Definition - Standard Brownian Motion -

X_t is a <u>stochastic process</u> $\left\{ \begin{array}{l} \text{family of R.V.} \\ \text{indexed by time, } \mathbb{T} = (0, \infty) \end{array} \right\}$

so that

(i) $X_0 = 0$

(ii) for $s_1 < t_1 \leq s_2 < t_2 \leq \cdots \leq s_n < t_n$

The varibles

$$(X_{t_1} - X_{s_1}) \; ; \cdots ; \; (X_{t_n} - X_{s_n})$$

are independent

(iii) for any $0 \leq s < t$

$X_t - X_s \sim N(0, t-s) \equiv$ normal w/mean 0 & variance $t-s$.

(iv) Realizations of the path are continuous:

$t \mapsto X_t^{\omega}$ is a continuous function, $\forall \; \omega \in \Omega$

$\Omega \equiv$ sample space

BROWNIAN MOTION IS A LIMIT OF RANDOM WALKS.

(Donsker's theorem)

Let $Y_1, Y_2, Y_3, \ldots$ be a seq. of i.i.d. r.v.
w/ mean 0 + variance 1.

$$S_n := \sum_{i=1}^{n} Y_i$$ is a Random Walk

$$W^{(n)}(t) := \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}$$ $n \Leftrightarrow$ step size is $\frac{1}{n}$ in time

By CLT – $$\frac{S_{\lfloor nt \rfloor}}{\sqrt{\lfloor nt \rfloor}} \to N(0,1) \quad \text{as } n \to \infty$$

$$\therefore \quad \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} \sim \sqrt{t}\, Z \quad \text{for } Z \sim N(0,1)$$

$$\therefore \quad W^{(n)}(t) \to X_t$$

As discussed last time: $\tau < t$

$$\operatorname{cov}\left(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}, \frac{S_{\lfloor n\tau \rfloor}}{\sqrt{n}}\right)$$

$$= \frac{1}{n}\,\mathbb{E}\left\{S_{\lfloor n\tau \rfloor}\left(\left(S_{\lfloor nt \rfloor} - S_{\lfloor n\tau \rfloor}\right) + S_{\lfloor n\tau \rfloor}\right)\right\}$$

$$= \frac{1}{n}\,\mathbb{E}\left\{S_{\lfloor n\tau \rfloor}^2\right\}$$

$$= \operatorname{var}\left(\frac{S_{\lfloor n\tau \rfloor}}{\sqrt{n}}\right) \cong \tau$$

$\therefore$ in the limit...

$$\operatorname{cov}\left(X_t \cdot X_\tau\right) = \min(\tau, t).$$

Can we differentiate / how to integrate functions of Brownian motion?

~~\#~~ The derivative of BM exists if

$$\lim_{s \to t} \frac{W_s - W_t}{s - t} \text{ exists...}$$

...But it does not

The paths are continuous but rough

* assumption (iv) tells us $\lim_{s \to t} W_s = W_t$.

* However $\mathrm{var}(W_t) = t \Rightarrow$ std deviation @ time t is $\sqrt{t}$

so 'Bulk' is only contained in $[\sqrt{t}, \sqrt{t}]$

- this is not smooth enough for derivatives

* $\sqrt{t}$ fluctuations ...

$\alpha \geq 1/2$ $\alpha < 1/2$

for any t

$$\lim_{\varepsilon \to 0} \sup_{|t-s|<\varepsilon} \frac{|W_t - W_s|}{|t-s|^\alpha} = \infty \; ; \; \lim_{\varepsilon \to 0} \sup_{|t-s|<\varepsilon} \left| \frac{W_t - W_s}{|t-s|^\alpha} \right| = 0$$