General theory for Market
w/ 2 securities.

$\Omega \equiv$ space of outcomes.

K_i, $i=1,2$; $K_i : (-1, \infty)$.

$\because \quad \frac{S_i(1) - S_i(0)}{S_i(0)} \geq \frac{0 - S_i(0)}{S_i(0)} \geq -1$.

Given $\{w : w_1 + w_2 = 1\}$.

$K_V = w_1 K_1 + w_2 K_2 = w^T K = (w_1 \ w_2)\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$

Def^n $\left.\begin{array}{l} \mu_i = \mathbb{E} K_i \\ c_i = Var(K_i) \end{array}\right\} i = 1, 2$

$c_{12} = cov(K_1, K_2)$

$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$, $\Sigma = \begin{pmatrix} c_1 & c_{12} \\ c_{12} & c_2 \end{pmatrix}$

$\mu_V = \mathbb{E} K_V = \mu^T \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$; $\sigma_V^2 = w^T \Sigma w$

$\mu_V = w_1 \mu_1 + w_2 \mu_2$

$\sigma_V^2 = w_1^2 C_1 + 2 w_1 w_2 C_{12} + w_2^2 C_2$

$w_1 = s \quad ; \quad w_2 = 1-s$

$\mu_V = s\mu_1 + (1-s)\mu_2 = \mu_2 + s(\mu_1 - \mu_2)$

$\sigma_V^2 = s^2 \sigma_1^2 + (1-s)^2 \sigma_2^2 + 2s(1-s) C_{12}$

Let us find the portfolio w/ min verizance.

$$0 = \frac{d(\sigma_V^2)}{ds} = 2s\sigma_1^2 - 2(1-s)\sigma_2^2 + 2(1-s)C_{12} - 2sC_{12}$$

$$s(\sigma_2^2 + \sigma_1^2 - 2C_{12}) = \sigma_2^2 - C_{12}$$

w which minimizes σ_V^2 is

$$W_0 = \begin{pmatrix} \dfrac{\sigma_2^2 - C_{12}}{\sigma_2^2 + \sigma_1^2 - 2C_{12}} \\ \\ \dfrac{\sigma_1^2 - C_{12}}{\sigma_2^2 + \sigma_1^2 - 2C_{12}} \end{pmatrix} = \begin{pmatrix} s \\ \\ 1-s \end{pmatrix}$$

But notice if $\sigma_1^2 = \sigma_2^2 = C_{12}$

$$\sigma_V^2 = \left\{ s^2 + (1-s)^2 + 2s(1-s) \right\} \sigma_1^2 = \sigma_1^2 = C_{12}.$$

We know graph of (σ_v, μ_v) is of form of hyperbola, but how do (μ_i, σ_i) fit in?

In the simplest case $|\rho_{12}| = \left|\frac{c_{12}}{\sigma_1 \sigma_2}\right| = 1$.

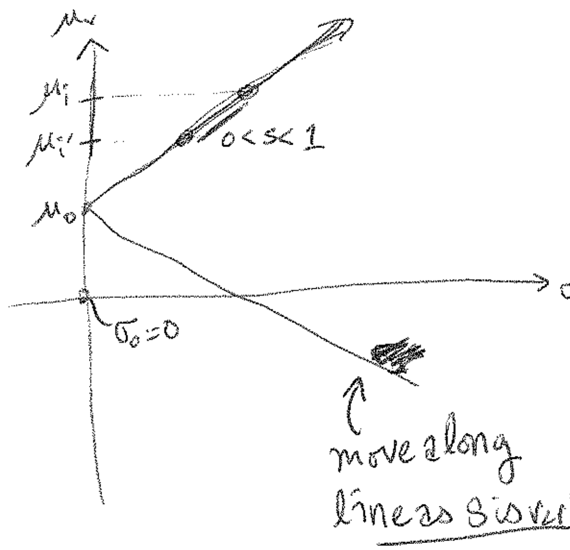
Consider these as archtypal examples.

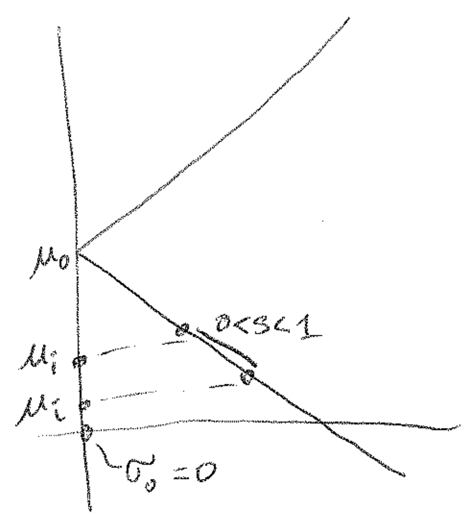
$\rho_{12}=1$: $\quad S_0 = \frac{\sigma_2^2 - \sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2} = \frac{\sigma_2}{(\sigma_2-\sigma_1)} \quad$ if $\begin{cases} * \ \sigma_2 > \sigma_1 : S_0 > 1 \\ * \ \sigma_1 > \sigma_2 : S_0 < 0 \end{cases}$

$c_{12} = \sigma_1\sigma_2$

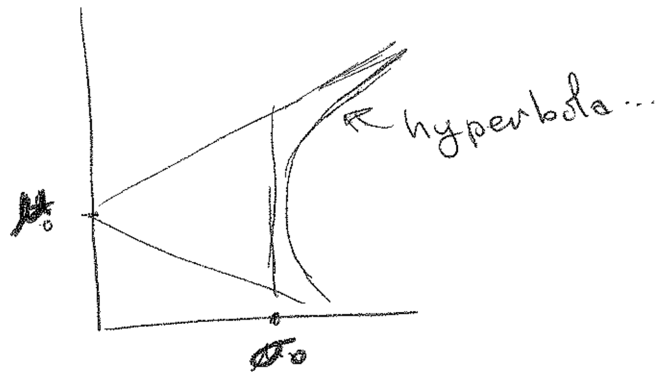
$\therefore \quad \mu_0 = S_0 \mu_1 + (1-S_0)\mu_2 \Rightarrow \mu_0 < \mu_1 \wedge \mu_2$ or $\mu_0 > \mu_1 \vee \mu_2$.

$$\sigma_0^2 = \frac{\sigma_1^2\sigma_2^2 - c_{12}^2}{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2} = 0$$





We know graph is of form



but we haven't said where (μ_i, σ_i) are
w/r/t σ_0, μ_0.

In the simplest case $|\rho_{12}| = \left|\frac{c_{12}}{\sigma_1 \sigma_2}\right| = 1$.

Let us consider this as the archtypal examples.

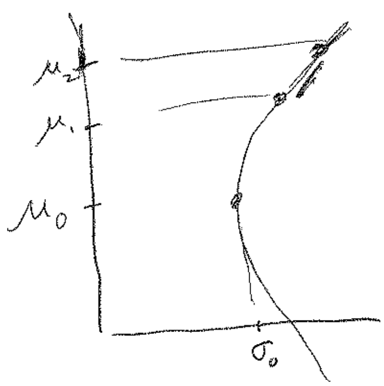
then

$\rho_{12}=1$: $\quad s_0 = \dfrac{\sigma_2^2 - \sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2} = \dfrac{\sigma_2}{(\sigma_2 - \sigma_1)}$ $\quad \therefore$ if $\sigma_2 > \sigma_1 \rightarrow s_0 > 1$

$\sigma_2 < \sigma_1 \rightarrow s_0 < 0$.

ie $\xrightarrow{\rho_{12}=1} \mu_0 = s_0 \mu_1 + (1-s_0)\mu_2$

$\mu_0 < \mu_1 \wedge \mu_2$ or $\mu_0 > \mu_1 \vee \mu_2$



Minimum Variance Portfolio : Risk + return.

$$\sigma_v^2(w_0) = w_0^T \sum w_0 =$$

$$= \frac{(\sigma_2^2 - c_{12})^2 \sigma_1^2 + (\sigma_1^2 - c_{12})^2 \sigma_2^2 + 2(\sigma_1^2 - c_{12})(\sigma_2^2 - c_{12}) c_{12}}{(\sigma_1^2 + \sigma_2^2 - 2c_{12})^2}$$

$$= \frac{\sigma_1^2 \sigma_2^2 - c_{12}^2}{\sigma_1^2 + \sigma_2^2 - c_{12} 2} \qquad \left(\begin{array}{c}\text{line (3.12)} \\ \text{in error}\end{array}\right)$$

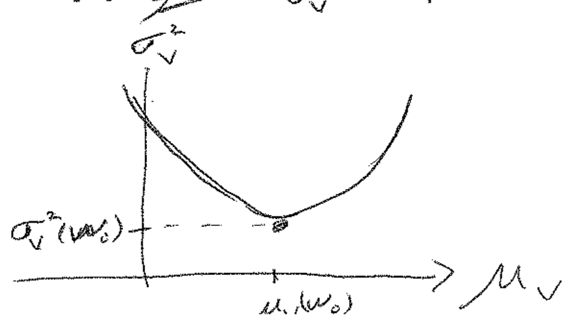
$$\mu_v(w_0) = \frac{\mu_1 \sigma_2^2 + \mu_2 \sigma_1^2 - (\mu_1 + \mu_2) c_{12}}{\sigma_1^2 + \sigma_2^2 - 2c_{12}}$$

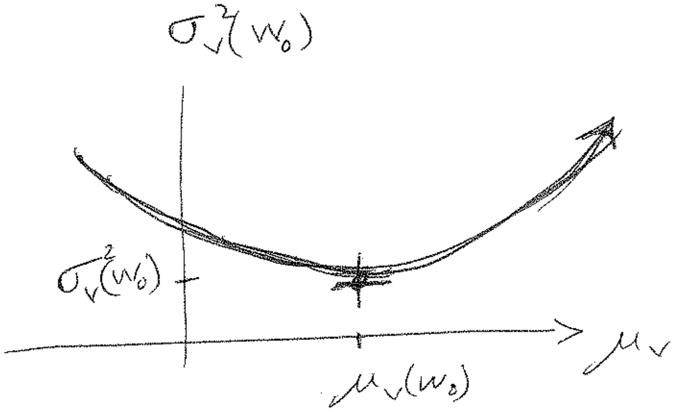
Notice $\sigma_v^2(s)$ is a parabola in terms of s.

but $\mu_v(s) = \mu_2 + s(\mu_1 - \mu_2)$

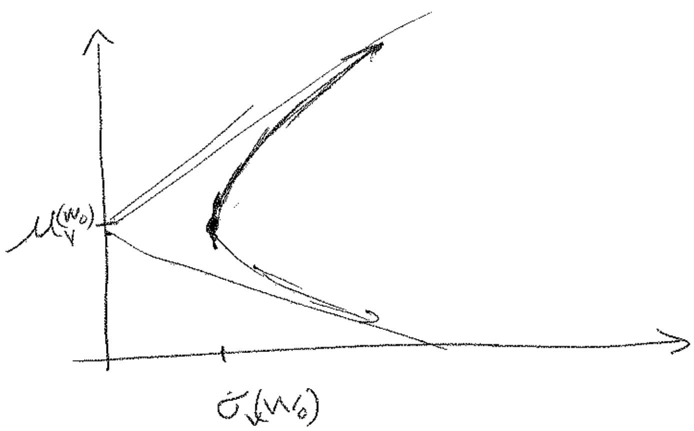
$\hookrightarrow \sigma_v^2(s) = \sigma_v^2\left(\frac{\mu_v - \mu_2}{\mu_1 - \mu_2}\right) = \tilde{\sigma}_v^2(\mu_v)$

$\therefore$ $\tilde{\sigma}_v^2$ is parabola in terms of μ_v





take $\sqrt{\ }$ invert axes. $\sqrt{\text{parabola}}$ = hyperbola.



$$\sigma_v^2 = \frac{(\mu_v-\mu_2)^2\sigma_1^2 + (\mu_v-\mu_1)^2\sigma_2^2 - 2(\mu_v-\mu_1)(\mu_v-\mu_2)c_{12}}{(\mu_1-\mu_2)^2}$$

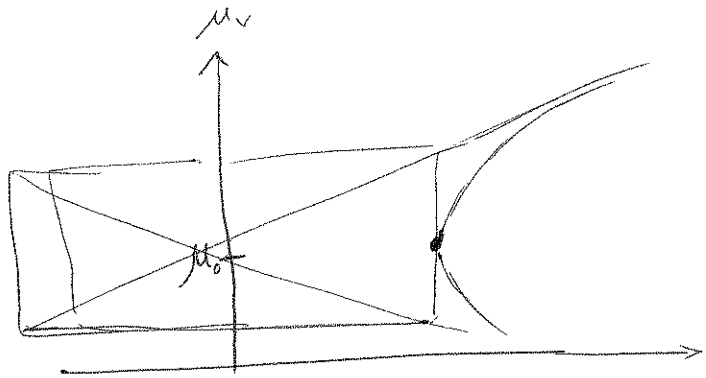
. . .

$$\mu_0 = \mu_v(w_0)\ ;\ \sigma_0^2 = \sigma_v^2(w_0)$$

$$\sigma_v^2 - A^2(\mu_v-\mu_0)^2 = \sigma_0^2$$

$$A^2 = \frac{\sigma_1^2+\sigma_2^2-2c_{12}}{(\mu_1-\mu_2)^2} > 0.$$

Recall graph of hyperbla $\sigma_v^2 - A^2(\mu_v - \mu_0)^2 = \sigma_0^2$



large σ_v & μ_v

$$\sigma_v^2 = \sigma_0^2 + A^2(\mu_v - \mu_0)^2$$

$$1 = \frac{\sigma_0^2}{\sigma_v^2} + \left(A\left(\frac{\mu_v}{\sigma_v} - \frac{\mu_0}{\sigma_v}\right)\right)^2$$

$$\hookrightarrow 1 \sim 0 + \left(A\left(\left(\frac{\mu_v}{\sigma_v}\right) - 0\right)\right)^2$$

$$\hookrightarrow \left|\frac{\sigma_v}{\mu_v}\right| \sim A$$

Asymptotes

$$\mu_v = \pm \frac{1}{A}\sigma_v + \mu_0$$

We know graph of (σ_v, μ_v) is in the form of hyperbola,
but how do (σ_i, μ_i) fit in?

In the simplest case $|\rho_{12}| = \left|\frac{c_{12}}{\sigma_1 \sigma_2}\right| = 1$

(In this case $\det(\Sigma) = 0 \Rightarrow$ it is not major importance
but it is 'archtypal' example
we can learn from)

$$\text{If } |\rho_{12}| = 1 \Rightarrow K_1 = t K_2 + p.$$

$$\sigma_1^2 = t^2 \sigma_2^2$$

$$C_{12} = t \sigma_2^2$$

plug into formula for m.v.p.

$$S_0 = \frac{\sigma_2^2 - t\sigma_2^2}{t^2\sigma_2^2 + \sigma_2^2 - 2t\sigma_2^2} = \frac{1-t}{(1-t)^2} = \frac{1}{1-t}.$$

$$\sigma_0^2 = \frac{t^2\sigma_2^4 - t^2\sigma_2^4}{\sigma_1^2 + \sigma_2^2 - 2c_{12}} = 0$$

$$\mu_1 = t\mu_2 + P$$

$$\mu_0 = \frac{\mu_1\sigma_2^2 + t^2\mu_2\sigma_2^2 - (\mu_1+\mu_2)t\sigma_2^2}{t^2\sigma_2^2 + \sigma_2^2 - 2t\sigma_2^2} = \frac{P}{1-t}$$

$\therefore \quad t > 0 \Rightarrow \quad S_0 \begin{cases} = \frac{1}{1-t} > 1 & \text{if } 0 < t < 1 \\ = \frac{1}{1-t} < 0 & \text{if } 1 < t \end{cases}$

$\Updownarrow$

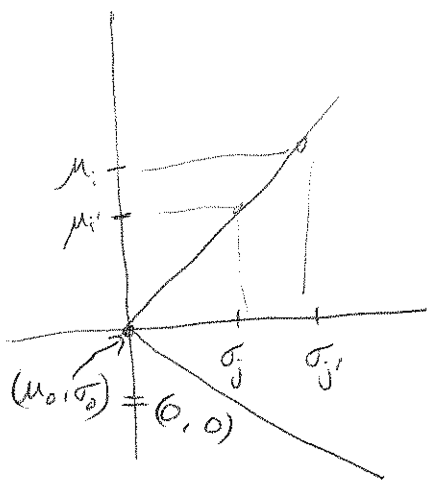
$\rho_{12} = 1$

$t < 0$

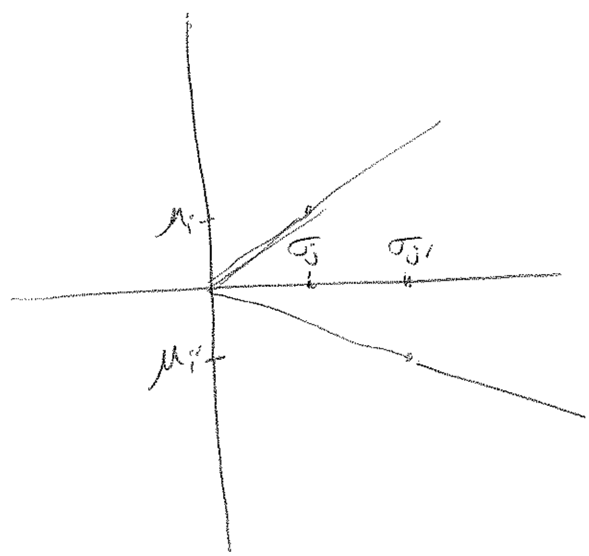
$\Updownarrow \quad \Rightarrow \quad S_0 = \frac{1}{1-t} \in (0, 1)$

$\rho_{12} = -1$

$\therefore$ @ $p = 0$



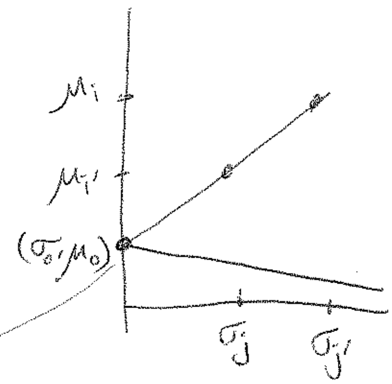
$t > 0$



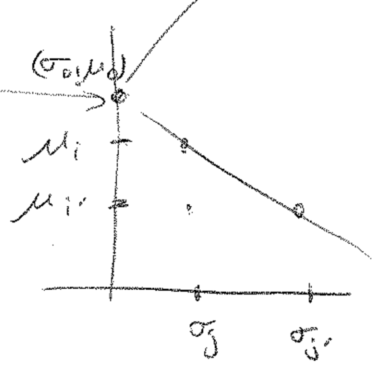
$t < 0$

$\therefore$ @ $\rho \neq 0$

$t > 0$

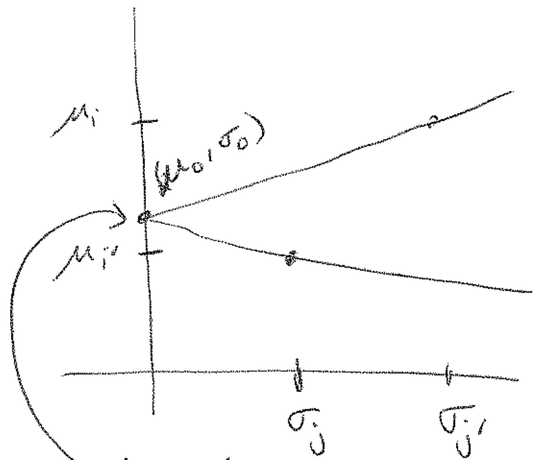


or –



outside because
$s_0 < 0$ or $s_0 > 1$

$t < 0$



inside
because $0 < s_0 < 1$.

But notice if $K_1 = tK_2 + p$

then is risk free w/ return p

$S_3 = S_1 - tS_2$

So it is not neccesary to consider
S_1 + S_2 as separate stocks.

Eg illustrates diff between

$\rho_{12} \sim 1$ mvp req shortsell

$\rho_{12} \sim -1$ mvp does not req shortsell

where is bdry?

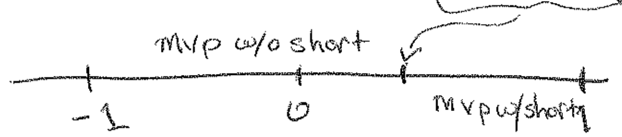
$0 < \delta_0 < 1 \leftarrow$ mvp w/o short sell.

$$\delta_0 = \frac{\sigma_2^2 - c_{12}}{\sigma_1^2 + \sigma_2^2 - 2c_{12}}$$

$\delta_0 < 1$	$\delta_0 > 0$
$\Rightarrow$ $\sigma_2^2 - c_{12} < \sigma_1^2 + \sigma_2^2 - 2c_{12}$	$\sigma_2^2 - c_{12} > 0$
$c_{12} < \sigma_1^2$	$c_{12} < \sigma_2^2$
$\rho_{12} < \dfrac{\sigma_1}{\sigma_2}$	$\rho_{12} < \dfrac{\sigma_2}{\sigma_1}$

$\therefore$ mvp w/ short sell is possible iff

$$\rho_{12} < \frac{\sigma_2}{\sigma_1} \wedge \frac{\sigma_1}{\sigma_2} = \rho^c$$



Suppose

$$\sigma_1^2 = 1/2 \quad . \quad \sigma_2^2 = 1/4 \quad . \quad c_{12} = 1/8 .$$

$$\rho_{12} = \frac{1/8}{1/\sqrt{2}\ 1/2} = \frac{\sqrt{2}}{4}$$

$$\rho^c = \frac{1/4}{1/2} = 1/2$$

$$\therefore \rho_{12} \not< \rho^c \Rightarrow \text{does not req short sell}$$

$$s_0 = \frac{1/2 - 1/8}{1/2 + 1/4 - 2\frac{1}{8}} = \frac{3/8}{1/2} = 3/4 \in (0,1).$$

$$\sigma_1^2 = \frac{1}{3} \qquad \sigma_2^2 = \frac{1}{9} \qquad c_{12} = \frac{1}{6}$$

$$\rho_{12} = \frac{1/6}{1/3\ 1/\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\rho^c = \frac{1/9}{1/3} = 1/3$$

$$\rho_{12} > \rho^c \Rightarrow \text{require short sell.}$$

indeed

$$s_0 = \frac{1/9 - 1/6}{1/9 + 1/3 - 2\,1/6} = \frac{2-3}{2+6-6} = \frac{-1}{2} < 0$$