Money Market: Default free Securities

- Under simplifying assumption of 'universal constant interest rate r' Bonds become equivalent & risk-free.

ZERO COUPON BOND.

BOND MAKES 1 payment, the face value F, at a given time T the maturity date.

typically the face value is $F = \$100$ (or another round #) & T is 1 year. So the value at the purchase date is

$$\cancel{V(0) = \beta_T 100}$$

$$V(0) = \beta_T F = (1+r_e)^{-1} F.$$

(or $(1+r_e)^{-T}$ for general T)

In reality trading value of Bond sets interest rate,

$$r_e = \frac{F}{V_0} - 1.$$

Let us simplify to $F = \$1^{00}$

β_T is the discount factor of \$. from Maturity time T to time 0

The discount value from maturity T to time $t>0$ is

β_{T-t}

Let us use notation $B(t,T) = \beta_{T-t}$ for this case (The distinction is not significant when interest rate is constant).

Thus we have, for general T

$$\beta_T = (1-r_e)^{-T}$$

$$\Rightarrow \beta_{T-t} = B(t,T) = (1-r_e)^{-(T-t)}.$$

$$\beta_{T-t} = e^{-r(T-t)}$$

or

$$\beta_{T-t} = \left(1+\frac{r}{m}\right)^{-m(T-t)}$$

Coupon Bonds,

Some Bonds pay a series of dispersements before Maturity, each payment is a fixed value called a Coupon

$\therefore$ Value of Coupons must be discounted + summed + added to discounted face value of Bond to find total value of Bond.

$\therefore$ Value of Bond @ time $t=0$ w/ maturity T, + payments @ times $\frac{1}{m}T, \frac{2}{m}T, \dots, \frac{m}{m}T$, $\tau = T/m$

$$V(0) = C\beta_\tau + C\beta_{2\tau} + \dots + C\beta_{m\tau} + \beta_T F.$$

Eg, Suppose Maturity $T = 5$ years, Coupons are worth \$10 + Face value is \$100

$$V(0) = 10e^{-r} + 10e^{-2r} + \dots + 10e^{-5r} + 100e^{-5r}$$

$$= 10e^{-r} + \dots + 10e^{-4r} + 110e^{-5r}.$$

Let us recall, for any $q \in \mathbb{R}$,

$$1 + q + \cdots + q^k = \frac{1 - q^{k+1}}{1 - q}.$$

$\therefore$ Value of Bond,

$$V(0) = Ce^{-r\tau} + \cdots + Ce^{-r\tau n} + Fe^{-rT}$$

$$= Ce^{-r\tau}\left(1 + \cdots + e^{-r\tau(n-1)}\right) + Fe^{-rT}$$

$$= Ce^{-r\tau}\left(\frac{1 - e^{-r\tau n}}{1 - e^{-r\tau}}\right) + Fe^{-rT}$$

if $\tau = 1$ year then

$$e^{r\tau} = e^{r} = 1 + r_e, \quad T = n\tau = n.$$

$$V(0) = C\frac{1}{r_e}\left(1 - (1 + r_e)^{-n}\right) + F(1 + r_e)^{-n}$$

$$= C \times PA(r_e, n) + F(1 + r_e)^{-n}$$

Time value of coupon bond

Let $\tau = 1$ year $C = \$10$

$T = 5$ year $F = \$100$

We have

$$V(0) = \underbrace{10e^{-r}}_{\text{year 1 coupon.}} + \cdots + \underbrace{10e^{-5r}}_{\text{year 5 coupon}} + \underbrace{100e^{-5r}}_{F}$$

At time $t=1$ the first coupon is paid so it is nolonger inclued in the value of the bond,

$$V(1) = \underbrace{10e^{-r}}_{\text{year 2 coupon}} + 10e^{-2r} + 10e^{-3r} + \underbrace{10e^{-4r}}_{\text{year 5 coupon}} + \underbrace{100e^{-4r}}_{F}$$

But for $t<1$ we have,

$$V(t) = 10e^{-r(1-t)} + 10e^{-r(2-t)} + \cdots + 10e^{-r(5-t)} + 100e^{-r(5-t)}$$

$$= e^{rt}\left\{10e^{-r} + 10e^{-r2} + \cdots + 10e^{-r5} + 100e^{-r5}\right\}$$

$$= e^{rt} V(0)$$

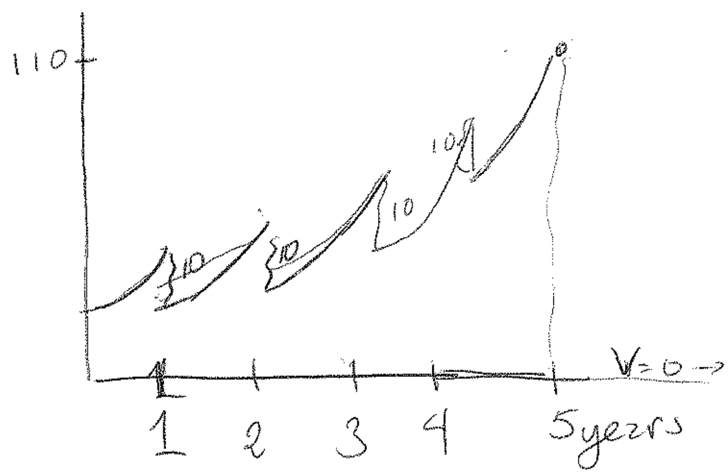
$\therefore$

$$\lim_{t\to 1} V(t) = 10 + 10e^{-r} + \cdots + 10e^{-r4} + 100e^{-r4}$$

$$\lim_{t\to 1} V(t) = 10 + V(1)$$

$\therefore$ So there is a discontinuity of $V(t)$ @ $t=1$ ...

Graph V(t)



Bonds at Par:

In a special case for Coupon Bonds, we have

$$V(0) = F.$$

here $F = ?$, $C = ?$

$\tau = 1$ year, $T = n$ years.

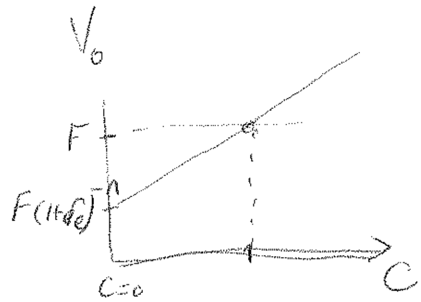
Notice the formula for the value is

$$V(0) = C \frac{1}{r_e}\left(1 - (1+r_e)^{-n}\right) + F(1+r_e)^{-n}$$

* the first term is increasing and second term is constant as C increases.

$$\therefore \quad V(0) = F(1+r_e)^{-n} < F \quad @ \; C = 0$$

$$\frac{d}{dC} V(0) = \frac{1}{r_e}\left(1 - (1+r_e)^{-n}\right) = PA(r_e, n)$$



Let us find C st $V(0) = F$.

$$C \frac{1}{r_e}\left(1 - (1+r_e)^{-n}\right) + F(1+r_e)^{-n} = F$$

$$C \frac{1}{r_e}\left\{1 - (1+r_e)^{-n}\right\} = F\left\{1 - (1+r_e)^{-n}\right\}$$

$$\Leftrightarrow \quad C = r_e F \checkmark.$$

Notice the formula of C does not depend on n,

$$V(0) = (r_e F)\, PA(r_e, n) + F(1+r_e)^{-n}$$

for $n > m > 0$

$$V(m) = (r_e F)\, PA(r_e, n-m) + F(1+r_e)^{-(n-m)}$$

$\therefore \ \forall \ i = 0, 1, \ldots, n-1$

$V(i) = F.$; $\lim_{t \nearrow i} V(t) = (1+r_e)F$

$V(t)$