Review Risk free assets.

Bonds, Annuity, Perpetuity

Interest, Effective Interest.

$$PA(r,n) = \frac{1}{1+r} + \cdots + \frac{1}{(1+r)^n} = \frac{1-(1+r)^{-n}}{r}$$

$$PA(r) = PA(r,\infty) = \sum_{k=1}^{\infty} \frac{1}{(1+r)^k} = \frac{1}{1-\frac{1}{1+r}} - 1 = \frac{1}{r}.$$

Eg. Effective Interest is $r_e = .04$, (4%).

Suppose Bond pays \$100 coupon every 2 months for 2 years. At 2 years Pays \$10000

What is present Value?

12 Coupon payments: 2 mo, 4 mo, 6 mo, 8 mo, ...

1 Face val payment 24 mo.

Interest in each 2 mo period $(1+r) = (1+r_e)^{1/6}$

$$1+r = (1.04)^{1/6} \to$$

$\hookrightarrow$ $r \approx .0066$.

$$V = 100\left(\frac{1}{.0066}\right)\left\{1-(1.0066)^{-12}\right\} + 10{,}000\left(\frac{1}{1.04}\right)^2$$

$$= \underbrace{100\left(\frac{1}{.0066}\right)}_{15151}\underbrace{\left\{1-\left(\frac{1}{1.04}\right)^2\right\}}_{.075} + 10{,}000\underbrace{\left(\frac{1}{1.04}\right)^2}_{.9246} = 1143 + 9246 = 10{,}388$$

Suppose you decide to sell Above asset @ 13 mo

how much do you sell it for?

find value@ 12 mo & markup 1 mo. + 6 remaining bonds
+ 1 Face val.

$$V(1) = 100 \frac{1}{.0066}\left\{1-(1.0066)^{-6}\right\} + 10{,}000\left(\frac{1}{1.04}\right)^{1}$$

$$= 100 \frac{1}{.0066}\left\{1-\frac{1}{1.04}\right\} + 10{,}000\ \frac{1}{1.04}$$

$$.0385 \qquad\qquad .9615$$

$$= 582 + 9615 = 10197.$$

Recall: value <u>decreases</u> w/ increasing r.

Compound interest ~ 1 year m intervals

$$\frac{1}{\beta_{01}} = \left(1+\frac{r}{m}\right)^m$$

fixed interest: increased compounding:

$$\left(1+\frac{r}{m}\right)^m < \left(1+\frac{r}{m+1}\right)^{m+1}$$

Effective interest

$$1+r_e = \cancel{\#}\, e^r = \lim_{m\to\infty}\left(1+\frac{r}{m}\right)^m.$$

Suppose bond A is compounded every 3 mo @ 2%
bond B ―――――――――――――― 4 mo @ 2%

Both have "Face value" of \$100

which is worth more today?

$$100\left(1+\frac{.02}{4}\right)^{-4} < 100\left(1+\frac{.02}{3}\right)^{-3}$$

3mo
4 intervals

4 mo
3 intervals.

2 security market.

$$V = x_1 S_1 + x_2 S_2$$

$$w_i = \frac{x_i S_i}{V}.$$

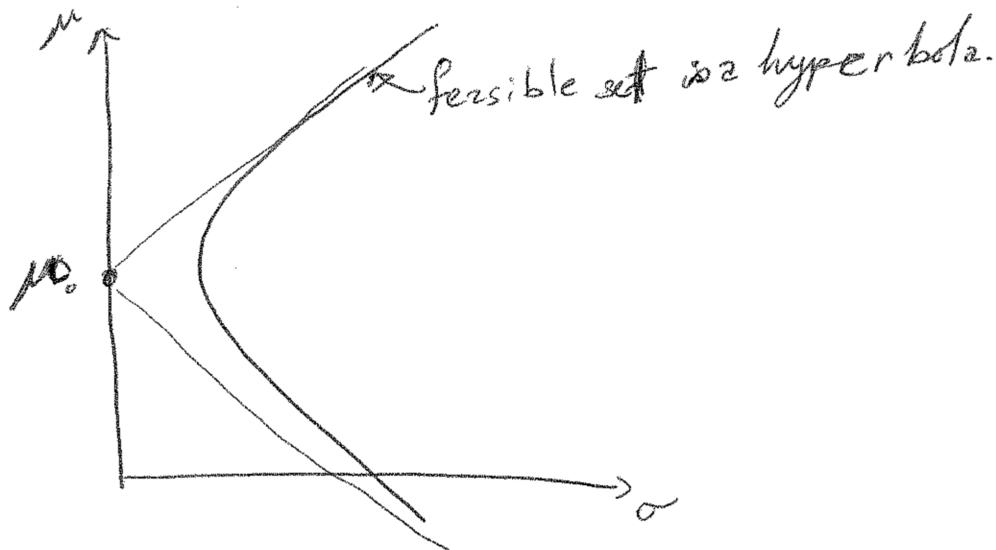
$$K_i = \frac{S_i(1) - S_i(0)}{S_i(0)}.$$

$$\mu_i = \mathbb{E}(K_i) \quad , \quad \sigma_i^2 = \mathrm{Var}(K_i)$$

$$m = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \qquad C_{12} = \rho_{12}\sigma_1\sigma_2 = \mathrm{Cov}(K_1, K_2)$$

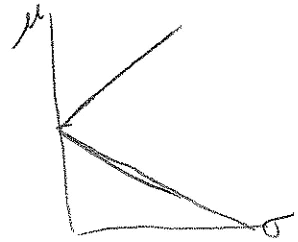
$$\mu_w = w^T m$$

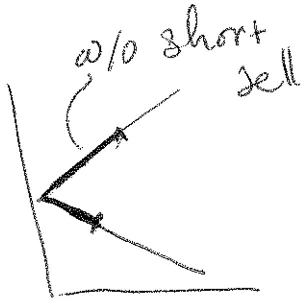
$$\sigma_w^2 = w^T \Sigma w \quad \text{for } \Sigma = \begin{matrix} \sigma_1^2 & C_{12} \\ C_{12} & \sigma_2^2 \end{matrix}$$



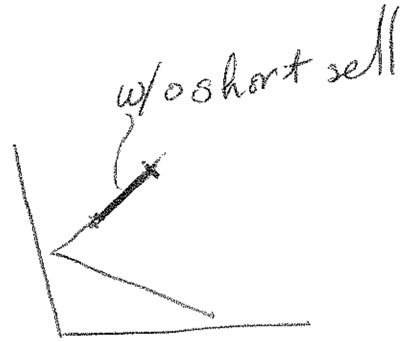
$$A^2 = \frac{\sigma_1^2 + \sigma_2^2 - 2C_{12}}{(\mu_1 - \mu_2)^2} > 0 \qquad y = \mu_0 \pm \frac{1}{A}$$

If $|\rho| = 1$ feasible set





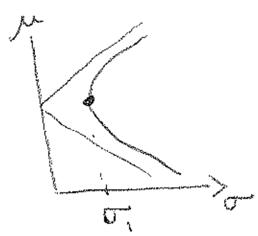
$\rho_{12} = -1$

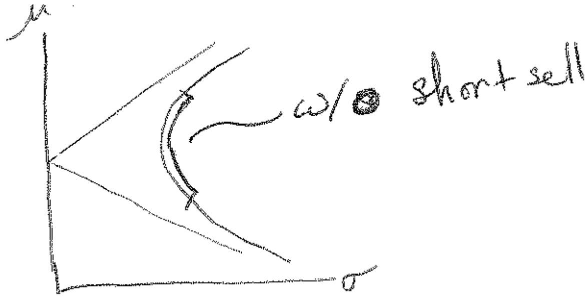


$\rho_{12} = +1.$

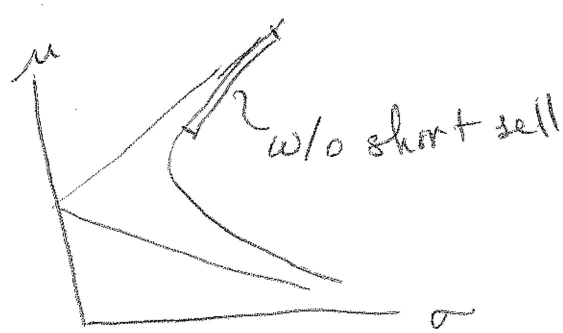
$\sigma_1 \le \sigma_2$

$\rho_{12} = \frac{\sigma_1}{\sigma_2}$





$-1 < \rho_{12} < \sigma_1/\sigma_2$



$\sigma_1/\sigma < \rho_{12} < 1.$

Several Securities –

$$\omega_i = \frac{x_i S_i(0)}{V(0)}.$$

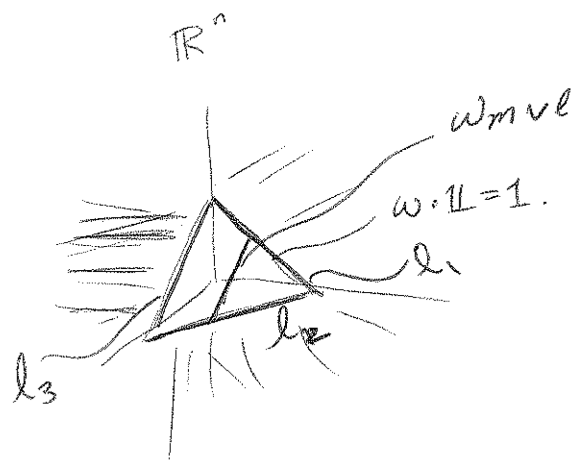
$$\omega \cdot \mathbb{1} = 1.$$

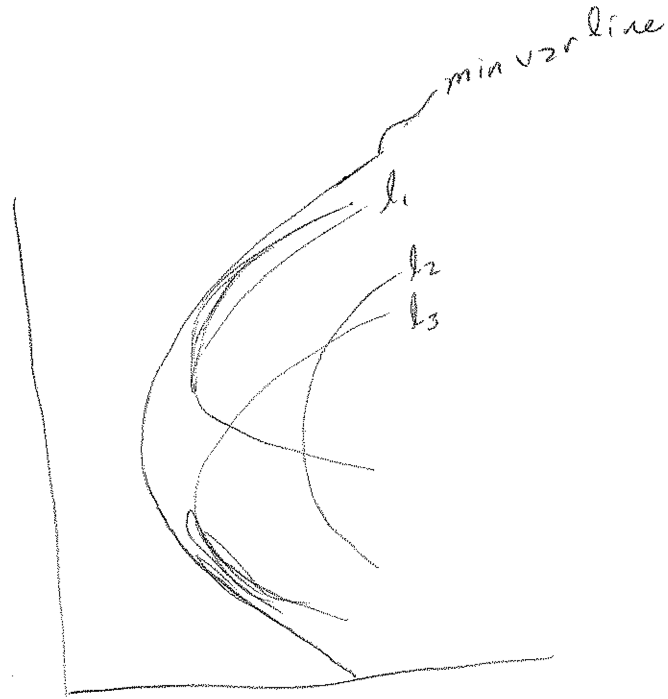
$$\Sigma_{ij} = \mathrm{cov}(K_i, K_j), \quad m_i = \mathbb{E} K_i$$

$$\mu_V = W \cdot m, \quad \sigma_V^2 = w^T \Sigma w$$

$$W_{mvp} = \frac{\Sigma^{-1}\mathbb{1}}{\mathbb{1}^T \Sigma^{-1} \mathbb{1}}.$$

$$W_{mvl} = \frac{\lambda_1}{2}\Sigma^{-1}\mathbb{1} + \frac{\lambda_2}{2}\Sigma^{-1} m = a\mu + b.$$





2 fund

Consider $\omega_{mvl}(\mu)$

let μ_1, μ_2 be two return values.

$$\omega^{(i)} = \omega(\mu_i)$$

eg $\begin{cases} \mu_1 = 0 \\ \mu_2 = 1/10 \end{cases}$

where

$$\omega(\mu) = \left((\mu \ \ 1)\, M^{-1} e_1 \right) \Sigma^{-1} m + \left((\mu \ 1)\, M^{-1} e_2 \right) \Sigma^{-1} \mathbb{1}$$

Define, return variables, for these portfolios

$$K_i^* = \omega^{(i)} \cdot K = \sum \omega_j^{(i)} K_j$$

$$K_j = \frac{S_j(1) - S_j(0)}{S_j(0)}.$$

$$(\sigma_i^*)^2 = \omega^{(i)} \Sigma \omega^{(i)}; \quad \mu_i = \mathbb{E}(\omega^{(i)} \cdot K) = \mathbb{E} K_i^{(*)}$$

$$c_{12}^* = \rho_{12}^* \sigma_1^* \sigma^* = \omega^{(1)} \Sigma \omega^{(2)}.$$

feasible set of 2 sec. syst.
=
min variance line of many security system.

FORWARD PRICE

Forward - agreement to purchase security at future date,

(1 div) $F(0,T) = \{S - \beta_{0\tau}\,\delta\}\,\dfrac{1}{\beta_{0T}}$

(n dividends)

$$F(0,T) = \{S - \textstyle\sum_i \beta_{0\tau_i}\,\delta_i\}\,\frac{1}{\beta_{0T}}$$

Continuous dividends —

$$F(0,T) = S\,\frac{e^{-\delta T}}{\beta_{0T}}\,, \qquad \delta \equiv \text{dividend rate.}$$

FORWARD, Change of CURRENCY.

$$F(0,T) = \rho_0\,\frac{B_f(0,T)}{B_d(0,T)}$$

$B_f(0,T) \equiv$ foreign discount ; $B_d(0,T) \equiv$ domestic discount

$\rho_0 \equiv$ exchange rate at time 0.

FORWARD w/ given EXCHANGE VALUE.

$X \equiv$ exchange value of security.

$F(t,T) \equiv$ forward contract exchange.

$$V_X(t) = \big(F(t,T) - X\big)\beta_{t,T}$$

Suppose $S(0) = 90$, $r_e = .04$.

(i) (Short forward @ 1 year) (Suppose @ 6 mo value $S(6/12) = 95$)

find value of Forward.

$$F(0,1) = 90\,(1.04)$$

$$F(1/2,1) = 95\,(1.04)^{1/2}$$

$$V_X(1/2) = \big(F(1/2,1) - F(0,1)\big)\,(1.04)^{-1/2}$$

$$= 90\,(1.04)^{1/2} - 95.$$

$$= -3.22$$

Options

Call / Put ; American / European

European ~

Put / Call parity ~

$$C_E(t) - P_E(t) = V_X(t)$$

Intrisic ~~\$~~ value ~ $\tilde{C}_E(t) = (S_t - X)^+$

$\tilde{P}_E(t) = (X - S_t)^+$

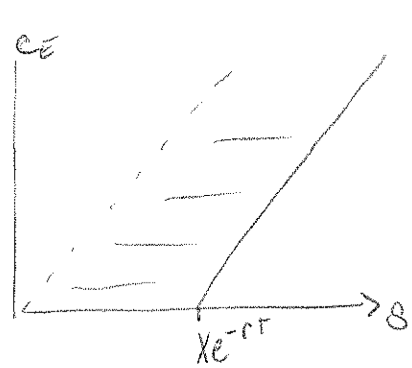
time value $\approx C_E(S,t) - (S-X)^+$

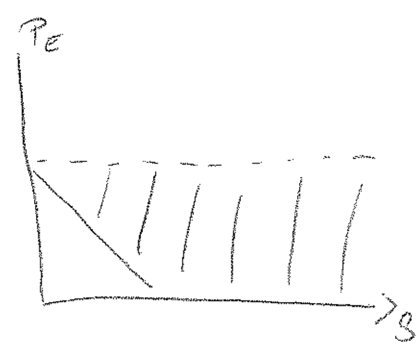
$P_E(S,t) - (X-S)^+$

time value goes to zero as $t \to T$.

$$(S(0) - X e^{-rT})^+ \le C_E \le S(0) \sim \{V_x(t)\}^+ \le C_E < V_x(t) + Xe^{-rT}$$

$$(Xe^{-rT} - S)^+ \le P_E \le Xe^{-rT} \sim \{-V_X(t)\}^+ \le P_E < S_0 - V_X(t)$$





American

$\tilde{C}_A(t) = (S_t - X)^+ \quad , \quad \tilde{P}_A = (X - S_t)^+$

Put call parity ~

$$S - Xe^{-rT} \geq C_A - P_A \geq S - X - \sum \delta_i B_0 \tau_i$$

$$S - Xe^{-rT} \geq C_A - P_A \geq Se^{-\delta T} - X$$

$\delta \equiv$ continuous div rate.

CALLS w/o DIVIDENDS

$$C_A = C_E.$$

Bounds ~

$$\{S - Xe^{-rT}\}^+ \leq C_A(S,0) \leq S(0)$$

$$\{X - S(0)\}^+ \leq P_A(S) < X.$$

Option Pricing Binomial model.

$$S_{t+1} = S_t (1 + M_{t+1})$$

$$M_{t+1} \in \{ m_1, m_2 \}.$$

Option w/ payoff value H_t. @ time t.
(may depend on path).

$$x_t(\mathcal{F}_{t-1}) \cancel{\;} = \frac{V_t(m_1 | \mathcal{F}_{t-1}) - V_t(m_2 | \mathcal{F}_{t-1})}{S_t(1+m_1) - S_t(1+m_2)}.$$

$$y_t(\mathcal{F}_t) = \frac{(1+m_1) V_t(m_2 | \mathcal{F}_{t-1}) - (1+m_2) V_t(m_1 | \mathcal{F}_{t-1})}{A(t+1) \{ S_t(1+m_1) - S_t(1+m_2) \}}$$

Replicating Portfolio value

$$V_t = x_t S_t + y_t A_t.$$

$$A_t = (1+r)^t$$

$$m_2 \le r \le m_1 \qquad m_1 \neq m_2.$$

Risk neutral measure

$$p_1^* = \frac{r - m_2}{m_1 - m_2}, \quad p_2^* = \frac{m_1 - r}{m_1 - m_2}, \quad p^* = \begin{pmatrix} p_1^* \\ p_2^* \end{pmatrix}$$

Martingale property.

$$S_t = \frac{1}{1+r} \mathbb{E}(S_{t+1} \mid \mathcal{F}_t)$$

$$= \frac{1}{1+r} \left\{ p_1^* S_t \frac{r - m_2}{m_1 - m_2} + p_2^* S_t \frac{m_1 - r}{m_1 - m_2} \right\}.$$

$\therefore$ $\tilde{S}_t = \left(\frac{1}{1+r}\right)^t S_t$ is a Martingale.

Similarly, for any portfolio $V_t \sim (x_t, y_t)$

$$\tilde{V}_t = \left(\frac{1}{1+r}\right)^t V_t$$

is a Martingale.

$\therefore$ for Replicating portfolio V_t st $V_T = H_T$ where H_T is value of option at maturity:

$$V_0 = \mathbb{E}\left\{V_T \frac{1}{(1+r)^T}\right\} \equiv \text{value @ time 0}$$

in time 0 dollars

$$V_t = \mathbb{E}\left(V_T \frac{1}{(1+r)^{T-t}} \Big| \mathcal{F}_t\right) \equiv \text{value @ time } t$$

in time t dollars.

Call value - European.

Cox Rubinstein Ross formula -

$$C_E(T) = (S_T - K)^+$$

$$S_T = S_0 (1+m_1)^k (1+m_2)^{T-k}$$

— there are k steps up T-k steps down

$\therefore$ # of paths $\binom{T}{k} = \frac{T!}{(T-k)!\,k!}$

$$C_E(0) = \left(\frac{1}{1+r}\right)^T \mathbb{E}(C_E(T))$$

$$= \left(\frac{1}{1+r}\right)^T \sum_{k=0}^{T} \binom{T}{k} (p_1^*)^k (p_2^*)^{T-k} (S_T - K)^+$$

Let $k_0 = \inf \{ k \in \mathbb{R} : S_0 (1+m_1)^k (1+m_2)^{T-k} \geq K \}$.

$$k_0 = \frac{\log \frac{X}{S_0 (1+m_2)^T}}{\log \left(\frac{1+m_1}{1+m_2}\right)}$$

$$C_E(0) = \sum_{T \geq k > k_0} q_1^k q_2^{T-k} \binom{T}{k} S_0 - \sum_{T \geq k > k} \frac{(p_1^*)^k (p_2^*)^{T-k}}{(1+r)^T} \binom{T}{k} X$$

Large T, Gaussian Approx:

$$C_E(0) = S_0 \underbrace{\mathbb{P}_q(k \geq k_0)}_{\frac{k - q_1 T}{\sqrt{T q_1 q_2}} \geq \frac{k_0 - q_1 T}{\sqrt{T q_1 q_2}}} - X \frac{1}{(1+r)^T} \underbrace{\mathbb{P}^*(k \geq k_0)}_{\frac{k - p_1 T}{\sqrt{T p_1 p_2}} \geq \frac{k_0 - p_1 T}{\sqrt{T p_1 p_2}}}$$

$$= S_0 \,\phi\left(-\frac{k_0 - q_1 T}{\sqrt{T q_1 q_2}}\right) - X \frac{1}{(1+r)^T} \phi\left(-\frac{k_0 - p_1 T}{\sqrt{T p_1 p_2}}\right)$$

ARBITRAGE.

Let $S_i(t)$ be a system of securities $t=0,1.,\ i=1,\dots,n$

Let outcomes be $\Omega = \{\omega_1,\dots,\omega_n\}$

Arbitrage: there exists $x_1,\dots,x_n$

So that $\sum_i x_i S_i^j(1) \geq \sum x_i S_i(0)$ for all outcomes ω_j

and for some outcome $\omega_{\hat{j}}$

$$\sum_i x_i S_i^{\hat{j}}(1) > \sum x_i S_i(0).$$

If Interest is added to model $A(t+1) = (1+r)\,A(t)$

define
$$K_{ij} = \frac{S_i^j(1) - (1+r)S_i(0)}{(1+r)S_i(0)}$$

Then, arbitrage is vector $(x_1,\dots,x_n)$ so that

$\cancel{(x_1,\dots,x_n)K \geq 0}$ ~~for all~~

for all j $\quad \sum_i K_{ij} x_i \geq 0$

and for some $\hat{j}$ $\sum_i K_{i\hat{j}} x_i > 0$.

First fundamental theorem of Arbitrage:
Exactly 1 of the two following possibilities occur -

(i) There is an Arbitrage opportunity,

(ii) There is a positive probability vector $p > 0$ that

$$Kp = 0$$

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ie to find if given system allows arbitrage,
find solutions $Kp = 0$ ~ if ~~any solution~~ all solution => require $p_i \leq 0$ for some i, then Arbitrage occurs.

Risk neutral measure ~

A risk neutral measure is a measure $\mathbb{P}^*$ on Ω

$p_j^* = \mathbb{P}^*(\omega_j)$ so that

$$S_i(0)(1+r) = \mathbb{E}^* S_i(1) = \{e_i K p^*\} S_i(0)$$

ie $\quad e_i K p^* = \mathbb{1}(1+r)$ where $\mathbb{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

A model is <u>complete</u> if for all functions H on Ω, there exists portfolio $((x_i), y)$ so that,

$$\sum_i x_i S_i^j(1) + y(1+r) = H(\omega_j)$$

for all outcomes j.

Second fundamental theorem:

A model is Arbitrage free + <u>complete</u>

if and only if

there exists a <u>unique</u> vector $p_j^* = \mathbb{P}^*(\omega_j)$.

Exercise: organize all models you can think of in terms of 1^{st} + 2^{nd} fundamental theorem.

Can you think of any others?