Homework 7.

(1)

$$m = \begin{pmatrix} .05 \\ .02 \\ -.01 \end{pmatrix}, \quad r = .01$$

risk neutral measures solve

$$\begin{pmatrix} 1 & 1 & 1 \\ m_1 - r & m_2 - r & m_3 - r \end{pmatrix} P^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ .04 & .01 & -.02 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ .06 & .03 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ .02 \end{pmatrix}$$

$$(.06)a + (.03)b = .02.$$

$$2a + b = 2/3 \longleftrightarrow 0 < a < 1/3$$

$$c = 1 - a - b$$
$$= \frac{1}{3} + a$$

$$P^* = \begin{pmatrix} a \\ \frac{2}{3}(1-3a) \\ \frac{1}{3} + a \end{pmatrix}, \quad 0 < a < \frac{1}{3}$$

2. Replicating Portfolio,

$$V(1) = x\begin{pmatrix} .05 \\ .02 \\ -.01 \end{pmatrix} + x\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1+r \\ 1+r \\ 1+r \end{pmatrix}$$

ie, rowspace of

$$\begin{pmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 6 & 3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

$\therefore$ find $\perp$ vector

$$\det\begin{pmatrix} i & j & k \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} = -i + 2j - k = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

$\therefore$ ~~Replication~~

Options which may be replicated/priced, are those $H = \left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right\}$ which solves

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = 0$$

Ito derivatives

3. $f(t,x)$, $\mathring{f} = n x^m t^{n-1}$, $f' = m x^{m-1} t^n$, $f'' = m(m-1) x^{m-2} t^n$.

$$Z_t = f(t, W_t)$$

$$dZ_t = \left\{ \mathring{f}(t, W_t) + \tfrac{1}{2} f''(t, W_t) \right\} dt + f'(t, W_t)\, dW_t .$$

$$= \left\{ n W_t^{m} t^{n-1} + \frac{m(m-1)}{2} W_t^{m-2} t^n \right\} dt$$
$$+ m W_t^{m-1} t^n\, dW_t$$

4. $f(t,x) = e^{ptx^2}$, $\mathring{f} = px^2 e^{ptx^2}$, $f' = 2ptx\, e^{ptx^2}$

$$f'' = \left\{ 2pt + 4p^2 t^2 x^2 \right\} e^{ptx^2}$$

$$dZ_t = \left(\left\{ p W_t^2 e^{ptW_t^2} \right\} + \left\{ 2pt + 4p^2 t^2 W_t^2 \right\} e^{ptW_t^2} \right) dt$$
$$+ 2pt W_t\, e^{ptW_t^2}\, dW_t .$$

5. $dX_t = (pX_t - rt)dt + \sigma dW_t$

Integrating factor $g = e^{-pt}$

$$d(g(t)\cancel{X}_t) = -rt\,g(t)\,dt + \sigma\,g(t)\,dW_t$$

$$= -rt e^{-pt}\,dt + \sigma e^{-pt}\,dW_t$$

$$g(t)X_t - g(0)X_0 = -\frac{r}{p^2} + \int_t^{\infty} r\cancel{s} e^{-ps}\,ds + \sigma\int_0^{\infty} e^{-ps}\,dW_s$$

$$X_t = e^{pt}\left\{X_0 - \frac{r}{p^2}\right\} + \int_t^{\infty} e^{p(t-s)} rs\,ds + \sigma\int_0^{\infty} e^{p(t-s)}\,dW_s.$$

6. $dX_t = (pX_t - rt^2)dt + t\sigma dW_t.$

Integrating factor $g = e^{-pt}$

$$d\{g(t)X_t\} = -rt^2 g(t)\,dt + t\sigma g(t)\,dW_t$$

$$g(t)X_t - X_0 = -\int_0^t r\cancel{s}^2 e^{-p\cancel{s}}\,d\cancel{s} + \int_0^t s\sigma e^{-ps}\,dW_s.$$

$$X_t = X_0 e^{pt} - e^{pt}\int_0^t rs^2 e^{-ps} + \int_0^t s\sigma e^{p(t-s)}\,dW_s.$$