \doteq \sum_{i=1}^{n} f(c_i)(\Delta(x))_i, \tag{4}$$

then:

$$\left|R - \int_a^b f(x)dx\right| < \epsilon. \tag{5}$$

Now choose a positive integer n such that $\Delta \doteq (b-a)/n < \delta$. Let $x_i = a+i\Delta, 1 \leq i \leq n$. Then $P : a = x_0 < ... < x_n = b$ is a partition of $[a, b]$, $(\Delta(x))_i \doteq x_i - x_{i-1} = \Delta, 1 \leq i \leq n$ and $||P|| = \max\{(\Delta(x))_i | 1 \leq i \leq n\} = \Delta < \delta$.

Let i be an integer such that $1 \leq i \leq n$. Since F is continuous on $[a, b]$ and $[x_{i-1}, x_i]$ is contained in $[a, b]$, F is continuous on $[x_{i-1}, x_i]$. Likewise, since F is differentiable on (a, b) and (x_{i-1}, x_i) is contained in (a, b), F is differentiable on (x_{i-1}, x_i).

It follows from the Mean Value Theorem for Derivatives that there exists $c_i \in (x_{i-1}, x_i)$ such that:

$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = F'(c_i). \tag{6}$$

Since F is an antiderivative for f on (a, b) and $c_i \in (x_{i-1}, x_i) \subset (a, b)$, it follows that $F'(c_i) = f(c_i)$. Hence, by equation (6) and the fact that $x_i - x_{i-1} = (\Delta(x))_i$, it follows that:

$$\frac{F(x_i) - F(x_{i-1})}{(\Delta(x))_i} = f(c_i) \tag{7}$$

which implies that:

$$F(x_i) - F(x_{i-1}) = f(c_i)(\Delta(x))_i. \tag{8}$$

Since equation (8) holds for each integer i with $1 \leq i \leq n$, it follows that:

$$R \doteq \sum_{i=1}^{n} f(c_i)(\Delta(x))_i = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) \tag{9}$$

$$= (F(x_1) - F(x_0)) + (F(x_2) - F(x_1)) + ...$$

$$+ (F(x_{n-1}) - F(x_{n-2})) + (F(x_n) - F(x_{n-1}))$$

$$= -F(x_0) + F(x_n) = F(x_n) - F(x_0) = F(b) - F(a).$$

Since $||P|| < \delta$, It follows from equations (5) and (9) that:

$$\left|(F(b) - F(a)) - \int_a^b f(x)dx\right| < \epsilon. \tag{10}$$

Since equation (10) holds for every positive real number ϵ, it follows that

$$\int_a^b f(x)dx = F(b) - F(a). \tag{11}$$

□

3. The Relationship of Parts 1 and 2

As mentioned earlier, Thomas' Calculus proves Part 2 by first proving Part 1 and then deducing Part 2 from Part 1.

Since in Section 2 we proved Part 2 without using Part 1, this raises the question, can we deduce Part 1 from Part 2?

The following theorem shows that the answer to this question is affirmative, provided that we have Corollary 2 on the existence of antiderivatives. Since the proof of Corollary 2 depends upon Part 1, this theorem falls short of demonstrating that Part 2 implies Part 1.

Theorem 4. *The Fundamental Theorem of Calculus Part 2 (i.e. Theorem 3) and Corollary 2 on the existence of antiderivatives imply the Fundamental Theorem of Calculus Part 1 (i.e. Theorem 1).*

Proof. Assume Part 2 and Corollary 2 and suppose that f is continuous on $[a, b]$. By Corollary 2, there exists a continuous function G on $[a, b]$ such that G is differentiable on (a, b) and G is an antiderivative for f on (a, b).

Suppose that $x \in [a, b]$; that is to say, $a \leq x \leq b$.

Suppose, on the one hand, that $x = a$. Then $F(x) \doteq \int_a^x f(t)dt = \int_a^a f(t)dt = 0$ and $G(x) - G(a) = G(a) - G(a) = 0$. Hence:

$$F(x) = \int_a^x f(t)dt = G(x) - G(a). \tag{12}$$

Suppose, on the other hand, that $x \neq a$. Then $a < x \leq b$. It follows that $[a, x] \subset [a, b]$ and $(a, x) \subset (a, b)$. Hence, since f is continuous on $[a, b]$, f is continuous on $[a, x]$; since G is continuous on $[a, b]$, G is continuous on $[a, x]$; since G is differentiable on (a, b), G is differentiable on (a, x); and since G is an antiderivative for f on (a, b), G is an antiderivative for f on (a, x). It follows from Part 2 that:

$$F(x) = \int_a^x f(t)dt = G(x) - G(a). \tag{13}$$

Hence, by equations (12) and (13), for all $x \in [a, b]$:

$$F(x) = G(x) - G(a). \tag{14}$$

Since G is continuous on $[a, b]$ and constant functions are continuous and a difference of continuous functions is continuous, it follows from equation (14) that F is continuous on $[a, b]$.

Since G is an antiderivative for f on (a, b), it follows from equation (14), the difference rule for derivatives, and the constant rule for derivatives that:

$$F'(x) = (G(x) - G(a))' = G'(x) - 0 = f(x) - 0 = f(x) \tag{15}$$

for all $x \in (a, b)$.

Hence, the function F on $[a, b]$ given by the rule $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and an derivative of f on (a, b).

□

4. Summary

In Thomas' Calculus, Part 1 is first proven without using Part 2 and then it is proven that Part 1 implies Part 2, thereby proving Part 2. In the converse direction, in Section 2, we proved Part 2 without using Part 1.

This raised a question; does Part 2 imply Part 1, thereby proving Part 1? In Section 3, we showed that the answer to this question is affirmative provided that we have Corollary 2 on the existence of antiderivatives; that is, we showed that Corollary 2 and Part 2 imply Part 1.

But, as previously observed, the proof of Corollary 2 uses Part 1. So the proof in Section 3 does not prove that Part 2 implies Part 1, thereby proving Part 1. If we had a proof of Corollary 2 which did not use Part 1, then the proof in Section 3 would prove that Part 2 implies Part 1, thereby proving Part 1.

In conclusion, it appears that Part 1 is the stronger of the two parts of the Fundamental Theorem of Calculus. Part 1, once established, not only gives us Corollary 2 on the existence of antiderivatives but also Part 2. In the converse direction, we have not been able to first establish Corollary 2, as well as Part 2, and thereby obtain Part 1.

This leaves us with the question, is it possible to prove Corollary 2 without using Part 1? Or, to put the question differently, is it possible to prove the existence of antiderivatives in the context of continuous functions on closed intervals without using the construction of antiderivatives given by Part 1, a construction that uses the concept of a definite integral? That is to say, must we define definite integrals before we can show that antiderivatives exist in this context?

Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027
E-mail address: mccarthy@math.msu.edu
URL: http://www.math.msu.edu/ mccarthy/