(1) Let X be a RV w/

$$\mathbb{P}(X=a) = p \quad ; \quad \mathbb{P}(X=b) = 1-p \qquad 0 \le p \le 1.$$

Let Y be RV, w/

$$\mathbb{P}(Y=0) = p \quad ; \quad \mathbb{P}(Y=1) = 1-p$$

then

$$\cancel{X = a + X(b-a)}$$

$$X = a + Y(b-a)$$

$$\mathbb{E}Y = p$$

$$\Rightarrow \mathbb{E}X = a + p(b-a)$$

$$\operatorname{Var} Y = p - p^2 = p(1-p)$$

$$\Rightarrow \operatorname{Var} X = (b-a)^2 p(1-p).$$

2. $X = U([a,b])$; X uniformly distributed on a to b.

$Y = U([0,1])$; Y uniformly dist on 0 to 1

$X = a + Y(b-a)$

PDF:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & ; \quad a \le x \le b \\ 0 & ; \quad o/w \end{cases}$$

CDF

$$F_X = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & ; a \le x \le b \\ 1 & ; \quad b < y. \end{cases}$$

$$\mathbb{E}(X) = a + \tfrac{1}{2}(b-a) = \tfrac{1}{2}(b+a).$$

$$\mathrm{Var}(Y) = \frac{1}{12}$$

$$\mathrm{Var}(X) = \tfrac{1}{12}(b-a)^2.$$

3. We must have

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1.$$

So we have

$$1 = \int_0^b f_X(x)\, dx = \int_0^b a x\, dx = \frac{a}{2} x^2 \Big|_0^b = \frac{a b^2}{2}$$

$$a = \frac{2}{b^2}$$

ie

$$f_X(x) = \begin{cases} \frac{2}{b^2} x & 0 \le x \le b \\ 0 & o/w. \end{cases}$$

$$\mathbb{E}(X) = \int_0^b \frac{2}{b^2} x^2\, dx$$

$$= \frac{2}{3b^2} x^3 \Big|_0^b = \frac{2}{3} b$$

$$\mathbb{E}(X^2) = \int_0^b \frac{2}{b^2} x^3\, dx = \frac{1}{2b^2} x^4 \Big|_0^b$$

$$= \frac{b^2}{2}.$$

$$\operatorname{Var} X = \frac{b^2}{2} - \frac{4}{9} b^2 = \frac{1}{18} b^2.$$

4. Define:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad ; \quad \text{Jacobian} \quad J_{uv} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Let $A = \{ (x,y) : \ 0 \le x ; \ 0 \le y ; \ x + 2y \le 2b \}$.

$$x = 0 \longrightarrow v = \frac{1}{2}u$$

$$y = 0 \longrightarrow v = -2u$$

$$f_{X_1 X_2}(x,y) = a(x+2y) = \tilde{f}(u,v) = au.$$

$$1 = \iint_A f_{X_1 X_2}(x,y)\, dx\, dy = \iint_A \tilde{f}(u,v)\, |\det(J_{uv})|\, du\, dv$$

$$= \int_0^{2b} \int_{-2u}^{\frac{1}{2}u} au \frac{1}{5}\, dv\, du$$

$$= \int_0^{2b} au \left(\frac{1}{2}u - (-2u)\right) \frac{du}{5}$$

$$= \int_0^{2b} \frac{a}{2} u^2\, du = \frac{a}{6} u^3 \Big|_0^{2b} = \frac{4}{3} a b^3$$

$$\Rightarrow \quad a = \frac{3}{4 b^3}$$

2)

$$f_{X_1 X_2}(x,y) = \begin{cases} \frac{3}{4b^3}(x+2y) & ; \ (x,y) \in A \\ 0 & ; \ x,y \notin A \end{cases}$$

$$f_{X_1}(x) = \int_{-\infty}^{\infty} f_{X_1 X_2}(x,y)\,dy = \int_0^{b-x/2} \frac{3}{4b^3}(x+2y)\,dy$$

$$= \frac{3}{4b^3}(b-x/2)(x+b-x/2) = \frac{3}{4b^3}\left(b^2-\frac{x^2}{4}\right)$$

$$f_{X_2}(y) = \int_0^{2(b-y)} \frac{3}{4b^3}(x+2y)\,dx$$

$$= \frac{3}{4b^3}\left((b-y)^2 + 4y(b-y)\right)$$

$$= \frac{3}{4b^3}\left(b^2+2by-3y^2\right)$$

b. expectation + variance — let us take $b=1$.
for simplicity.

$$f_{X_1}(x) = \begin{cases} \frac{3}{4}\left(1 - \frac{x^2}{4}\right) & : 0 < x < 2 \\ 0 & o/w \end{cases}$$

$$f_{X_2}(y) = \begin{cases} \frac{3}{4}\left(1 + 2y - 3y^2\right) & : 0 < y < 1 \\ 0 & o/w \end{cases}$$

$$\mathbb{E}(X_1) = \int_0^2 \frac{3}{4}\left(x - \frac{x^3}{4}\right) dx = \frac{3}{4}\left(\frac{x^2}{2} - \frac{x^4}{4^2}\right)\Bigg|_0^2$$

$$= \frac{3}{4}\left(\frac{4}{2} - \frac{2^4}{4^2}\right) = \frac{3}{4}.$$

$$\mathbb{E}(X_2) = \int_0^1 \frac{3}{4}\left(y + 2y^2 - 3y^3\right) dy$$

$$= \frac{3}{4}\left(\frac{y^2}{2} + \frac{2y^3}{3} - \frac{3y^4}{4}\right)\Bigg|_0^1 = \frac{3}{4}\left(\frac{1}{2} + \frac{2}{3} - \frac{3}{4}\right) = \frac{15}{48} ✓$$

$$= \frac{5}{16}$$

b <u>cont.</u>

$$\mathbb{E}X_1^2 = \int_0^2 \frac{3}{4}\left(x^2 - \frac{x^4}{4}\right)dx = \frac{4}{5}$$

$$\operatorname{Var} X_1 = \frac{4}{5} - \left(\frac{3}{4}\right)^2 = \frac{19}{80} \; ; \; \sigma_{X_1} = .487$$

$$\mathbb{E}X_2^2 = \int_0^1 \frac{3}{4}\left(y^2 + 2y^3 - 3y^4\right)dy$$

$$= \frac{3}{4}\left(\frac{y^3}{3} + \frac{y^4}{4} - \frac{3y^5}{5}\right\Big|_0^1 = \frac{11}{60}$$

$$\operatorname{Var} X_2 = \frac{11}{60} - \frac{25}{256} \sim .0857$$

$$\sigma_{X_2} = .293$$

c.

$$f_{X_1 X_2}(x,y) = \frac{3}{4}(x+2y).$$

$$\mathrm{cov}(X_1 X_2) = \mathbb{E}\left\{\left(X_1 - \frac{3}{4}\right)\left(X_2 - \frac{5}{16}\right)\right\}$$

$$= \int_0^b \int_0^{2(b-y)} \left(x - \frac{3}{4}\right)\left(y - \frac{5}{16}\right)\frac{3}{4}(x+2y)\,dx\,dy$$

$$= \int_0^b \left(y - \frac{5}{16}\right)\frac{3}{4} \int_0^{2(b-y)} \left(x^2 + \left(2y - \frac{3}{4}\right)x - \frac{3}{2}y\right) dx \quad dy$$

$$= \int_0^b \left(y - \frac{5}{16}\right)\frac{3}{4}\left\{\frac{(2(b-y))^3}{3} + \frac{\left(2y - \frac{3}{4}\right)}{2}\,2(b-y) - 3y(b-y)\right\} dy$$

... etc ...

d. $Z = X_1 - X_2$

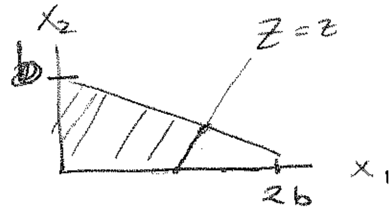
to find marginal - hold Z constant & integrate f.

ie

$X_1 - X_2 = z \quad \longleftrightarrow \quad x - y = z \longrightarrow x = z + y$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X_1, X_2}(z+y, y)\, dy$$

Let us refer to picture of set A



$$f_Z(z) = \begin{cases} 0 & z < -2b \\ \int_0^{\frac{2b-z}{3}} f_{X_1,X_2}(z+y, y)\, dy & -2b < z \le 0 \\ \int_z^{\frac{2b-z}{3}} f_{X_1,X_2}(z+y, y)\, dy & 0 < z \le b \\ 0 & z > b \end{cases}$$

e

$$f_{X_1}(x \mid Z = b/2) = ?$$

$X_1 = x$

$+$

$Z = b/2 \Rightarrow X_1 - X_2 = b/2 \Rightarrow X_1 - b/2 = X_2$

$b = 1 \Rightarrow x - \frac{1}{2} = X_2 = y$

$X_1 = x$

$$f_{X_1}(x \mid Z = 1/2) = \frac{f_{X_1 X_2}(x, x - \frac{1}{2})}{f_Z(1/2)}.$$

f. $E(X_1|X_2) = \int_{-\infty}^{\infty} x \frac{f_{X_1,X_2}(x, X_2)}{f_{X_2}(X_2)} dx$

$$= \int_0^{2X_2} \frac{x(x+2X_2)}{1+2X_2-3X_2} dx$$

$$= \frac{12}{3} \cdot \frac{X_2^3}{1+2X_2-3X_2}$$

$$= 4 \frac{X_2^3}{1+2X_2-3X_2}$$

$$E(Z|X_1) = E(X_1 - X_2|X_1)$$

$$= E(X_1|X_1) - E(X_2|X_1)$$

$$= X_1 - E(X_2|X_1)$$

where $E(X_2|X_1) = \int_0^{\frac{1}{2}X_1} \frac{y(X_1+2y)}{\left(1-\frac{X_1}{4}\right)} dy \quad \ldots$