Continuous time stock model.

Let us consider constructing a model for stock prices on the set $\mathbb{T} = [0,T] \subset \mathbb{R}$.

* Do this by taking the limit of an N step binomial model.

Let $S_0 \equiv$ value of security @ $t=0$

we will define variables S_t at $t = 0, \frac{T}{N}, \dots, n\frac{T}{N}, \dots, T$.

Let m_u + m_d be multiplicative factors shifting the value of the security up or down.

$$\Omega_n = \{ \omega_1 \cdots \omega_n : \ \omega_i = u \text{ or } d \}$$

$$m_u = m_u^N ; \ m_d = m_d^N$$

then

$$S_t = S_0 (1 + m_{\omega_1}) \cdots (1 + m_{\omega_n})$$

We study the Logarithmic Return of S_t ... $\left(t = n\frac{T}{N}\right)$

$$\log \frac{S_t}{S_0} = \log(1+m_{\omega_1}) + \cdots + \log(1+m_{\omega_n})$$

For N steps, take the values of m to be,

$$m_u = r\frac{T}{N} + a\sqrt{\frac{T}{N}} \quad ; \quad m_d = r\frac{T}{N} - b\sqrt{\frac{T}{N}} .$$

For r defining the bond rate:

$$A_t = A_0\left(1 + r\frac{T}{N}\right)^n$$

The risk free measure is defined by weights.

$$\tilde{p}_u = \frac{r\frac{T}{N} - m_d}{m_u - m_d} = \frac{b}{a+b}$$

$$\tilde{p}_d = \frac{m_u - r\frac{T}{N}}{m_u - m_d} = \frac{a}{a+b}$$

Calculate return, variance + cov of log-return.

Recall

$$\log(1+x) = \sum_{k=1}^{\infty} -\frac{(-1)^k}{k} x^k = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$$

$$\therefore \tilde{E} \log(1+m) = \frac{b}{a+b}\log(1+m_u) + \frac{a}{a+b}\log(1+m_d)$$

$$= \frac{b}{a+b}\left\{\left(r\frac{T}{N} + a\sqrt{\frac{T}{N}}\right) - \frac{1}{2}\left(r\frac{T}{N} + a\sqrt{\frac{T}{N}}\right)^2\right\} + \frac{a}{a+b}\left\{\left(r\frac{T}{N} - b\sqrt{\frac{T}{N}}\right) - \frac{1}{2}\left(r\frac{T}{N} - b\sqrt{\frac{T}{N}}\right)^2\right\} + O\left(\sqrt{N}^{-3}\right)$$

$$= r\frac{T}{N} - \frac{1}{2}\frac{T}{N}\frac{ba^2+ab^2}{a+b} + O\left(\sqrt{N}^{-3}\right) =$$

$$= \left(r - \frac{1}{2}ab\right)\frac{T}{N} + O\left(\frac{1}{N^{+3/2}}\right)$$

$i \neq j$

$$\tilde{E}\left\{\log(1+m_i)\log(1+m_j)\right\} = \mathbb{E}\log(1+m_i)\ \mathbb{E}\log(1+m_j)$$

$$= \left(r - \frac{1}{2}ab\right)^2 \frac{T^2}{N^2} + O\left(\frac{1}{N^{5/2}}\right)$$

$i=j$

$$\tilde{\mathbb{E}}\left\{ \log^2(1+m) \right\} =$$

$$= \frac{b}{a+b}\left(r\frac{T}{N} + a\sqrt{\frac{T}{N}}\right)^2 + \frac{a}{a+b}\left(r\frac{T}{N} - b\sqrt{\frac{T}{N}}\right)^2 + O\left(\frac{1}{N^{3/2}}\right)$$

$$= \frac{ba^2 + ab^2}{a+b}\left(\frac{T}{N}\right) + O\left(\frac{1}{N^{3/2}}\right)$$

$$= ab\left(\frac{T}{N}\right) + O\left(\frac{1}{N^{3/2}}\right)$$

$$\tilde{\mathbb{E}} \log \frac{S_t}{S_0} = \sum_{i=1}^{n} \tilde{\mathbb{E}} \log(1+m_i) =$$

$$= \left(r - \frac{1}{2}ab\right) n \frac{T}{N} + O\left(\frac{1}{N^{1/2}}\right) = \left(r - \frac{1}{2}ab\right)t + O\left(\frac{1}{N^{1/2}}\right)$$

$\cancel{\tilde{\mathbb{E}} \log}$

let $\tau = k\frac{T}{N}$; $t = n\frac{T}{N}$; $k \le n$

$$\tilde{\mathbb{E}} \log\frac{S_t}{S_0} \log\frac{S_\tau}{S_0} = \tilde{\mathbb{E}}\left\{\left(\sum_{i=1}^{n} \log(1+m_i)\right)\left(\sum_{j=1}^{k} \log(1+m_j)\right)\right\}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{k} \tilde{\mathbb{E}} \log(1+m_i)\log(1+m_j)$$

$$= \left\{\sum_{j=1}^{k} \tilde{\mathbb{E}} \log^2(1+m_j)\right\} + \left\{\left(\sum_{i=1}^{n} \tilde{\mathbb{E}} \log(1+m_i)\right)\left(\sum_{j=1}^{k} \tilde{\mathbb{E}} \log(1+m_j)\right)\right\}_{i \ne j}$$

$$= k\left\{ab\frac{T}{N} + O\left(\frac{1}{N^{3/2}}\right)\right\} + \left\{\left[\left(r - \frac{1}{2}ab\right)\frac{T}{N} + O\frac{1}{N^{3/2}}\right]\left[\left(r - \frac{1}{2}ab\right)\frac{T}{N} + O\frac{1}{N^{3/2}}\right] nk\right\}$$

$$= ab\tau + \left(r - \frac{1}{2}ab\right)^2 \tau t + O\frac{1}{N^{1/2}}$$

on the other hand - -

$$\tilde{\mathbb{E}} \log\frac{S_t}{S_0}\, \tilde{\mathbb{E}} \log\frac{S_\tau}{S_0} = \left\{\left(r - \frac{1}{2}ab\right)t + O\frac{1}{N^{1/2}}\right\}\left\{\left(r - \frac{1}{2}ab\right)\tau + O\frac{1}{N^{1/2}}\right\}$$

$$= \left(r - \frac{1}{2}ab\right)^2 t\tau + O\frac{1}{N^{1/2}}.$$

$$\therefore \quad \widetilde{\mathrm{cov}}\left(\log\frac{S_t}{S_0}, \log\frac{S_\tau}{S_0}\right) = ab\tau + O\frac{1}{N^{1/2}}$$

$$\& \quad \widetilde{\mathrm{var}}\left(\log\frac{S_\tau}{S_0}\right) = \widetilde{\mathrm{cov}}\left(\log\frac{S_t}{S_0}, \log\frac{S_\tau}{S_0}\right).$$

Thus,

$$\log \frac{S_t}{S_0} = \left(r - \tfrac{1}{2}ab\right)t + \sqrt{ab}\, W_t$$

* $W_t \equiv$ Brownian motion which ~~a~~ are Gaussian R.V. so that W_t has mean 0 + variance t.

* Gaussian Processes are determined by their covariances, covariances of BM is

$$\mathrm{cov}(W_t, W_\tau) = \min(\tau, t).$$

We construct stock prices desired variance -

$\sigma^2 = ab.$

$$\log \frac{S_t}{S_0} = \left(r - \tfrac{1}{2}\sigma^2\right)t + \sigma W_t .$$

A discrete model w/ N steps + variance $\sigma^2 t$ is constructed by steps of

$$m_u = r\frac{T}{N} + \sigma\sqrt{\frac{T}{N}} \; ; \; m_d = r\frac{T}{N} - \sigma\sqrt{\frac{T}{N}}$$