Model for general securities Market.
+ Fixed Interest.

$S_i(t) \equiv$ price of i^{th} security @ time t. $i=1,\dots,m$

We can also denote bonds as $A(t)$, but sometimes it is easier to suppress that notation.

Let $x_i(t) \equiv$ # of securities held at begining of timestep. $t-1 \to t$ ($\therefore$ $x_i(t)$ depends only on $\mathcal{F}_{t-1}$).

$y \equiv \$$ in bonds

$\therefore$ Value of initial portfolio @ time 0:

$$V(0) = \left(\sum_{i=1}^{n} x_i(1)\, S_i(0)\right) + y(1).$$

$S_i(0)$ given, $S_i(t)$ "random" at times $k \leq t-1$.
$S_i(t)$ value in $\mathcal{F}_t \equiv$ info up to time t.

$$V(1) = \left(\sum_{i=1}^{n} x_i(1)\, S_i(1)\right) + y(1)(1+r)$$

"self financing" $\longrightarrow$

$$= \left(\sum_{i=1}^{n} x_i(2)\, S_i(1)\right) + y(2)$$

As usual we assume $x_i(t)$ + $y(t)$ may be any real $\mathbb{R}$ number.

And we assume $S_i(t) > 0$

The sequence of portfolios $(x_1(t), \ldots, x_n(t), y(t))$ are the investment strategy. (holdings)

Weath at time t,

$$V(t) = \sum_{i=1}^{n} x_i(t+1)\, S_i(t) + y(t+1)$$

$$= x_S^T(t+1) \cdot S(t) + y(t+1)$$

$(x(t+1), y(t+1))$ given by $\mathcal{F}_t$ ~ predictable

(we cannot base current holdings on future outcomes).

Note that given a sequence of holdings $x(t)$ we can always create a self financing portfolio by balancing w/ the bond $y(t)$.

ie

$$y(t+1) = x^T(t) \cdot S(t) + y(t)(1+r) - x^T(t+1) \cdot S(t)$$

$$= \left(x^T(t) - x^T(t+1)\right) \cdot S(t) + y(t)(1+r).$$

The sequence is admissible if $V(t) \geq 0 \;\forall t$.

No Arbitrage: There is no Admissible strategy st $V(0) = 0$ & $\mathbb{P}(V(t) > 0) > 0$.

Risk Neutral Measure (Arbitrage free measure)

Each time step:

$$S_i(t+1) = S_i(t)\left(1 + M_i(t+1)\right)$$

$$\mathbb{P}(M_i(t+1) = m_{i,j}) = p_j \qquad j = 1, \dots, n$$

Return Matrix.

K so that $K_{ij} = m_{ij} - r$

Recall: No Arbitrage if and only if there exists p^* so that $Kp^* = 0$

for positive probability vector p^*.

(First Fundamental Theorem)

Thus if such p^* exists:

$$\begin{aligned}
\mathbb{E}^* S_i(1) &= \mathbb{E}^* S_i(0)(1 + M_i(1)) \\
&= S_i(0) + r S_i(0) - \mathbb{E}^* S_i(0)(M_i(1) - r) \\
&= S_i(0) + r S_i(0) - S_i(0) \underbrace{\mathbb{E}^*(M_i(1) - r)}_{Kp^* = 0} \\
&= (1 + r) S_i(0).
\end{aligned}$$

As we have seen, not all models have unique risk neutral measures (when they even exist!).

Binomial model ~ No risk neutral measure or unique risk neutral measure.

Trinomial model ~ No risk neutral measure or ∞'ly many risk neutral measure.

Trinomial w/ 2 securities may have unique risk neutral measure.

This relates to issue of European claimes being replicable:

We say a model is <u>complete</u> if all European claims can be replicated.

Recall: Replicating ~~optio~~ claim: (one step)

$\exists\ x_1, \ldots, x_m, y$ st

for any $\omega \in \Omega$

$$x \cdot S(1) + y(1+r) = H(\omega)$$

where $H(\omega)$ is the ~~value~~ payoff value of the claime at time 1.

Second Fundamental Theorem,

A model is Arbitrage free + complete

if and only if

there is a unique risk neutral probability. (RNM)

Proof:

Suppose the model is complete + Arbitrage free,

Show there is unique (RNM).

There exists some RNM As it is Arbitrage free,

we have to show uniqueness.

For any $\omega \in \Omega$ let H_ω be defined as; for $\eta \in \Omega$

$$H_\omega(\eta) = 1_{\{\omega\}}(\eta) = \begin{cases} 1 & \text{if } \eta = \omega \\ 0 & \text{otherwise.} \end{cases}$$

Let (x_ω, y_ω) be the portfolio

so that

$$x_\omega \cdot S(1) + y_\omega(1+r) = H_\omega \quad \left(\text{for all outcomes } \eta \in \Omega\right)$$

Thus

$$p_j^* = \mathbb{E}^*(H_{\omega_j}) = \mathbb{E}^*\left(x_{\omega_j} \cdot S(1) + y_{\omega_j}(1+r)\right)$$

$$= \sum_i x^i_{\omega_j} \, \mathbb{E}^* S_i(1) \quad + y_{\omega_j}(1+r)$$

$$= \sum_i x^i_{\omega_j} \, S_i(0)(1+r) \quad + y_{\omega_j}(1+r)$$

No arbitrage $\Rightarrow$ any $\hat{x}_{\omega_j}, \hat{y}_{\omega_j}$ w/ same payoff has same value at time zero $\Rightarrow p^*$ is unique.

Now assume RNM is unique, p^*, $0<p_i^*<1$

Suppose H is a ~~security~~ Derivative w/ values depending only on $S(1)$ but has no Replicating portfolio.

The space of all replicable ~~portfolios are given by~~ Derivatives are given by:

$\cancel{A_{0j} = 1+r}$ $\cancel{A_{ij}}$

$A_{ji} = S_i(0)(1+m_j^i)$ i^{th} sec, j^{th} outcome.

RNM implies $(p^*)^T A = \begin{pmatrix} S(0)(1+r) \\ (1+r) \end{pmatrix}$ $\longrightarrow$ $A_{jm+1} = 1+r$

A "Value matrix"

$$\mathcal{R} = \left\{ A\binom{x}{y}, \ x \in \mathbb{R}^m, \ y \in \mathbb{R} \right\}$$

$\mathcal{R} \in \mathbb{R}^n$ is space of all payoffs which can be replicated

H as vector in $\mathbb{R}^n$ ($H_i = H_{\omega_i} \equiv$ value under i^{th} outcome)

$H \notin \mathcal{R}^n$.

Notice $\mathbb{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathcal{R}$, $\mathbb{1}$ is vector of all 1's. portfolio $x=0, y=\frac{1}{1+r}$.

Let v be a vector $\perp$ to $\mathcal{R}$. Then $v \cdot \mathbb{1} = 0$, ie $\sum_j v_j = 0$.

Let $\max_j |v_j| \le 1$; $\delta \le \min p_j^*, \min 1-p_j^*$

let, $\varepsilon < \delta/2$.

then $\tilde{p}_j^* = p_j^* + \varepsilon v_j$ is a RNM.

Note: $(\tilde{p}^*)^T A = \begin{pmatrix} S(0)(1+r) \\ (1+r) \end{pmatrix}$ since $v^T A = 0$

Thus, We have

No arbitrage iff $Kp^* = 0$ for some p^*

& then $\mathbb{E}^* S(1) = (1+r)\, S(0)$.

If the p^* is unique then and only then can all claims be replicated.

2 stock binomial

$$S_1(1) = S_1(0)\,(1+r \pm \varepsilon)$$

$$S_2(1) = S_2(0)\,(1+r \pm \delta)$$

$$p^* = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tfrac{1}{4}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \varepsilon & \varepsilon & -\varepsilon & -\varepsilon \\ \delta & -\delta & \delta & -\delta \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}$$

$\cancel{p_1 = p_2 \quad p_3 = p_4}$

$p_1 + p_3 = p_2 + p_4$

$p_1 + p_2 = p_3 + p_4$ $\quad \cancel{2p_1 = 2p_3 = 0}$ $\quad$ ~~∴ RNM IS UNIQUE~~

$p_3 - p_2 = p_2 - p_3$

$p_3 = p_2$

$p_1 = p_4$

$$p = \begin{pmatrix} s \\ t \\ t \\ s \end{pmatrix}$$

∴ RNM not unique
not all options can be priced.