OPTIONS

Options are contracts which allow purchasers to exercize a trade at a future date.

Unlike Forwards, the holder of the Option is not obligated to exercise the ~~trade~~.

Therefore: the value of the contract/option is never negative.

Options come in various flavors

(1) Call options: The contract allows the purchaser to purchase a stock at ~~[illegible]~~ an arranged price X.

(2) Put options: The contract allows the holder to sell a stock at a prearranged price X.

(3) European Options: Option may only be exercized at a specific expiry date.

(4) American Options: Option may be exercized at any date $t \leq T \equiv$ the expiry date.

Value of European Call.

Euro Call allows holder to purchase security for strike price X at time T.

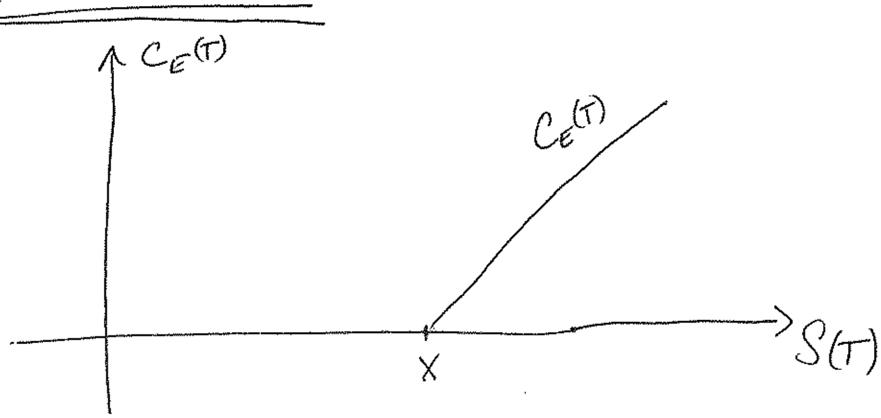
* If $S(T) \le X$, the security may be purchased for less than strike price – so the option will not be exercised. $\therefore$ Value is 0.

* If $X < S(T)$, holder will exercise the option and purchase the security for X. Then the holder will turn around ~~[illegible]~~ and sell it for $S(T)$
  $\therefore$ Value is $S(T) - X$.

Let x^+ be a notation defining: $x^+ = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \le 0. \end{cases}$

The value of the call at expiry T:

$$\underline{\underline{C_E(T) = \left(S(T) - X\right)^+}}$$



Value of European Put

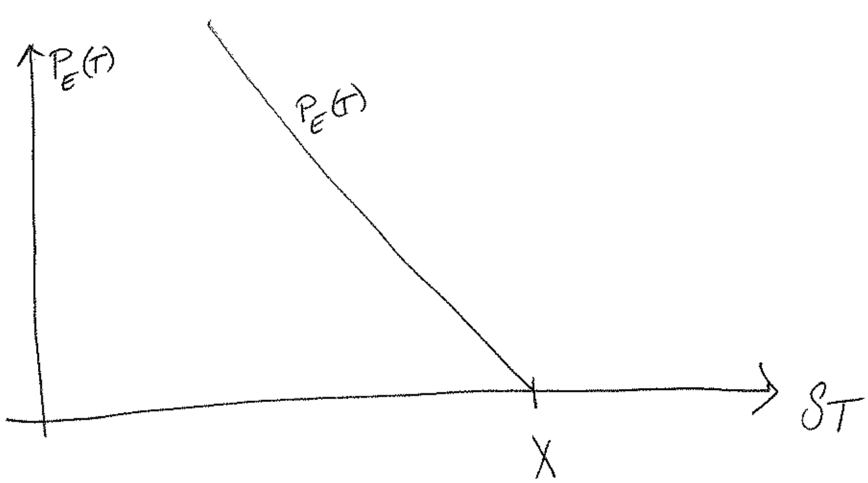
Euro Put allows holder to ~~buy~~ sell security for strike price X at expiry T.

* If $S(T) \geq X$, the security may be sold on the market for more than the strike price - so the option will not be exercised
  $\therefore$ Value is 0.

* If $S(T) < X$, the holder will exercise the contract.
  The holder may buy the security for $S(T)$ and sell it for X.
  $\therefore$ Value is $X - S(T)$

The value of the put at expiry:

$$P_E(T) = \left(X - S(T)\right)^+$$



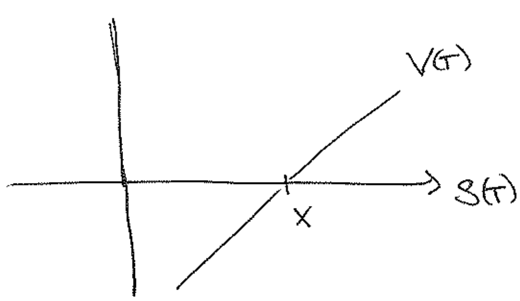
We can not find the values of the options at time $t=0$ directly : $C_E(0)$, $P_E(0)$

But we can find the value of portfolios constructed from these options which we do know how to price.

Take portfolio of 1 call with strike X - Long
1 put with strike X - Put

Value of portfolio at expiry T :

$$V(T) = C_E(T) - P_E(T)$$
$$= (S(T)-X)^+ - (X-S(T))^+$$
$$= S(T) - X$$



$\therefore$ Value of portfolio at time T is the same as value forward w/ strike price X at maturity time T.

* Thus value of portfolio at time 0 is the same as value of forward contract.

$\therefore$ Value of portfolio $C_E - P_E$ @ time $t=0$

is

* In the case of security w/o dividend

$$V_X(0) = S(0) - X\,B(0,T)$$

* In the case of discrete dividends paying δ_i @ τ_i for $i=1,\dots,n$

$$V_X(0) = S(0) - \sum \delta_i B(0,\tau_i) - X\,B(0,T)$$

* In the case of continuous dividends at rate r_δ

$$V_X(0) = S(0)\,e^{-r_\delta T} - X\,B(0,T)$$

$\therefore$ At time $t=0$

$$C_E(0) - P_E(0) = V_X(0)$$

This is called the Put Call parity Eq.

If forward has strike price $X = \frac{1}{B(0,T)} S(0)$ (w/dividend)

then

$$C_E(0) = P_E(0)$$

This highlights apparent paradox:

One would expect $C_E(0)$ ~~the~~ to be higher valued than $P_E(0)$ if the security is expected to increase.

Infact $C_E(0)$ & $P_E(0)$ do not depend on Expected return, instead they depend on variance of stock.

~~This is the same~~

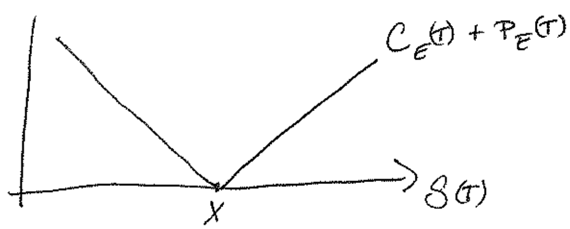
The same priciple holds for all X, indeed

$$C_E(0) = P_E(0) + \underbrace{S(0) - X B(0,T)}_{\text{const}}$$

So $C_E(0)$ increases with $P_E(0)$.

To illustrate, Consider portfolio Long on Call
Long on Put.

The payoff is:



$\therefore$ If variance is small, payoff is small.

If variance is large $\rightarrow$ payoff $|S(T) - X|$ is $\sim$ linear in σ (for large σ)

FUNDAMENTAL BOUNDS ON Options

Let us find an upper bound on the Call price.

* In the case of No Dividend $C_E(0) < S(0)$

* In the case of Discrete Dividend $C_E(0) < S(0) - \sum \delta_i B(0, \tau_i)$

* In the case of Continuos Dividend

$$C_E(0) < S(0)\, e^{-r_\delta T}$$

LET US PROVE The FIRST CASE.

Suppose $C_E(0) \geq S(0)$, We will show an arbitrage profit.

At $t=0$ * Short option - Collect $C_E(0)$

* Buy stock - cost $S(0)$

* Portfolio $(C_E, S, \$) = (-1, 1, C_E(0) - S(0))$

At $t=T$ * $\begin{cases} \text{If } X \geq S(T), \text{ Call is not exercised, sell stock.} \\ \text{If } S(T) > X, \text{ call is exercised, clear obligation. collect } X. \end{cases}$

* Collect bond $(C_E(0) - S(0)) \frac{1}{B(0,T)}$

$\therefore$ value @ time T: $V_T = \min(X, S(T)) + (C_E(0) - S(0)) \frac{1}{B(0,T)}$

$\therefore$ $V_T > 0$ w/ positive probability

& $V_T \leq 0$ w/ zero probability $\Rightarrow$ Arbitrage.

Let us find the lower bound on $C_E(0)$

* Clearly $C_E(0) \geq 0$.

* Another lowerbound follows from the P-C parity equation:

$$C_E(0) - P_E(0) = V_X(0)$$

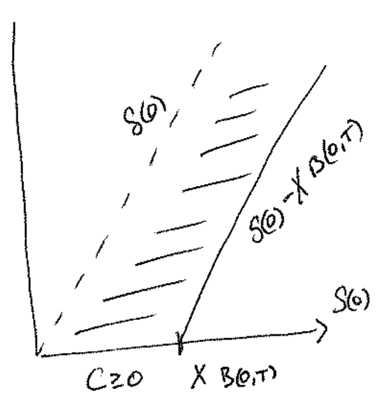
$$\Updownarrow$$

$$C_E(0) = V_X(0) + P_E(0) \geq V_X(0)$$

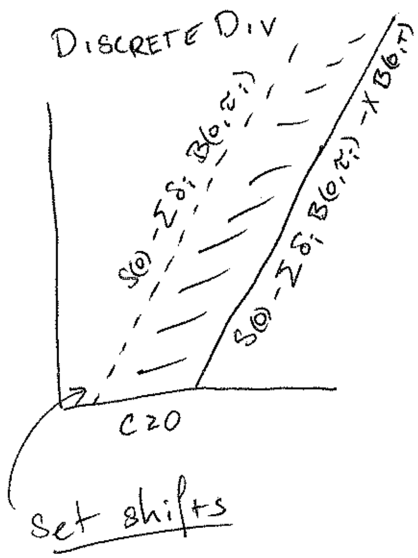
Thus Bounds on $C_E(0)$ are:

$$\max\{0, V_X(0)\} \leq C_E(0) < F(0,T)\ B(0,T)$$

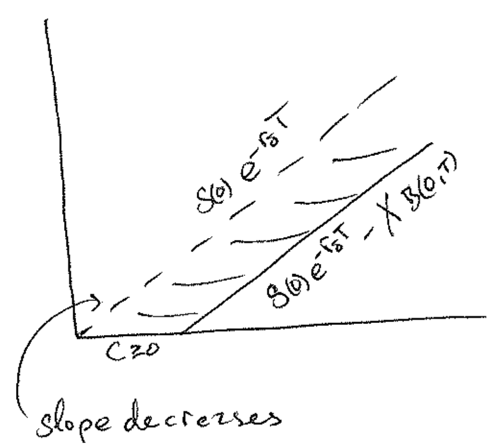
No Dividend:



DISCRETE DIV



CONTINUOUS DIV



BOUNDS FOR EUROPEAN PUT OPTIONS:

UPPER BOUND: $P_E(0) \times B(0,T)$

Proof: SUPPOSE: $P_E(0) \geq X\,B(0,T)$.

Let us find an Arbitrage Profit.

At $t=0$ * Short Put $P_E(0)$ } Portfolio: $(P, \$) = (-1, P_E(0))$

* Invest in bond

At $t=T$ * $\begin{cases} \text{If } X \leq S(T), \text{ put is not exercized - collect bond} \\ \text{If } X > S(T), \text{ Put is exercized - buy stock@ } X \\ \qquad\qquad\qquad\qquad \text{- sell stock for } S(T). \end{cases}$

$$\therefore\ V_T = \begin{cases} P_E(T)\,\frac{1}{B(0,T)} & S(T) \geq X \\ S(T) & S(T) < X \end{cases}$$

$\therefore$ there is zero probability of loss and positive probability of profit $\Rightarrow$ Arbitrage.

Lower bound: $* P_E(0) \geq 0$

* From PC parity:

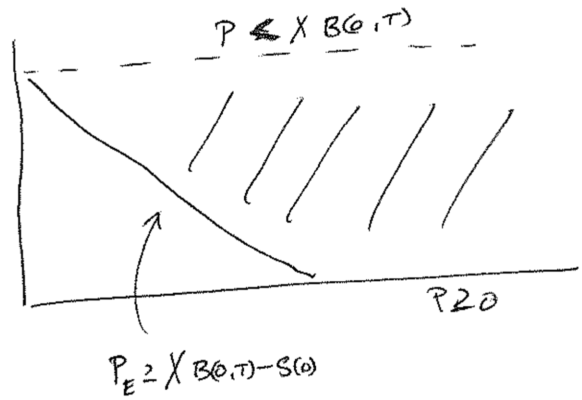
$C_E(0) - P_E(0) = V_X(0) \Rightarrow 0 \leq C_E(0) = V_X(0) + P_E(0)$

$\Rightarrow\ P_E(0) \geq V_X(0)$

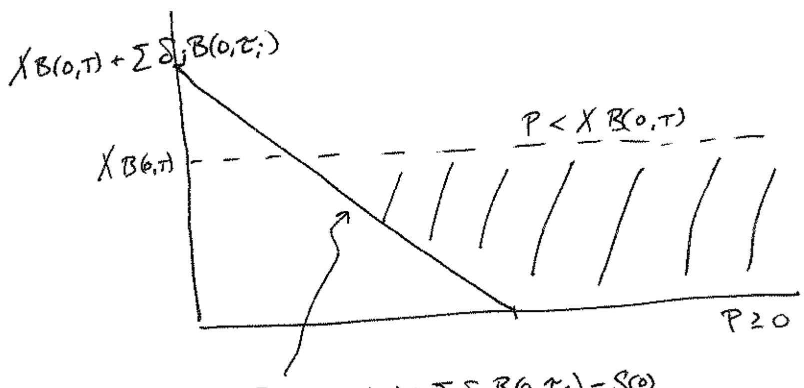
BOUNDS ON EUROPUT:

$$\text{Max}\{0, V_X(0)\} \leq P_E(0) < X B(0,T)$$

No dividend:



Discrete Dividend:



CONTINUOUS DIVIDEND