Regularity of call + put

$C_E(0)$ + $P_E(0)$ depend on $S(0)$, X, $r \equiv$ interest,

$T \equiv$ expiry,

$$K = \frac{S(T) - S(0)}{S(0)}$$

As discussed in previous lecture Call + Put price does not depend on $\mathbb{E}(K)$

However it does depend on higher moments
- In most models it may be understood as dependence on σ.

Let us discuss regularity on dependence on these parameters.

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(1) $C_E(0)$ + $P_E(0)$ are convex in strike price X.

(2) If $\{S_x\}$ are a family of securities w/ $S_x(0) = x$

~~(2) If S_1 + S_2 are securities w/ given $S_1(0)$, $S_2(0)$ so that $S_1(1) = S_1(0)(1+K)$~~

and $S_x(1) = x(1+K)$ for shared return K,

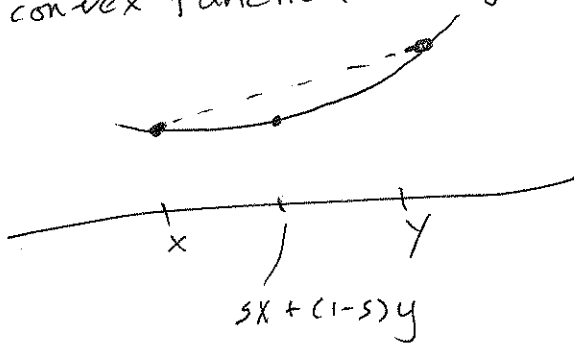
then $C_E(0)$ + $P_E(0)$ are convex in x.

What is Convexity?

* A function f is convex on $I \subset \mathbb{R}$ if for all $x, y \in I$ and s so that $0 < s < 1$, we have:

$$f(sx + (1-s)y) \le s f(x) + (1-s) f(y).$$

* IE: A convex function always lies below its avg.



* If f is convex & second differentiable then $f'' \ge 0$.

* If f is convex in I, then f has both left & right 1 sided derivatives at every point interior to I.

left derivative: $f'_-(x) = \lim\limits_{\substack{s \to 0 \\ s > 0}} \dfrac{f(x) - f(x-s)}{s}$

right derivative: $f'_+(x) = \lim\limits_{\substack{s \to 0 \\ s > 0}} \dfrac{f(x+s) - f(x)}{s}$

* The left & right derivative agree at "almost every" point. Eg $f(x) = |x|$ is convex. $f'_-(0) = -1$; $f'_+(0) = 1$

for $x \ne 0$ $f'_-(x) = f'_+(x)$ ∎

Demonstration that C_E is convex wrt X.

Let $X' < X''$ & $X = \alpha X' + (1-\alpha) X''$ for $0 < \alpha < 1$.

Let C_E be price for call w/ strike X

C_E' ——— X'

C_E'' ——— X''

Suppose $C_E > \alpha C_E' + (1-\alpha) C_E''$ ie C_E is overval'd so let us short & find arbitrage.

Let us short C_E + long α contracts of C_E' + $(1-\alpha)$ contracts of C_E''.

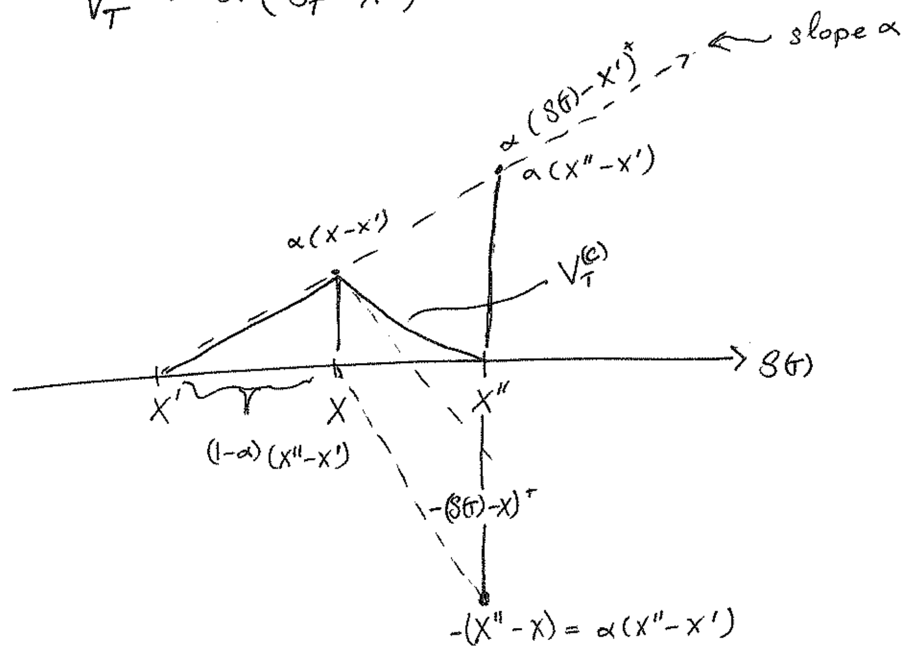
At $t=0$ portfolio: $(\$, C_E, C_E', C_E'') = (C_E - \alpha C_E' - (1-\alpha) C_E'', 1, \alpha, -(1-\alpha))$

At $t=T$ * collect earnings from Bond

* Clear payoffs of calls:

$$V_T^{(c)} = \alpha (S_T - X')^+ + (1-\alpha)(S_T - X'')^+ - (S_T - X)$$

Observe:



$$V_T = V_T^{(c)} + \left(C_E - \alpha C_E' - (1-\alpha) C_E''\right) \frac{1}{B(0,T)}$$

Let us determine "Shape" of C_E wrt X.

It is helpful to use bounds derived last time, Convexity & PC parity.

(i) $S(0) - X\,B(0,T) \le C_E < S(0)$

$\therefore$ as $X \downarrow 0 \quad C_E \to S(0)$.

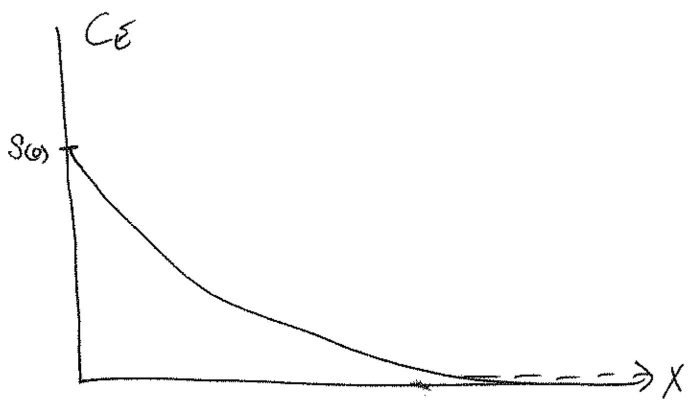
(ii) $P_E \ge X\,B(0,T) - S(0) \quad \therefore$ as $X \to \infty \quad \frac{d}{dX} P_E \gtrsim B(0,T) - \varepsilon$

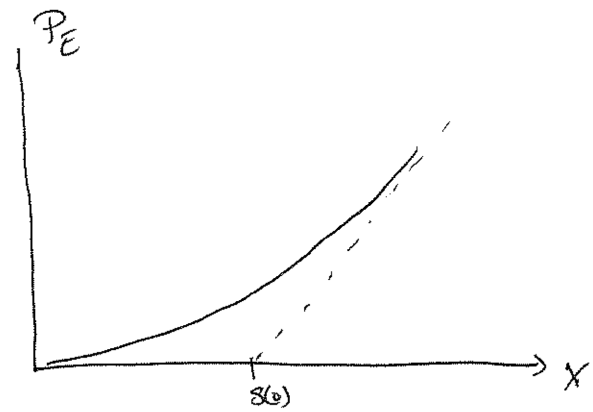
(iii) $\frac{d}{dX}(P_E - C_E) = \frac{d}{dX}\left(X\,B(0,T) - S(0)\right) = B(0,T)$

$\therefore$ ~~as $X \to \infty$ C_E~~ as $X \to \infty \quad \frac{d}{dX} C_E \to 0$.

(iv) $C_E - P_E = S(0) - X\,B(0,T) \Rightarrow P_E \to 0$ as $X \downarrow 0$.

(v) C_E is decreasing wrt X
P_E is increasing wrt X } clear from payoff { $C_E = (\cancel{S} - \cancel{X})^+$, $P_E = (\cancel{X} - \cancel{S_T})$





INTRINSIC VALUE / EXTRINSIC VALUE /
Speculative/Time Value

At time of expiry $\equiv T$ the exercize value is

$$C_E(T) = (S(T) - X)^+$$
$$S_E(T) = (X - S(T))^+$$

Intrinsic value: value determined by exercizing immediately:

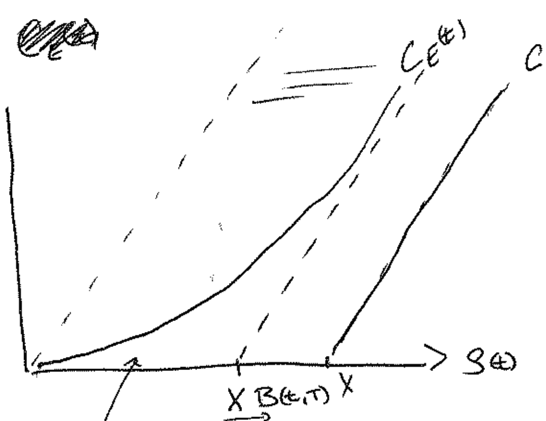
$$C_E^*(t) = (S(t) - X)^+ \; ; \; P_E^* = (X - S(t))^+$$

Total value of option (Premium/trading value) is generally greater than intrinsic value.

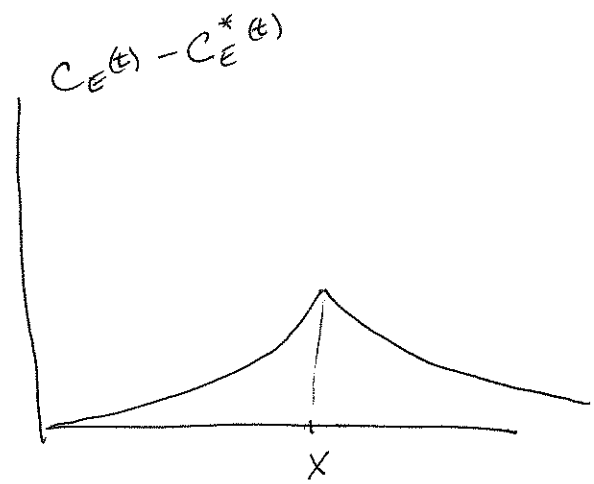
$$C_E(t) > C_E^*(t). \; ; \; P_E(t) > P_E^*(t).$$

Time value (Extrinsic/speculative) value is the difference in price:

$$\text{time value} = C_E(t) - C_E^*(t).$$



value of call in this set.



$\therefore$ TIME VALUE $\to 0$ as $t \to T$.