

Stochastic Integration Introduction

Abstract

Stochastic integration summary.

1 Integration by Taylor theorem and Reimann sums

Review Taylor's theorem in calculus (non-stochastic case). Let f be defined on $\mathbb{R} \times \mathbb{T}$ with time set $\mathbb{T} = [0, T]$. We create a discrete Riemann sum, for function $t \rightarrow x(t)$ for $t \in \mathbb{T}$ then

$$f(T, x(T)) = f(0, x(0)) + \sum_{k=1}^N f(t_k, x(t_k)) - f(t_{k-1}, x(t_{k-1}))$$

where we take (for example) $\Delta t = \frac{1}{N}$ and $t_k = k/N$. We also write $x_k = x(t_k)$. Let f have one derivative in space and one derivative in time, then

$$\begin{aligned} f(t_k, x(t_k)) - f(t_{k-1}, x(t_{k-1})) &= f_t(t_{k-1}, x_{k-1})(t_k - t_{k-1}) + f_x(t_{k-1}, x_{k-1})(x_k - x_{k-1}) \\ &\quad + o(t_k - t_{k-1}) + o(x_k - x_{k-1}) \end{aligned}$$

Approximation Here the notation o indicates a term vanishing with respect to the argument, for example $y_\epsilon = o(\epsilon)$ if say $|y_\epsilon| < C\epsilon^\alpha$ for some $\alpha > 1$ and $C < \infty$. Moreover, $x_k - x_{k-1} = x'_{k-1}(t_k - t_{k-1}) + o(t_k - t_{k-1})$. The important point is $\sum_k (t_k - t_{k-1})^\alpha \rightarrow 0$, as $N \rightarrow \infty$.

From this we have that

$$\begin{aligned} f(T, x(T)) &= f(0, x(0)) + \sum_{k=1}^{NT} f_t(t_{k-1}, x_{k-1})(t_k - t_{k-1}) + f_x(t_{k-1}, x_{k-1})(x_k - x_{k-1}) \quad (1.1) \\ &= f(0, x(0)) + \int_0^T [f_t(s, x(s)) + f_x(s, x(s))x'(s)] ds \end{aligned}$$

2 Stochastic integration by similar approach of Taylor's theorem and Riemann sums

We attempt to reproduce this calculation for functions $f(t, W_t)$ for Wiener process W_t .

In the deterministic case, fluctuations of $x(t)$ are proportional to x' in time Δt . However, we know that brownian motion W_t fluctuations are of order $\sqrt{\Delta t}$ in a time step Δt . Therefore, if we consider only one spatial derivative we come across the following issue

$$\begin{aligned} u(t + \Delta t, W_{t+\Delta t}) &= u(t, W_t) + \dot{u}(t, W_t)[\Delta t] + u'(t, W_t)[W_{t+\Delta t} - W_t] \\ &\quad + o(\Delta t) + o(W_{t+\Delta t} - W_t) \end{aligned}$$

the second term of little o remainders cannot be counted upon to sum up to something finite.

Ito's formula for functions of Brownian processes We therefore take 2 derivatives, and using Taylor's theorem we have

$$u(t + \Delta t, W_{t+\Delta t}) = u(t, W_t) + \dot{u}(t, W_t)[\Delta t] + u'(t, W_t)[W_{t+\Delta t} - W_t] + \frac{1}{2}u''(t, W_t)[W_{t+\Delta t} - W_t]^2 + o(\Delta t) + o([W_{t+\Delta t} - W_t]^2) \quad (2.1)$$

putting these terms into a summation like (1.1) we have,

$$u(T, W_T) = u(0, W_0) + \sum_{i=0}^{NT-1} \dot{u}(t_i, W_{t_i})[t_{i+1} - t_i] + \sum_{i=0}^{NT-1} u'(t_i, W_{t_i})[W_{t_{i+1}} - W_{t_i}] + \sum_{i=0}^{NT-1} \frac{1}{2}u''(t_i, W_{t_i})[W_{t_{i+1}} - W_{t_i}]^2 \quad (2.2)$$

we have dropped the little o terms for convenience. It is reasonable to expect that $[W_{t_{i+1}} - W_{t_i}]^2 \sim (t_{i+1} - t_i)$ as $t_{i+1} - t_i \rightarrow 0$ by the 1/2-space time scaling. This is in fact true, although it is not a trivial point to show, but we will assume it here for convenience. Therefore, we have Ito's formula,

$$u(T, W_T) = u(0, W_0) + \int_0^T \dot{u}(t, W_t)dt + \frac{1}{2} \int_0^T u''(t, W_t)dt + \int_0^T u'(t, W_t)dW_t \quad (2.3)$$

this formula is equivalent to the differential form

$$du(t, W_t) = [\dot{u}(t, W_t) + \frac{1}{2}u''(t, W_t)] dt + u'(t, W_t)dW_t \quad (2.4)$$

Ito's formula for functions of Stochastic processes measurable with respect to Brownian motion Now that we understand the equation

$$dZ_t = X_t dt + Y_t dW_t \quad (2.5)$$

we can reproduce (2.3), replacing $W_t \rightarrow Z_t$. To see how this plays out, return to (2.1)

$$u(t + \Delta t, Z_{t+\Delta t}) = u(t, Z_t) + \dot{u}(t, Z_t)[\Delta t] + u'(t, Z_t)[Z_{t+\Delta t} - Z_t] + \frac{1}{2}u''(t, Z_t)[Z_{t+\Delta t} - Z_t]^2 + o(\Delta t) + o([Z_{t+\Delta t} - Z_t]^2) \quad (2.6)$$

the differences in (2.6) are a discrete version of (2.5) i.e.

$$[Z_{t+\Delta t} - Z_t] \sim X_t[\Delta t] + Y_t[W_{t+\Delta t} - W_t]$$

and

$$[Z_{t+\Delta t} - Z_t]^2 \sim X_t^2[\Delta t]^2 + 2X_t Y_t[(\Delta t)(W_{t+\Delta t} - W_t)] + Y_t^2[W_{t+\Delta t} - W_t]^2$$

as before, $[W_{t+\Delta t} - W_t]^2 \sim \Delta t$ but the other terms vanish. Therefore, carrying out (2.2) we have the following version of Ito's formula,

$$u(T, Z_T) = u(0, Z_0) + \int_0^T \dot{u}(t, Z_t)dt + \frac{1}{2} \int_0^T u''(t, Z_t)Y_t^2 dt + \int_0^T u'(t, Z_t)dZ_t \quad (2.7)$$

which is equivalent to the differential form

$$du(t, Z_t) = [\dot{u}(t, Z_t) + \frac{1}{2}u''(t, Z_t)Y_t^2] dt + u'(t, Z_t)dZ_t \quad (2.8)$$

In both (2.7) and (2.8) the term dZ_t can be replaced by (2.5).

3 Product rule

To derive product rule, let's remind ourselves again about the product rule in the deterministic case.

$$\Delta[f(t)g(t)] = f(t + \Delta t)g(t + \Delta t) - f(t)g(t)$$

Adding zero to this equation gives us,

$$\begin{aligned} \Delta[f(t)g(t)] &= f(t + \Delta t)g(t + \Delta t) - f(t + \Delta t)g(t) + f(t + \Delta t)g(t) - f(t)g(t) \\ &= [f(t + \Delta t)g(t + \Delta t) - f(t)g(t + \Delta t) - f(t + \Delta t)g(t) + f(t)g(t)] \\ &\quad + [f(t + \Delta t)g(t) - f(t)g(t)] + [f(t)g(t + \Delta t) - f(t)g(t)] \\ &= [f(t + \Delta t) - f(t)] \cdot [g(t + \Delta t) - g(t)] \\ &\quad + [f(t + \Delta t) - f(t)]g(t) + [g(t + \Delta t) - g(t)]f(t) \end{aligned} \tag{3.1}$$

This obtains for small Δt ,

$$\Delta[f(t)g(t)] = [f'(t)g'(t)](\Delta t)^2 + [f'(t)g(t) + f(t)g'(t)](\Delta t)$$

Thus the $[\Delta t]^2$ term vanishes in the ratio

$$[f(t)g(t)]' = \frac{\Delta[f(t)g(t)]}{[\Delta t]} = f'(t)g(t) + f(t)g'(t)$$

Of course this is easily extended to $f_1(t, x(t)) = f(t)$.

Now do the same for stochastic processes for $i = 1, 2$,

$$dZ_t^{(i)} = X_t^{(i)} dt + Y_t^{(i)} dW_t$$

Reverting again to the discrete version,

$$\begin{aligned} \Delta[Z_t^{(1)} Z_t^{(2)}] &= [Z_{t+\Delta t}^{(1)} - Z_t^{(1)}] \cdot [Z_{t+\Delta t}^{(2)} - Z_t^{(2)}] \\ &\quad + [Z_{t+\Delta t}^{(1)} - Z_t^{(1)}] Z_t^{(2)} + [Z_{t+\Delta t}^{(2)} - Z_t^{(2)}] Z_t^{(1)} \end{aligned} \tag{3.2}$$

Again we determine which are the higher order terms which vanish, the differences on the second line in (3.2) become our \mathbf{d} terms and on the first line,

$$\begin{aligned} [Z_{t+\Delta t}^{(1)} - Z_t^{(1)}] \cdot [Z_{t+\Delta t}^{(2)} - Z_t^{(2)}] &= X_t^{(1)} X_t^{(2)} [\Delta t]^2 + [X_t^{(1)} Y_t^{(2)} + Y_t^{(2)} X_t^{(1)}] (\Delta t) [W_{t+\Delta t} - W_t] \\ &\quad + Y_t^{(1)} Y_t^{(2)} [W_{t+\Delta t} - W_t]^2 \end{aligned}$$

and the only term which survives is the final term. There fore the product rule is

$$d[Z_t^{(1)} Z_t^{(2)}] = Z_t^{(1)} dZ_t^{(2)} + Z_t^{(2)} dZ_t^{(1)} + Y_t^{(1)} Y_t^{(2)} dt$$

4 Examples

Let's find the derivative of some terms.

4.1 A stock price with log Brownian forcing

Consider $u(t, x) = e^{at+bx}$. The derivatives are

$$\dot{u} = ae^{at+bx} = au, \quad u' = be^{at+bx} = bu \quad \text{and} \quad u'' = b^2e^{at+bx} = b^2u.$$

Therefore, for $Z_t = u(t, W_t)$, and applying (2.4) on u we have,

$$dZ_t = du(t, W_t) = (a + \frac{1}{2}b^2)u dt + budW_t = (a + \frac{1}{2}b^2)Z_t dt + bZ_t dW_t$$

Notice if we set $b = 0$ we have a deterministic process increasing exponentially at rate a . On the other hand setting $a = 0$ gives Brownian forcing proportional to the value of the process. This forms the stochastic model of a stock with forcing independent on disjoint intervals.

Thus, if we expect the value of the stock to drift at a rate of μ at a given time in proportion to its current value and fluctuate at intensity σ relative to its current value the stochastic differential equation of the stock is

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

which has solution

$$S_t = e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}.$$

Stock with continuous dividends Suppose the stock pays $\delta \frac{1}{N} S(t)$ on each time step. Then in the continuous model we have

$$dS_t = (r - \delta)dt + \sigma dW_t$$

So that

$$S_t = S(0)e^{(r - \delta - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

4.2 holdings of stock and bond

Suppose we have holdings of $x(t), y(t)$ so that our port folio is

$$V(t) = x(t)S(t) + y(t)A(t)$$

Suppose $x(t)$ satisfies

$$dx(t) = u(t, S(t))dt + v(t, S(t))dS(t) = [u(t, S(t)) + rv(t, S(t))S(t)]dt + \sigma v(t, S(t))S(t)dW(t)$$

Then,

$$d[x(t)S(t)] = S(t)dx(t) + x(t)dS(t) + \sigma^2 S^2(t)v(t, S(t))dt$$

Suppose $y(t) = z(t)/A(t)$, with $A(t) = e^{rt}$ and

$$dz(t) = \xi(t, S(t))dt + \zeta(t, S(t))dS(t) = [\xi(t, S(t)) + r\zeta(t, S(t))S(t)]dt + \sigma\zeta(t, S(t))S(t)dW(t)$$

Note,

$$dy(t) = ry(t)dt + \frac{1}{A(t)}dz(t)$$

Then,

$$d(y(t)A(t)) = -ry(t)A(t)dt + A(t)dy(t)$$

5 Portfolios

At time t , holding is $x(t), y(t)$ portfolio has value,

$$V(t, S(t)) = x(t)S(t) + y(t)A(t)$$

For $S(t)$ value of stock and $A(t) = e^{rt}$ the value of the bond.

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \quad dA(t) = re^{rt}dt$$

European option/replicating portfolio Let V replicate a European option.

Recall in discrete version: $S^\pm(t) = S(t)(1 + r\frac{1}{N} \pm \sigma\frac{1}{\sqrt{N}})$

$$x(t) = \frac{V(t + 1/N, S^+(t)) - V(t + 1/N, S^-(t))}{S^+(t) - S^-(t)}$$

that is $x(t)$ is derivative of V with respect to S value,

$$x(t) = V'(t, S(t)).$$

On the other hand, in the discrete version,

$$y(t) = \frac{S^+(t)V(S^-(t)) - S^-(t)V(S^+(t))}{A(t + 1/N)(S^+(t) - S^-(t))}$$

then ‘adding by zero’

$$y(t) = \frac{S^+(t)V(S^-(t)) - S^+(t)V(S^+(t)) + S^+(t)V(S^+(t)) - S^-(t)V(S^+(t))}{A(t + 1/N)(S^+(t) - S^-(t))}$$

Thus, taking the limit we have

$$y(t) = \frac{z(t)}{A(t)} = \frac{V(t, S(t)) - S(t)V'(t, S(t))}{A(t)}$$

Stochastic formula From Ito formula,

$$dx(t) = dV'(t, S(t)) = [\dot{V}' + \frac{1}{2}\sigma^2 S^2(t)V''']dt + V''dS(t)$$

or

$$dx(t) = dV'(t, S(t)) = [\dot{V}' + rS(t)V'' + \frac{1}{2}\sigma^2 S^2(t)V''']dt + \sigma S(t)V''dW(t)$$

Differential Change in value, use Ito formula,

$$dV(t) = x(t)dS(t) + S(t)dx(t) + \sigma^2 S^2(t)V''dt + y(t)dA(t) + A(t)dy(t).$$

Assume self financing:

$$0 = S(t)dx(t) + \sigma^2 S^2(t)V''dt + A(t)dy(t)$$

Then

$$dV(t) = x(t)dS(t) + y(t)dA(t) = rV(t, S(t))dt + \sigma V'(t, S(t))S(t)dW(t)$$

Thus

$$d(e^{-rt}V(t)) = (-re^{-rt}V(t) + re^{-rt}V(t))dt + \sigma e^{-rt}S(t)V'dW(t) = \sigma e^{-rt}S(t)V'dW(t)$$

It follows that $\tilde{V}(t) = e^{-rt}V(t)$ is a martingale so that

$$\tilde{V}(0) = \mathbb{E}(\tilde{V}(t))$$

for any t .