Martingale

The Arbitrage Free measure has the special property that the discounted stock value is a Martingale wrt the Arbitrage free measure.

$$\frac{1}{1+r}\mathbb{E}^*\left(S_{t+1}\,\middle|\,\mathcal{F}_t\right) = \frac{1}{1+r}\mathbb{E}^*\left((1+M_{t+1})\,S(t)\,\middle|\,\mathcal{F}_t\right)$$

$$M_t = m_1 \text{ or } m_2 .$$

$$= \frac{1}{1+r}\mathbb{E}^*\left(1+M_{t+1}\right)\, S(t)$$

$$= \frac{1}{1+r}\left\{(1+m_1)\frac{m_2-r}{m_1-m_2} + (1+m_2)\frac{r-m_1}{m_1-m_2}\right\} S(t)$$

$$= 1 * S(t) = S(t) \quad \checkmark$$

Defn

A Martingale (wrt measure μ) is a random process X_t st

$$\mathbb{E}_\mu\left(X_t \mid \mathcal{F}_{t-1}\right) = X_t .$$

Let Y_t , $t=0,1,2,\ldots$ be a sequence of R.V. w/ final value Y_N.

eg option value.

We can easily construct a ~~■~~ Martingale by:

$$X_t = \mathbb{E}^*(Y_N \mid \mathcal{F}_t)$$

then

$$\mathbb{E}^*(X_{t+1} \mid \mathcal{F}_t) = \mathbb{E}^*\left(\mathbb{E}^*(Y_N / \mathcal{F}_{t+1}) / \mathcal{F}_t\right) = \mathbb{E}^*(Y_N \mid \mathcal{F}_t) = X_t.$$

integrat everything outside $\mathcal{F}_{t+1}$ (after time $t+1$)

integrate everything outside $\mathcal{F}_t$ (after time t)

✓.

Purpose of martingale in Math finance:

Value of ~~martingale~~ Option sometimes given by expected value of Mg under given stopping time

optimal

Example

~~"The betting martingale"~~
~~The martingale betting strategy.~~

play game of flipping coin

* heads gain \$1
* tails pay \$1

} I.e. buy bets for \$1 if you win collect \$2

Let Y_k be the number of bets purchased on the k^{th} flip.

Let X_k be the payment of a bet due to flip, ie

$X_k = 1$ if heads & $X_k = -1$ if tails.

Then value at N^{th} flip is

$$V_N = X_1 Y_1 + X_2 Y_2 + \cdots + X_N Y_N .$$

Note

$$\begin{aligned} \mathbb{E}(V_{k+1} \mid \mathcal{F}_k) &= X_1 Y_1 + \cdots + X_k Y_k + \mathbb{E}(X_{k+1} Y_{k+1} \mid \mathcal{F}_k) \\ &= X_1 Y_1 + \cdots + X_k Y_k + Y_{k+1} \mathbb{E}(X_{k+1}) \\ &= X_1 Y_1 + \cdots + X_k Y_k \end{aligned}$$

we assume
Y_{k+1} depends only on info in $\mathcal{F}_k$

Thus for any such seq Y_k, (V_k) is Mg.

Moreover, for any N

$$\mathbb{E}(V_N) = V_0 = 0 \therefore \text{"fair"}$$

Constant N is most basic stopping time (play for fixed number of bets)

We want to see where/how to extend to more stopping times.

Now let us consider
"Martingale Betting Strategy"

Point: Not all Mg are "created equal".

Strategy: * first flip Buy 1 Bet
& if you lose do not stop betting,

* On the k^{th} flip Buy 2^{k-1} bets
if you lose do not stop betting.

* As soon as you win, stop betting.
* This step τ is "stopping time".
- find V_τ.

At k flips if $\tau > k$

$$V_k = -1-2-4 \cdots\cdots 2^{k-1} = -2^k+1$$

At k flips $\tau = k$

$$V_k = -1-2-4 \cdots\cdots 2^{k-2} + 2^k = 1$$

$\therefore$ for any sequence (X_i) we have

$$V_\tau = 1.$$

Thus in this case

$$0 = V_0 \neq \mathbb{E}(V_\tau) = 1$$

When Mg is "well behaved" we can determine value at end of Mg.

Ie. exercize time in American Option is example of "stopping time".

"Def^n" A stopping time is $\tau : \Omega \to \mathbb{T}$ st if $\tau = t$ then info is in $\mathcal{F}_t$.

(Ie we only need to know information up to time t to decide if we should stop @ t.)

Techincally Mg betting strategy is stopping time.

Optimal Sampling theorem.

Suppose $X_0, X_1, X_2, \cdots$ is Mg. wrt $\mathcal{F}_t$

+ τ is some stopping time

w/ $\mathbb{P}(\tau < \infty) = 1$

$\mathbb{E}(|X_\tau|) < \infty$

$\lim\limits_{n \to \infty} \mathbb{E}(X_t \mathbb{1}_{\{\tau > t\}}) = 0$.

then $\mathbb{E}(X_\tau) = \mathbb{E}(X_0)$

Eg walk on line.

Let $X_0 = 1, \dots, N-1$ be initial position of "Random Walk"

$$\mathbb{P}(X_{t+1} = X_t + 1) = 1/2$$
$$\mathbb{P}(X_{t+1} = X_t - 1) = 1/2$$

Let τ be first time t st $X_t = 0$ or $X_t = N$

$$\tau = \min\{t : X_t = 0, N\}.$$

Apply O.S.T. $q = \mathbb{P}(X_\tau = N)$

$$X_0 = q N + (1-q)\, 0$$

$$q = \frac{X_0}{N}.$$

Notice in Mg betting $|V_k| = 2^k - 1$ if $\tau > k$

$+ \; \mathbb{P}(\tau > k) = (1/2)^k$

$\therefore \; \mathbb{E}(V_k \mathbb{1}\{\tau > k\}) = (\tfrac{1}{2})^k (2^k - 1) \sim 1.$

thus ~~Optional st~~

Optimal stopping does not apply.