Market Portfolio + BOND

We now consider combining the portfolio over risky securities w/ the risk free security.

Ie: S_i $i = 1, \ldots, n$ are risky securities.

$S_i(0)$ given.

$S_i(1)$ a Random Variable.

$$K_i = \frac{S_i(1) - S_i(0)}{S_i(0)} \quad \text{Random variables.}$$

$x_i \equiv$ amount of ith security purchased at time 0.

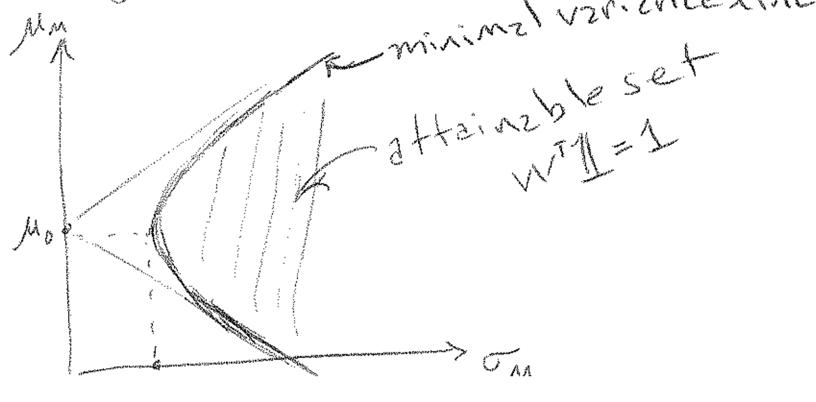
$$V_M(0) = x_1 S_1(0) + \cdots + x_n S_n(0)$$

$$w_i = \frac{x_i S_i(0)}{V_M(0)} \equiv \text{porportion in } i^{th} \text{ security}$$

$$w_M^T K_M = w_1 K_1 + \cdots + w_n K_n \equiv \text{return from market security}$$

$$\sigma_M = w_M^T \Sigma w_M \quad ; \quad \mu_M = w^T m$$

$$\Sigma_{ij} = \mathrm{cov}(K_i, K_j) \quad ; \quad m_i = \mathbb{E} K_i$$



Now Consider portfolio combining K_M w/ risk free Bond $K_B \equiv$ const return.

$\therefore$ for Bond $V_B(0)$, $V_B(1)$ given

& given market portfolio we create combined portfolio:

Let $w_M \in \mathbb{R}^n$ be market portfolio

$$\underset{\substack{\uparrow \\ \text{total portfolio.}}}{V_p} = \underset{\substack{\uparrow \\ \text{bond}}}{V_B(0)} + \underset{\substack{\uparrow \\ \text{market}}}{V_M(0)}$$

$$s = \frac{V_B(0)}{V_p(0)} \quad ; \quad (1-s) = \frac{V_M(0)}{V_p(0)}.$$

$$w_p = (s,\ (1-s)\, w_M) \in \mathbb{R}^{n+1}$$

$$w_p^T \mathbb{1} = s + (1-s)(w_1 + \cdots + w_n) = 1 \checkmark$$

$$K_p = \frac{V_p(1) - V_p(0)}{V_p(0)}$$

$$= s K_B + (1-s)\, w_M K_M$$

RISK + RETURN.

$$\mu_p = \mathbb{E} K_p = \mathbb{E}\left(s K_B + (1-s) w_M K_M\right)$$

$$= s\mu_B + (1-s)\, w^T m$$

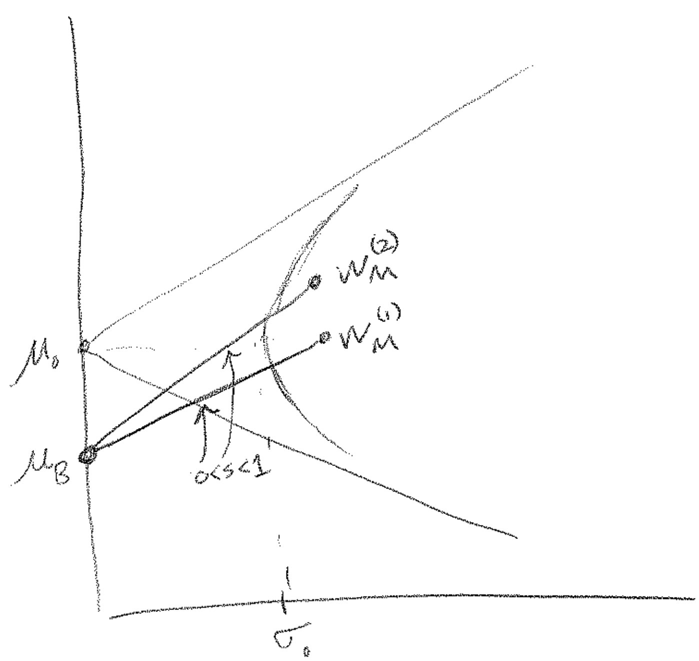
(here we wrote $\mu_B \equiv K_B$).

$$\sigma_p^2 = \text{Var } K_p = \text{Var}\left(s K_B + (1-s) w_M K_M\right)$$

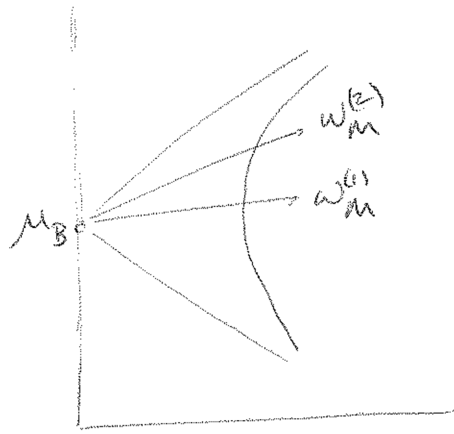
$$= (1-s)^2 \ \text{Var}(w_M K_M)$$

$$= (1-s)^2 \ w_M^T \Sigma w_M = (1-s)^2 \sigma_M^2$$

$\hookrightarrow \sigma_p = (1-s)\,\sigma_M$



$\left(s\mu_B + (1-s)\mu_M,\ (1-s)\sigma_M\right)$ is line between $(\mu_B, 0)$ and (μ_M, σ_M)



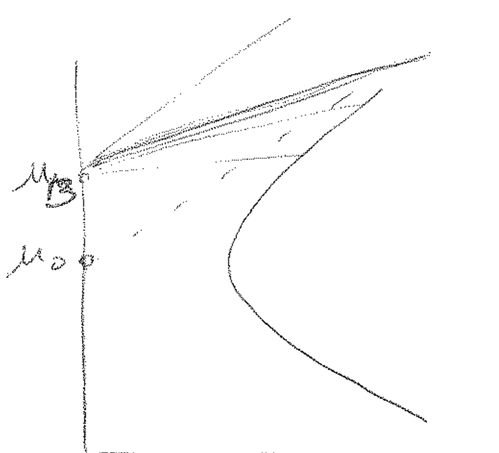
line going to $w_M^{(2)}$ is prefferable because points on this line dominate points on the other line.

$\therefore$ Best line is line "furthest to the left"

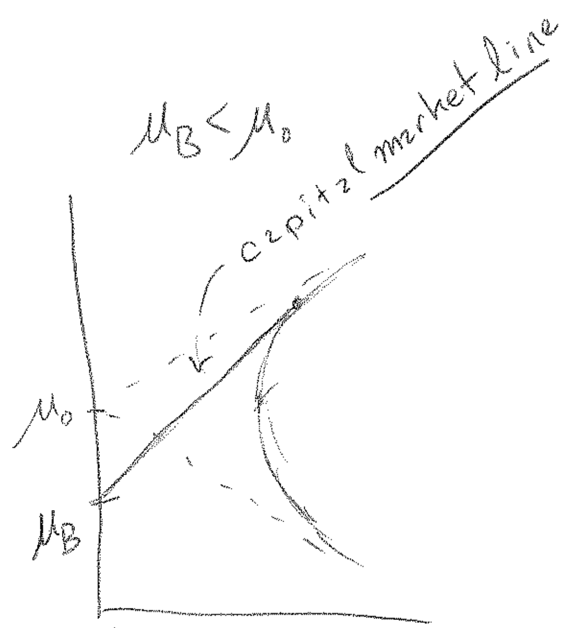
$\therefore$ Best line is line w/ the greatest slope.

2 scenarios:

$\mu_B \geq \mu_0 =$ mvp return



$\mu_B < \mu_0$



for every line 'below' line parallel to asymptote their is some w_M which obtains line — there is an 'optimal' market portfolio

line tangent to minimal variance line + intersects $(\mu_B, 0)$ implies optimal market portfolio.

Suppose $\mu_B < \mu_0$. let us find optimal portfolio:

~~[illegible]~~ line between points

$(0, \mu_B)$, $(\sqrt{w^T \Sigma w^T}, w^T m)$

$$\therefore \quad \text{slope} = \frac{w^T m - \mu_B}{\sqrt{w^T \Sigma w}}$$

$$(\text{slope})^2 = \frac{(w^T m - \mu_B)^2}{\left(\sqrt{w^T \Sigma w}\right)^2} = \frac{(w^T m - \mu_B)^2}{w^T \Sigma w}$$

Lagrange Multiplier

maximize $(\text{slope})^2$ wrt $w^T \mathbb{1} = 1$.

$$F(w, \lambda) = \frac{(w^T m - \mu_B)^2}{w^T \Sigma w} - \lambda (w^T \mathbb{1} - 1)$$

$$0 = \nabla F = \begin{pmatrix} \dfrac{2(m)(w^T m - \mu_B)}{w^T \Sigma w} - \dfrac{2(w^T m - \mu_B)^2}{(w^T \Sigma w)^2} \Sigma w - \lambda \mathbb{1} \\ w^T \mathbb{1} - 1 . \end{pmatrix}$$

Multiply top eq^n on Left by w^T

$w^T \mathbb{1} = 1$ ~~[illegible]~~ $\therefore$

$$0 = \frac{2 w^T m (w^T m - \mu_B)}{w^T \Sigma w} - \frac{2(w^T - \mu_B)^2}{w^T \Sigma w} - \lambda$$

$$= \frac{2 \mu_B (w^T m - \mu_B)}{w^T \Sigma w.} - \lambda .$$

Replace λ into top eq$^{\underline{n}}$:

$$\frac{(m-\mu_B\mathbb{1})(w^T m-\mu_B)}{w^T\Sigma w} = \frac{(w^T m-\mu_B)^2}{(w^T\Sigma w)^2}\,\Sigma w$$

$$\hookrightarrow \quad (m-\mu_B\mathbb{1})\,\frac{w^T\Sigma w}{(w^T m-\mu_B)} = \Sigma w \qquad (*)$$

Σ^{-1}

$$\Sigma^{-1}(m-\mu_B\mathbb{1})\,\frac{w^T\Sigma w}{w^T m-\mu_B} = w$$

$\mathbb{1}^T$

$$\mathbb{1}^T\Sigma^{-1}(m-\mu_B\mathbb{1})\,\frac{w^T\Sigma w}{w^T m-\mu_B} = 1.$$

$$\hookrightarrow \quad \frac{w^T\Sigma w}{w^T m-\mu_B} = \frac{1}{\mathbb{1}^T\Sigma^{-1}(m-\mu_B\mathbb{1})}$$

Insert into $(*)$ & multiply by Σ^{-1}:

$$\hookrightarrow \quad w_M = \frac{\Sigma^{-1}(m-\mu_B\mathbb{1})}{\mathbb{1}\,\Sigma^{-1}(m-\mu_B\mathbb{1})}$$

w_M is called the market portfolio.

Maximal slope:

$$\text{Slope} = \frac{w^T m - \mu_B}{\sqrt{w^T \Sigma w}}$$

$$w^T \Sigma w = \frac{(m-\mu_B \mathbb{1})^T \Sigma^{-1}}{\mathbb{1}^T \Sigma^{-1} (m-\mu_B \mathbb{1})} \Sigma \frac{\Sigma^{-1}(m-\mu_B \mathbb{1})}{\mathbb{1} \Sigma^{-1} (m-\mu_B \mathbb{1})}$$

$$= \frac{(m-\mu_B \mathbb{1})^T \Sigma^{-1} (m-\mu_B \mathbb{1})}{\left[\mathbb{1}^T \Sigma^{-1} (m-\mu_B \mathbb{1})\right]^2}$$

$$\cancel{\text{Slope} = \{ m^T \Sigma^{-1} (m-\mu_B \mathbb{1}) - \mu_B \}}$$

$$\text{Slope} = \frac{\dfrac{m^T \Sigma^{-1} (m-\mu_B \mathbb{1})}{\mathbb{1} \Sigma^{-1} (m-\mu_B \mathbb{1})} - \mu_B}{\dfrac{\sqrt{(m-\mu_B \mathbb{1}) \Sigma^{-1} (m-\mu_B \mathbb{1})}}{\mathbb{1} \Sigma^{-1} (m-\mu_B \mathbb{1})}}$$

$$= \frac{(m^T - \mu_B \mathbb{1}^T) \Sigma^{-1} (m-\mu_B \mathbb{1})}{\sqrt{(m^T - \mu_B \mathbb{1}^T) \Sigma^{-1} (m-\mu_B \mathbb{1})}}$$

$$= \sqrt{(m^T - \mu_B \mathbb{1}^T) \Sigma^{-1} (m-\mu_B \mathbb{1})}$$

Now suppose $u \in \mathbb{R}^n$, $u^T \mathbb{1} = 1$ is a portfolio.

then Risk + return is:

$$(\mu_u, \sigma_u) = \left(u m, \sqrt{u^T \Sigma u}\right)$$

Consider 2 security market ~~of~~ u + w_M.

$$(\mu_M, \sigma_M) = \left(\frac{m^T\Sigma^{-1}(m-\mu_B\mathbb{1})}{\mathbb{1}^T\Sigma^{-1}(m-\mu_B\mathbb{1})}, \frac{\sqrt{(m-\mu_B\mathbb{1})^T\Sigma^{-1}(m-\mu_B\mathbb{1})}}{\mathbb{1}\Sigma^{-1}(m-\mu_B\mathbb{1})}\right)$$

$$C_{\cancel{Mu}} = u\Sigma w_M = \frac{u(m-\mu_B\mathbb{1})}{\mathbb{1}\Sigma^{-1}(m-\mu_B\mathbb{1})} = \frac{\mu_u-\mu_B}{\mathbb{1}\Sigma^{-1}(m-\mu_B\mathbb{1})}$$

2 market portfolio:

$$\sigma_p = s^2\sigma_u^2 + (1-s)^2\sigma_M^2 + 2s(1-s)C_{\cancel{Mu}}$$

$$\mu_p = s\mu_u + (1-s)\mu_M$$

$$\frac{d}{ds}\sigma_p^2 = 2s\sigma_u^2 + 2(1-s)(-1)\sigma_M^2 + 2(1-s)C_{Mu} - 2sC_{Mu}$$

$$\left.\frac{d}{ds}\right|_{s=0}\sigma_p^2 = 2C_{Mu} - 2\sigma_M^2$$

$$\left.\frac{d}{ds}\right|_{s=0}(\sigma_p^2)^{1/2} = \frac{1}{2}\frac{1}{(\sigma_p^2)^{1/2}}\left.\frac{d}{ds}\right|_{s=0}(\sigma_p^2) = \frac{C_{Mu}-\sigma_M^2}{\sigma_M}$$

$$\left.\frac{d}{ds}\right|_{s=0}\mu_p = \mu_u - \mu_M$$

$$\frac{\mu_u-\mu_M}{\left(\frac{C_{Mu}-\sigma_M^2}{\sigma_M}\right)} = \frac{\mu_M-\mu_B}{\sigma_M}$$

$$\frac{\mu_u - \mu_M}{\left(\frac{C_{Mu} - \sigma_M^2}{\sigma_M}\right)} = \frac{\mu_M - \mu_B}{\sigma_M}$$

$$\mu_u - \mu_M = \frac{c_{Mu} - \sigma_M^2}{\sigma_M^2}(\mu_M - \mu_B) = \left(\frac{c_{Mu}}{\sigma_M^2} - 1\right)(\mu_M - \mu_B)$$

$$\mu_u = \mu_B + \frac{c_{Mu}}{\sigma_M^2}(\mu_M - \mu_B).$$

$\therefore$ We define for any seurity u the β factor:

$$\beta_u = \frac{\mathrm{cov}(K_u, K_M)}{\sigma_M^2}$$

Application of CAPM.

Suppose we write contract to pay value H @ time 1.

put aside money @ time $t=0$ to pay it off.

ie Put $V(0)$ into portfolio $\begin{cases} \text{amt: } w_B V(0) \text{ in Bonds} \\ \text{amt: } w_M V(0) \text{ in the Market} \end{cases}$

Value at time 1 is $V(1) = V(0)\left(w_M(1+K_M) + w_B(1+R)\right)$

Ideally $V(1) = H(1)$ to clear obligation.

But @ time 1 we have to correct the error:

$$\varepsilon = H(1) - V(1).$$

We try to minimize the error.

ie 1st set expectation to zero:

$$\mathbb{E}(\varepsilon) = 0 = \mathbb{E}(H(1)) - V(0)\left(W_M(1+\cancel{K_M}) + W_B(1+R)\right)$$

☆ Let us write $H(1) = V(0)(1+K_H)$

then $\mu_H = \mathbb{E} K_H$

$$\Longrightarrow \quad \cancel{K_H} \quad \mu_H = W_M \mu_M + W_B R.$$

$$\mathrm{Var}(\varepsilon) = \mathrm{Var}(H(1) - V(1)) = \mathrm{Var}\left(V(0)\{1+K_H\} - V_0[W_M(1+\mu_M) + W_B(1+R)]\right)$$

$$= V(0)^2 \; \mathrm{Var}(K_H - \{W_M \cancel{K_M} + W_B R\})$$

$$= V(0)^2 \left\{ \mathrm{var}\, K_H + \mathrm{var}(W_M K_M + W_B R) - 2\,\mathrm{cov}(K_H, W_M K_M + W_B R) \right.$$

$$= V(0)^2 \left\{ \mathrm{var}\, K_H + W_M^2 \,\mathrm{var}\, K_M - 2 W_M \,\mathrm{cov}(K_H, K_M) \right\}$$

$$\therefore \quad Var(\varepsilon) = V^2(0) \left\{ var(K_H) + w_M^2 \, var K_M - 2 w_M \, cov(K_H \cdot K_M) \right\}.$$

$$0 = \frac{d}{dw_M} var(\varepsilon) = V^2(0) \left\{ 2 w_M \, var K_M - 2 \, cov(K_H \cdot K_M) \right\}$$

$$\hookrightarrow \quad w_M = \frac{cov(K_H \cdot K_M)}{var K_M} = \beta_H .$$

$\therefore$ Minimum Risk:

$$\sigma_{min}^2 = V^2(0) \left\{ \sigma_H^2 + \underbrace{\beta_H^2 \sigma_M^2 - 2 \beta_H c_{MH}}_{\frac{cov^2}{\sigma_M^2} - 2 \frac{cov^2}{\sigma_M^2}} \right\}$$

$$\frac{\sigma_{min}^2}{V^2(0)} = \sigma_H^2 \left\{ 1 - \frac{cov^2}{\sigma_H^2 \sigma_M^2} \right\}$$

$$= \sigma_H^2 \left\{ 1 - \rho^2 \right\}$$

$$= \sigma_H^2 \left\{ 1 - \frac{\beta_H^2}{\sigma_H^2} \right\}.$$

β_H measures risk of portfolio relative to risk of market.

$\beta_H \lesssim 1$ low risk relative to market.

$\beta_H > 1$ high risk relative to market.