Portfolio of Risky Securities —

Now let us introduce an instrument S whose value @ time zero is given + the value @ time t = 1 is $S(1)$ a random variable.

(Keep in mind there are still Bonds w/ fixed interest rate r out there, but these securities allow us to earn more than a rate r of return).

Let Ω be outcomes of 'world' at time $t=1$.

"Ie what is situation of company @ time $t=1$? Any situation of the company will define the price of the stock at time $t=1$"

* Then $S(1)$ is a map from Ω to $[0,\infty)$ - possible values of the stock.

* If we think of situations the company can be in at time $t=1$ having some probabilities (defined by measure $\mathbb{P}$)

then $S(1)$ the random variable 'inherits' these probabilities.

E.g. Suppose Ω is the space of

* Earnings per year/quarter $= X$ (Per share)
* Projected Growth $= Y$ ~ some parameter to measure growth.

~~Value~~ Value of stock

Suppose the value of the stock is given by the formula: (at time 1 year)

$$S(1) = (20 + Y)X$$

Suppose (X, Y) are uniformly distributed on A.

for $A = \{ (x,y) : 2 \le x \le 4 \,;\ 0 \le y \le 2 \} = [2,4] \times [0,2]$.

$\therefore U_i \sim U[0,1] \,;\ X = 2 + 2U_1 \,;\ Y = 2U_2$.

Suppose the current value of the stock is

$S(0) = 52.5$ corresponding to $x = 2.5$, $y = 1$.

What is the expected Value of the stock in 1 year?

$$\mathbb{E}(S(1)) = \mathbb{E}((20+Y)X) = \mathbb{E}\,20X + \mathbb{E}XY$$

$$= 20\,\mathbb{E}X + \mathbb{E}X\,\mathbb{E}Y = 20 \cdot 3 + 3 \cdot 1 = 63.$$

Now let us define the Return, (In the same way we defined return under Interest)

$$K_1 = \frac{S(1) - S(0)}{S(0)} \equiv \text{the return on } S \text{ @ } t=1.$$

$$K_T = \frac{S(T) - S(0)}{S(0)}.$$

The expected return is

$$\mathbb{E}(K_1) = \frac{(\mathbb{E}\, S(1)) - S(0)}{S(0)} = \frac{63 - 52.5}{52.5} = .2.$$

Let $\mu = \mathbb{E}(K_1)$

We can rewrite $S(1)$ as,

$$S(1) = S(0)(1 + K_1)$$

$$\mathbb{E}\, S(1) = S(0)(1 + \mu K_1).$$

- we will focus on modeling an ensemble of K variables in this section, since it will not matter what value $S(0)$ is - We assume we can cut up stocks and buy any fractional portion of stocks for simplicity.

Note we will allow ourselves to purchase any fractional amount of stocks,

ie suppose price of stock is \$100

+ ~~$~~ risk free interest is r,

if we have \$100 in hand we are permitted to split investment between stock + bond

ie $\forall \alpha$, $0 < \alpha < 1$

we may buy α portion of stock +

invest $\$ (1-\alpha)100$ in risk free interest.

thus @ time 0 we have $V(0) = \$100$

pay $\$ \alpha S(0) = \$ \alpha \times 100$ for stock

invest $\$(1-\alpha)(100)$ in bonds.

@ time 1 we liquidate and the value is

$$V(1) = \alpha S(1) + (1-\alpha)\, 100\, e^{r}$$

Similarly we allow negative purchase of bonds, ie borrow to finance purchases,

And even negative purchase of stocks. - Short selling.

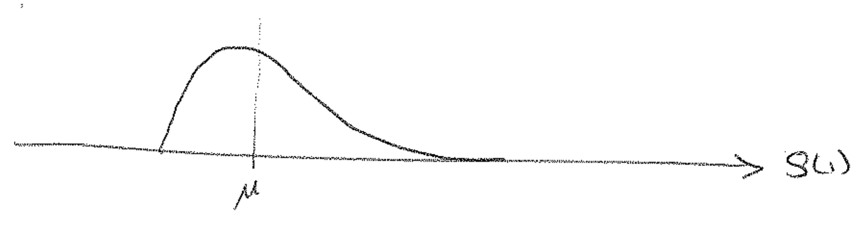
Measuring Risk - Standard Deviation.

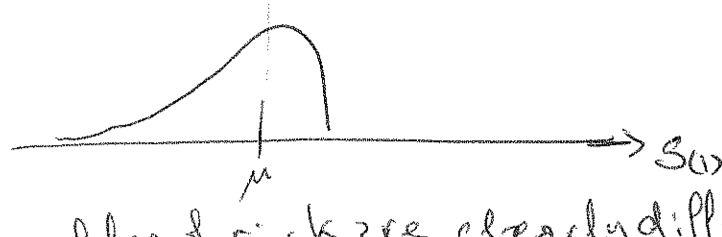
Suppose we think of risk as variation from the expected return, then the proper measure of the risk is the variance — more precisely

std dev $= \sqrt{\text{Variance}}$

* This suits us for now since we will be taking long (buying) and short (selling what we dont have) positions on stocks, so any deviation from the expectation can lead to a loss.

But if we buy a stock: And the PDF of the return is either





these profiles of risk are clearly different ... an issue considered in the measure Value at Risk. (VaR)

Let us define risk on return as,

$$\sigma_K = \sqrt{Var(K)}$$

$$Var\,K = Var\,\frac{S(T)-S(0)}{S(0)} = \frac{1}{S(0)^2} Var(S(T)-S(0))$$

$$= \frac{1}{(S(0))^2} Var\,S(T).$$

$$\sigma_K = \frac{\sqrt{Var\,S(T)}}{S(0)} = \frac{\sigma_{S(T)}}{S(0)}$$

<u>Eg</u> RETURN TO <u>EXAMPLE 1</u>

$$S(1) = (20+Y)X \qquad X = 2(1+U_1) \quad ; \; U_i = \mathcal{U}[0,1].$$
$$Y = 2U_2$$

$$Var\,S(1) = \mathbb{E}(S(1))^2 - (\mathbb{E}S(1))^2 = \mathbb{E}(20+Y)^2\,\mathbb{E}X^2 - 63^2$$

$$= \mathbb{E}(20+2U_2)^2\,\mathbb{E}2^2(1+U_1)^2 - 63^2$$

$$= \int_{20}^{22} y^2\,dy \; 4\int_1^2 x^2\,dx - 63^2$$

$$= \frac{4}{9}(22^3-20^3)(2^3-1) - 63^2$$

$$= 4269 + \frac{2}{9} = (64.34)^2$$

$$\sigma_{S(1)} = 64.34 \;; \qquad \sigma_K = \frac{64.34}{52.5} = 1.245$$