CapM, 2 securities.

$\Omega \equiv$ sample spaces.

$S_1(0)$, $S_2(0)$ given.

$S_1(1)$, $S_2(1)$: $\Omega \to (0, \infty)$

If $\Omega = \{1, \dots, m\}$ ← m possible outcomes,

write: $S_i^j(1) \equiv$ value of i^{th} stock under outcome j.

Return variables for each stock, are defined as:

$$K_i = \frac{S_i(1) - S_i(0)}{S_i(0)} \quad ; \quad K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$$

We need only concern ourselves with the parameters

$$\mu_i = \mathbb{E} K_i \qquad i = 1, 2.$$

$$\sigma_i^2 = C_i = \operatorname{var} K_i$$

$$C_{12} = \operatorname{cov}(K_1, K_2)$$

$$\mu_K = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \quad ; \quad \Sigma_K = \begin{pmatrix} C_1 & C_{21} \\ C_{21} & C_2 \end{pmatrix}$$

PARAMETERIZE the portfolios by the set

$$W = \left\{ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbb{R}^2 : \ w_1 + w_2 = 1 \right\}.$$

* let us use notation $\mathbb{1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ so $w^T \mathbb{1} = w_1 + w_2 = 1$.

For given portfolio $w \in W$

$$K_w = (w_1 \ \ w_2) \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = w_1 K_1 + w_2 K_2 = w^T K.$$

The expected return of the portfolio is then:

$$\mu_w = E(K_w) = \mathbb{E}(w_1 K_1 + w_2 K_2) =$$

$$= w_1 \mu_1 + w_2 \mu_2 = w^T \mu_K.$$

The variance (risk)^2 is

$$\sigma_w^2 = \mathrm{var}(K_w) = \mathrm{var}(w_1 K_1 + w_2 K_2)$$

$$= w_1^2 \sigma_1^2 + 2 w_1 w_2 c_{12} + w_2^2 c_2$$

$$= w^T \Sigma_K w$$

It is helpful to write the portfolio weight as $w_1 = s$
$w_2 = 1-s$.

$$\mu_w = \mu_s = s\mu_1 + (1-s)\mu_2 = \mu_2 + s(\mu_1 - \mu_2).$$

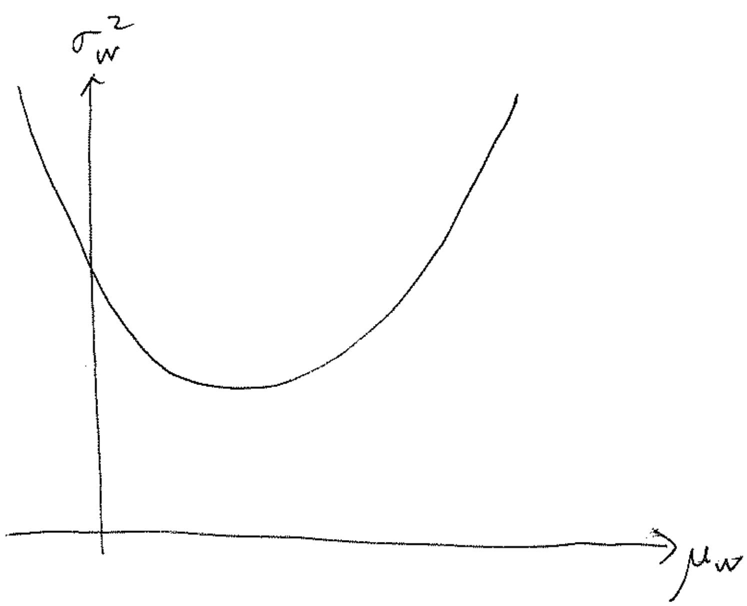
$$\sigma_w^2 = \sigma_s^2 = s^2\sigma_1^2 + (1-s)^2\sigma_2^2 + 2s(1-s)\sigma_{12}.$$

Notice μ_w is linear in s $\rightarrow$ s is linear in μ_w.
$\updownarrow$
$s = \dfrac{\mu_w - \mu_2}{\mu_1 - \mu_2}$.

σ_w^2 is quadratic in s

$\therefore$ σ_w^2 is quadratic in μ_w –

$$\sigma^2(s) = \sigma^2\left(\frac{\mu_w - \mu_2}{\mu_1 - \mu_2}\right).$$



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MINIMIZE $\sigma_W^2 \ldots$

$$\frac{d\sigma_W^2}{ds} = 2s\sigma_1^2 + \boxed{2(1-s) - 2s} c_{12} - 2(1-s)\sigma_2^2 = 0$$

(1) s minimizing σ_W^2 is (provided $\left\{ \begin{array}{l} \sigma_1 \neq \sigma_2 \\ -\sigma_1^2 \leq c_{12} < \sigma_1^2 \text{ if } \sigma_1 = \sigma_2 \end{array} \right.$)

$$s = \frac{\sigma_2^2 - c_{12}}{\sigma_1^2 - 2c_{12} + \sigma_2^2}$$

$\therefore$ portfolio minizing σ_W^2 is

$$(\dagger) \qquad W = \begin{pmatrix} s \\ 1-s \end{pmatrix} = \begin{pmatrix} \dfrac{\sigma_2^2 - c_{12}}{\sigma_1^2 - 2c_{12} + \sigma_2^2} \\[2ex] \dfrac{\sigma_1^2 - c_{12}}{\sigma_1^2 - 2c_{12} + \sigma_2^2} \end{pmatrix} = \frac{\Sigma_K^{-1} \mathbb{1}}{\mathbb{1}^T \Sigma_K^{-1} \mathbb{1}}.$$

(2) if $\sigma_1 = \sigma_2$ & $c_{12} = \sigma_1^2 = \sigma_2^2$.

$$\begin{aligned} \sigma_W^2 &= s^2\sigma_1^2 + (1-s)^2\sigma_2^2 + 2s(1-s)c_{12} \\ &= \left(s^2 + (1-s)^2 + 2s(1-s)\right)\sigma_1^2 \\ &= \left(s + (1-s)\right)^2 \sigma_1^2 = \sigma_1^2 \end{aligned}$$

$\therefore$ σ_W^2 is constant.

Notice the formula ($\dagger$) may imply $s < 0$ or $1-s < 0$
this would require short selling to find
minimal risk.

Find σ_w as a function of μ_w

$$\delta = \frac{\mu_w - \mu_2}{\mu_1 - \mu_2}.$$

$$\sigma_w^2 = \left(\frac{\mu_w - \mu_2}{\mu_1 - \mu_2}\right)^2 \sigma_1^2 + \left(\frac{\mu_1 - \mu_w}{\mu_1 - \mu_2}\right)^2 \sigma_2^2 - 2c_{12} \frac{(\mu_w - \mu_2)(\mu_w - \mu_1)}{(\mu_1 - \mu_2)^2}$$

write: $\mu_m = \mu_w(w_m)$; $\sigma_m^2 = \sigma_w^2(w_m)$

$$\sigma_w^2 - A^2(\mu_w - \mu_m)^2 = \sigma_m^2$$

$$\sigma_w = \left(A^2(\mu_w - \mu_m)^2 + \sigma_m^2\right)^{1/2}.$$

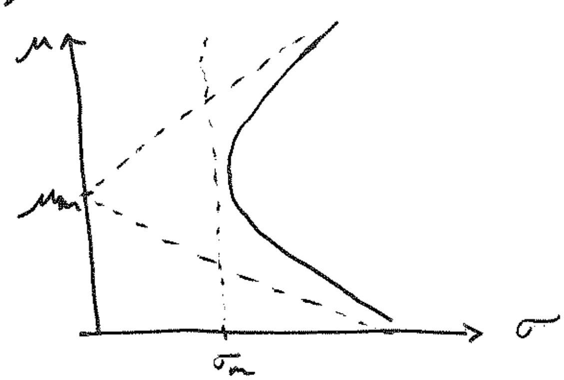
Asymptotes:

as $\mu_w \to \pm\infty$, $\sigma_w \sim A(\mu_w - \mu_m)$

hyperbola is centered @ $(\mu_m, 0)$.

Asym. lines

$$\mu = \mu_m \pm A\sigma \quad ; \quad A^2 = \frac{\sigma_1^2 + \sigma_2^2 - 2c_{12}}{(\mu_1 - \mu_2)^2} > 0$$



Let us consider simplest case first,

that $|C_{12}| = \sigma_1 \sigma_2 \iff |\rho_{12}| = \left|\frac{C_{12}}{\sigma_1 \sigma_2}\right| = 1$.

In this case,

$$K_1 = t K_2 + p .$$

So $\sigma_1^2 = t^2 \sigma_2^2$.

Assume $t \neq 1$ (in this case σ_W^2 is constant).

Find w for minimum variance.

$$W = \begin{pmatrix} S_0 \\ 1 - S_0 \end{pmatrix} = \cdots$$

$$S_0 = \frac{\sigma_2^2 - t\sigma_2^2}{t^2\sigma_2^2 + \sigma_2^2 - 2t\sigma_2^2} = \frac{1-t}{(1-t)^2} = \frac{1}{1-t} .$$

$$W = \begin{pmatrix} \frac{1}{1-t} \\ \frac{-t}{1-t} \end{pmatrix}$$

7b

∴ (i) $\rho_{12} = 1$

$t > 0 \Rightarrow$ $\quad S_0 = \frac{1}{1-t} < 0$ if $t > 1$

$1 - S_0 = \frac{-1}{1-t} < 0$ if $0 < t < 1$.

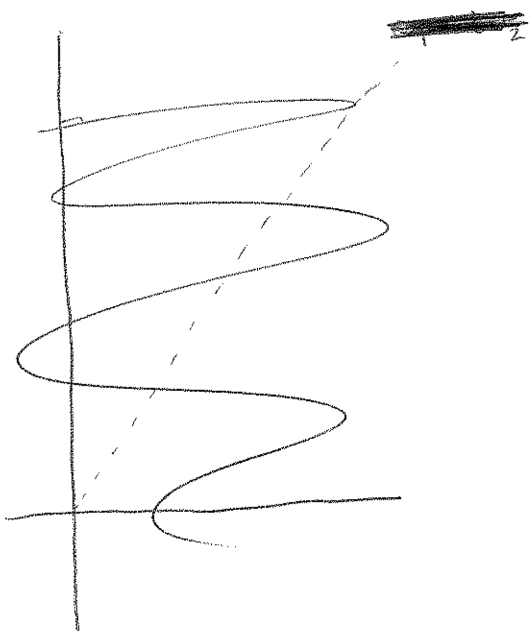
∴ finding minvariance requires short selling.

(ii) $\rho_{12} = -1$

$t < 0$

$$\left.\begin{aligned} S_0 &= \frac{1}{1-t} \\ 1 - S_0 &= \frac{-t}{1-t} \end{aligned}\right\} \text{both belong to } (0,1)$$

∴ minimal variance doesnot require short selling.



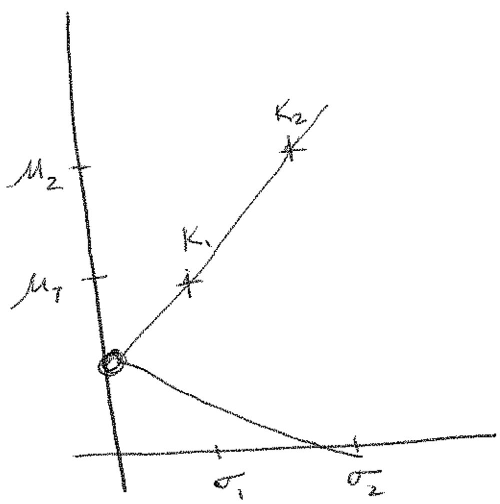
Notice

$$K_1 = t K_2 + p$$

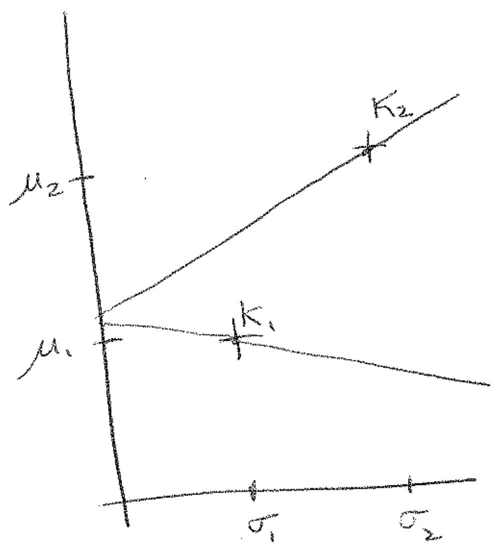
$$\Rightarrow \mu_1 = t \mu_2 + p .$$

$$\sigma_1 = |t| \sigma_2 .$$

$t > 0$



$t < 0$



min variance, $\sigma_W = 0$.

$\rho_{12} \sim 1$ mvp req shortsell

$\rho_{12} \approx -1$ mvp does not req shortsell

where is bdry?

$0 < \delta_0 < 1$ ← mvp w/o short sell.

$$\delta_0 = \frac{\sigma_2^2 - c_{12}}{\sigma_1^2 + \sigma_2^2 - 2c_{12}}$$

$\delta_0 < 1$

$\Rightarrow$

$\sigma_2^2 - c_{12} < \sigma_1^2 + \sigma_2^2 - 2c_{12}$

$c_{12} < \sigma_1^2$

$\rho_{12} < \frac{\sigma_1}{\sigma_2}$

$\delta_0 > 0$

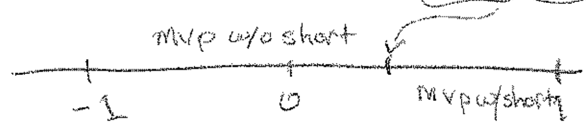
$\sigma_2^2 - c_{12} > 0$

$c_{12} < \sigma_2^2$

$\rho_{12} < \frac{\sigma_2}{\sigma_1}$

$\therefore$ mvp w/ short sell is possible iff

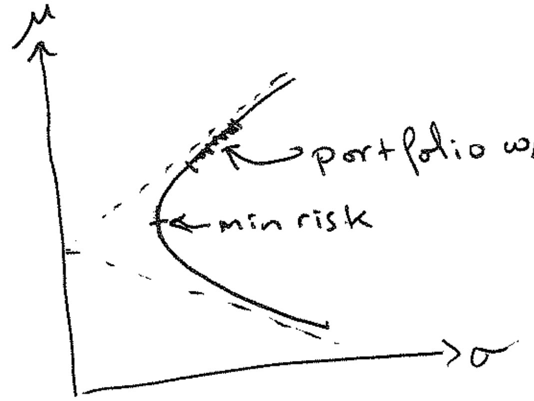
$$\rho_{12} < \underbrace{\frac{\sigma_2}{\sigma_1} \wedge \frac{\sigma_1}{\sigma_2}}_{} = \rho^c$$



$\therefore$

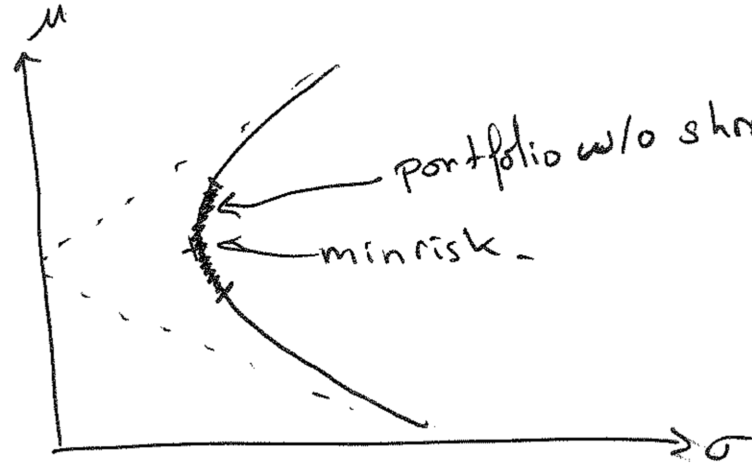
$$1 > \rho > \frac{\sigma_2}{\sigma_1} \wedge \frac{\sigma_1}{\sigma_2} = \min\left\{ \frac{\sigma_2}{\sigma_1}, \frac{\sigma_1}{\sigma_2} \right\}.$$

min variance portfolio requires short.



$$-1 < \rho < \frac{\sigma_2}{\sigma_1} \wedge \frac{\sigma_1}{\sigma_2}$$

min variance portfolio w/o short



Suppose

$$\sigma_1^2 = 1/2 \, . \quad \sigma_2^2 = 1/4 \, . \quad c_{12} = 1/8 \, .$$

$$\rho_{12} = \frac{1/8}{1/\sqrt{2}\;1/2} = \frac{\sqrt{2}}{4}$$

$$\rho^{c} = \frac{1/4}{1/2} = 1/2$$

$$\therefore \; \rho_{12} < \rho^{c} \;\Rightarrow\; \text{does not req short sell}$$

$$S_0 = \frac{1/2 - 1/8}{1/2 + 1/4 - 2\,\frac{1}{8}} = \frac{3/8}{1/2} = 3/4 \, . \in (0,1).$$

$$\sigma_1^2 = \frac{1}{3} \qquad \sigma_2^2 = \frac{1}{9} \qquad c_{12} = \frac{1}{6}$$

$$\rho_{12} = \frac{1/6}{1/3 \;\; 1/\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\rho^{c} = \frac{1/9}{1/3} = 1/3$$

$$\rho_{12} > \rho^{c} \;\Rightarrow\; \text{require short sell.}$$

indeed

$$S_0 = \frac{1/9 - 1/6}{1/9 + 1/3 - 2\,1/6} = \frac{2-3}{2+6-6} = \frac{-1}{2} < 0$$