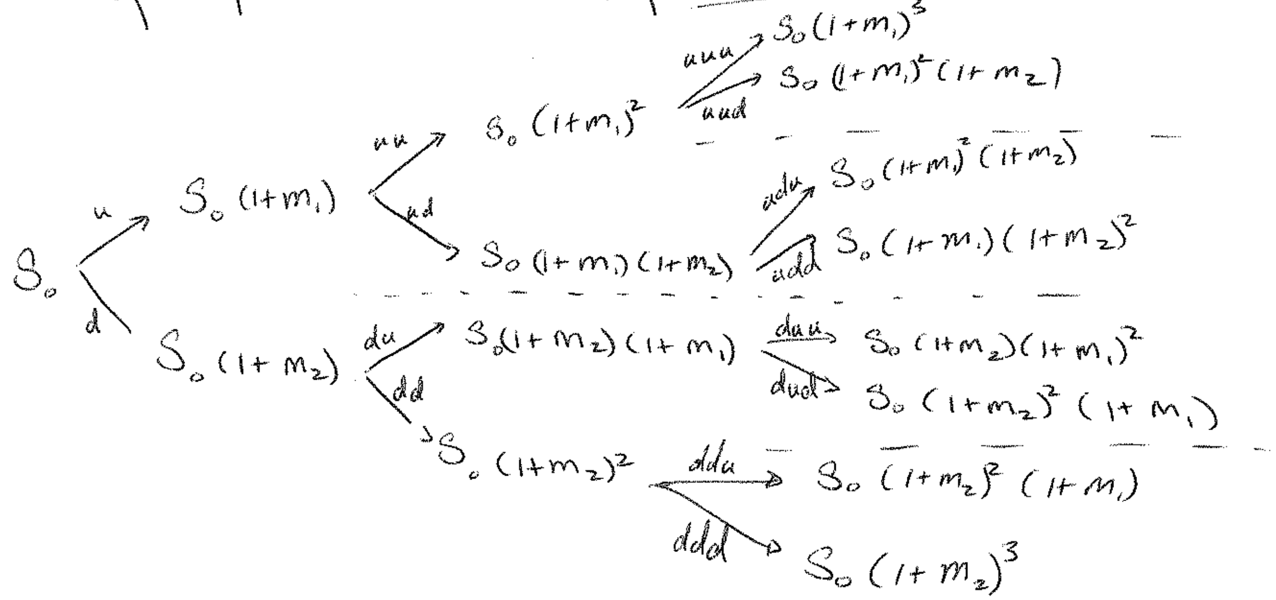
Let us consider the sequence of 'experiment's

in a three step model.

At each step the experiment is the outcome of the price on the next step. At any time step prob of moving up is p.



Outcome at first steps: u, d

——— second ——: (uu, ud) (du, dd)

$M_1 = u$ $M_1 = d$

——— third —:

$M_2 = uu$ $M_2 = ud$

uud uuu udu udd

$M_2 = du$ $M_2 = dd$

duu dud ddu ddd

'reverse grouping'

$\therefore$ Let outcome at first step be $B_u = (uuu, uud, udu, udd)$

$B_d = (duu, dud, ddu, ddd)$

Outcome at second step is

$$B_{uu} = \{uuu, uud\}$$

$$B_{ud} = \{udu, udd\}$$

$$B_{du} = \{dud, duu\}$$

$$B_{dd} = \{ddu, ddd\}$$

Outcome at third step is

B_{uuu}

B_{uud}

B_{udu}

B_{udd}

B_{duu}

B_{dud}

B_{ddu}

B_{ddd}.

$\Omega = \{uuu, uud, udu, udd, duu, dud, ddu, ddd\}$.

Ω can be partitioned into subsets of 1^{st} outcomes, 2^{nd} outcomes or 3^{rd} outcomes

Let $\mathcal{F}_t$ be the partition of Ω up to t^{th} outcome.

thus, $\mathcal{F}_1 = \{B_u, B_d\}$

$\mathcal{F}_2 = \{B_{uu}, B_{ud}, B_{du}, B_{dd}\}$

$\mathcal{F}_3 = \ldots$

The measure $\mathbb{P}$ can be restricted to any of these sets.

ie $\left.\begin{array}{l}\mathbb{P}(B_u) = p \\ \mathbb{P}(B_d) = 1-p\end{array}\right\}$ outcomes of step 1.

we formulate the outcomes of the first experiment w/ notation:

$$(\Omega, \mathcal{F}_1, \mathbb{P})$$

Next for the second step

$\mathbb{P}(B_{uu}) = p^2$; $\mathbb{P}(B_{ud}) = \mathbb{P}(B_{du}) = p(1-p)$; $\mathbb{P}(B_{dd}) = (1-p)^2$

these outcomes are $(\Omega, \mathcal{F}_2, \mathbb{P})$

finally on the last step we have

$\mathbb{P}(B_{uuu}) = p^3$ $\mathbb{P}(B_{uud}) = pp(1-p) \cdots$

outcomes $(\Omega, \mathcal{F}_3, \mathbb{P})$

Let us extend this definition to the general model on N steps, thus, $\mathbb{T} = \{0, 1, \dots, N\}$.

The space of outcomes is $\Omega = \{ w_1 \cdots w_N : w_i = u, d \text{ for } i = 1, \dots, N \}$.

ie length N words in letters u, d.

Let $v_i = u, d$ for $i = 1, 2, \dots, n$ for $n \leq N$.

then let

$$B_{v_1 v_2 \cdots v_n} = \{ w_1 w_2 \cdots w_N \in \Omega : w_i = v_i \text{ for } i = 1, \dots, n \}.$$

The partition at timestep n is

$$\mathcal{F}_n = \{ B_{v_1 v_2 \cdots v_n} : v_i = u, d \text{ for } i = 1, \dots, n \}.$$

Iε these are set of sets of words so that first n letters are fixed

Then measure space are $(\Omega, \mathcal{F}_n, \mathbb{P})$.

If the prob to move up is p_u
down is p_d.

$$\mathbb{P}(B_{v_1 v_2 \cdots v_n}) = p_{v_1} p_{v_2} \cdots p_{v_n}.$$

(~~Notation~~) In a minute we introduce arb. free measures. before this let us do simple example of expectation in given (apriori) probability measure.

Conditional expectation of 1 step price of stock:

$$\frac{1}{1+r}\mathbb{E}(S(1)) = \frac{1}{1+r}\left(p_1(1+m_1)S(0) + p_2(1+m_2)S(0)\right)$$

$$\frac{1}{(1+r)^2}\mathbb{E}(S(2)) = \left(\frac{1}{1+r}\right)^2\left(p_1^2(1+m_1)^2 + 2p_1p_2(1+m_1)(1+m_2) + p_2^2(1+m_2)^2\right)S_0.$$

$$\frac{1}{(1+r)^2}E(S(2)|\mathcal{F}_1) = \frac{1}{(1+r)^2}\,E\big((1+\underbrace{M(2)}_{m_1 \text{ or } m_2})S(1) \mid \mathcal{F}_1\big)$$

fixed wrt $\mathcal{F}_1$
$\therefore$ factor it out.

$$\left(\frac{1}{1+r}\right)^2 E\,S(2) = \left(\frac{1}{1+r}\right)^2 E(1+M(2)|\mathcal{F}_1)\,S(1)$$

$$= \left(\frac{1}{1+r}\right)^2\left(p_1(1+m_1) + p_2(1+m_2)\right)S(1).$$

Eg. $p_1 = \frac{1}{4}$, $p_2 = \frac{3}{4}$, $m_1 = .2$, $m_2 = 0$, $r = .1$

$S(0) = 100$,

$$\frac{1}{(1+r)^2}E(S(2)/\mathcal{F}_1) = \frac{S(1)}{1.1^2}\left((1.2)\tfrac{1}{4} + \tfrac{3}{4}\right) = S(1)(.87)$$

$$\left(\frac{1}{1+r}\right)^2\mathbb{E}(S(2)) = \frac{S(0)}{1.1^2}\left((1.2)^2\left(\tfrac{1}{4}\right)^2 + 2(1.2)\tfrac{1}{4}\tfrac{3}{4} + \left(\tfrac{3}{4}\right)^2\right) = 91.1$$

Now let us consider an option for N step Binomial model.

$S(0)$ given, $S(t+1) = S(t)(1+m)$, $m = m_1 \cdot m_2$.

$$\mathbb{P}\big(S(t+1) = S(t)(1+m_i)\big) = p_i$$

Possible outcomes for model $\Omega = \{\omega_1\omega_2 \cdots \omega_N : \omega_i = u, d\}$

where $\omega_i = u$ indicates increase on i^{th} step.

let $m_1 > r > m_2$.

Thus we have for any $\omega_1\omega_2 \cdots \omega_n$

$$\mathbb{P}\left(B_{\omega_1\omega_2\cdots\omega_n u} \,\middle|\, B_{\omega_1\omega_2\cdots\omega_n}\right) = p_1$$

$$\mathbb{P}\left(B_{\omega_1\omega_2\cdots\omega_n d} \,\middle|\, B_{\omega_1\omega_2\cdots\omega_n}\right) = p_2 .$$

Let H be payoff value $\therefore$ $H : \Omega \to \mathbb{R}^+$

$H(\bar{\omega}) \geq 0 \quad \forall \; \bar{\omega} \in \Omega$.

~~Eg.~~ if ω has k u's and $N-k$ d's we have

$\cancel{\mathbb{P}(\omega) = \mathbb{P} \; \mathbb{P} - H(\omega)}$

$$\mathbb{P}(\omega) = p_1^{k} \, p_2^{N-k} .$$

Thus we know value of contract w/payoff H @ N.
find value at $t=0$.

~European Case: Only may collect value @ maturity

$\therefore$ let $V(t)$ be value of the contract w/payoff H at time N. then $V(N)=H$.

As in 2 step ... find $V(N-1)$.

We find $V(N-1)$ for each outcome $\omega_1\omega_2\cdots\omega_{N-1}$.

that is, find $V(N-1)$ given B

But this is just 1 step Binomial since $V(N-1)$ has payoff depending on movement of stock.

$\therefore$ Let us write $V^{\omega_1\cdots\omega_n}$ for given outcome at time n.

then

$$V^{\omega_1\cdots\omega_{N-1}} = \frac{1}{1+r}\,\mathbb{E}^*\left(V(N)\mid B_{\omega_1\cdots\omega_{N-1}}\right)$$

$$= \frac{1}{1+r}\left(p_1^*\,V^{\omega_1\cdots\omega_N} + p_2^*\,V^{\omega_1\cdots\omega_N}\right)$$

$$p_1^* = \frac{r-m_2}{m_1-m_2}\;;\quad p_2^* = \frac{m_1-r}{m_1-m_2}.$$

More compactly $V(N-1) = \frac{1}{1+r}\,\mathbb{E}^*\left\{V(N)\mid \mathcal{F}_{N-1}\right\}$

Iterate

$$V(n) = \frac{1}{1+r}\,\mathbb{E}^*\left\{V(n+1) \middle| \mathcal{F}_n\right\}. \qquad n = N-1, \cdots, 1, 0.$$

Then we have,

$$V(0) = \left(\frac{1}{1+r}\right)^N \sum_{\bar{\omega} \in \Omega} \mathbb{P}^*(\bar{\omega})\, V^{\bar{\omega}}$$

Let Ω_k be set of $\bar{\omega}$ s.t. there are k steps up and $N-k$ steps down

$$V(0) = \left(\frac{1}{1+r}\right)^N \sum_{k=0}^{N} (p_1^*)^k (p_2^*)^{N-k} \sum_{\bar{\omega} \in \Omega_k} V^{\bar{\omega}}$$

This formula is true for all options (Asian Call) path dependent or pat independent.

What if options is path independent?

ie $H^{\bar{\omega}} = H(S(N))$. (Euro call)

If $H^{\bar{\omega}} = H(S(N)) = H(S^{\bar{\omega}})$.

path independent assm.

"same"

Then $\bar{\omega}, \bar{\omega}' \in \Omega_k \Rightarrow H^{\bar{\omega}} = H^{\bar{\omega}'}$

$V^{\bar{\omega}} = V^{\bar{\omega}'}$

how many $\bar{\omega} \in \Omega_k$? N total, k in first gp $N-k$ in 2nd gp

$\therefore$ (N choose k) $= \binom{N}{k} = \dfrac{N!}{k!(N-k)!}$

$\therefore$ path independent option formula —

$$V(0) = \left(\frac{1}{1+r}\right)^N \sum_{k=0}^{N} (p_1^*)^k (p_2^*)^{N-k} \binom{N}{k} H\left(S(0)(1+m_1)^k (1+m_2)^{N-k}\right)$$

Euro Call

given $S(0)$ + N + strikprice X

find formula:

$$H(S(N)) = (S(N) - X)^+$$

$\therefore$ find k st

$$S(0)(1+m_1)^k (1+m_2)^{N-k} > X$$

$$\left(\frac{1+m_1}{1+m_2}\right)^k > \frac{X}{S(0)\,(1+m_2)^N}$$

$$k > k_0 = \frac{\log \frac{X}{S(0)(1+m_2)^N}}{\log\left(\frac{1+m_1}{1+m_2}\right)}$$

$$C_E(0) = V(0) = \left(\frac{1}{1+r}\right)^N \sum_{N \geq k > k_0} (p_1^*)^k (p_2^*)^{N-k} \binom{N}{k} \left\{ S(0)(1+m_1)^k (1+m_2)^{N-k} - X \right\}$$

$$= \sum_{N \geq k > k_0} q_1^k q_2^{N-k} \binom{N}{k} S(0) - \sum_{N \geq k > k_0} \frac{(p_1^*)^k (p_2^*)^{N-k}}{(1+r)^N} \binom{N}{k} X$$

$$q_i = p_i^* \left(\frac{1+m_i}{1+r}\right)$$

Euro Put $C_E(0) - P_E(0) = S(0) - X\,B(0,T)$

$$\approx$$

$T = N$

$$C_E(0) - P_E(0) = S(0) - X \frac{1}{(1+r)^N}$$

$P_E(0) = C_E(0) - S(0) + X\frac{1}{(1+r)^N}$ from PC parity or,

$$P_E(0) = \frac{1}{(1+r)^N} \sum_{0 \le k \le k_0} (p_1^*)^k (p_2^*)^{N-k} \binom{N}{k} \left\{ X - S(0)\,(1+m_1)^k (1+m_2)^{N-k} \right\}$$

$$= \frac{X}{(1+r)^N} \sum_{0 \le k < k_0} \binom{N}{k} (p_1^*)^k (p_2^*)^{N-k} \; - \; S(0) \sum_{0 \le k \le k_0} q_1^k q_2^{N-k} \binom{N}{k}$$

Euro option, Call

$$C_E(0) = \sum_{N \ge k > k_0} q_1^k q_2^{N-k} \binom{N}{k} S(0) - \sum_{N \ge k > k_0} X \frac{(p_1^*)^k (p_2^*)^N}{(1+r)^N} \binom{N}{k}$$

$$q_i = p_i^* \left(\frac{1+m_i}{1+r} \right)$$

$$k_0 = \frac{\log\left(\frac{X}{S(0)(1+m_2)^N}\right)}{\log\left(\frac{1+m_1}{1+m_2}\right)} \qquad p_1^* = \frac{r-m_2}{m_1-m_2}, \quad p_2^* = \frac{m_1-r}{m_1-m_2}$$

Eg.

$S(0) = 100, \quad X = 115$

$m_1 = .03, \quad m_2 = 0. \quad r = .01$

maturity: $N = 6$

$$k_0 = \frac{\log 1.15}{\log 1.03} = 4.73$$

$$p_1^* = 1/3, \quad p_2^* = 2/3$$

$$q_1 = \frac{1.03}{3.03} \quad q_2 = \frac{2}{3.03}$$

$$C_E(0) = \sum_{k=5,6} q_1^k q_2^{N-k} \binom{N}{k} 100 - \sum_{k=5,6} \frac{115}{(1.01)^6} (1/3)^k (2/3)^{N-k} \binom{N}{k}$$

$$= 1.798 + .154 - 1.783 - .149 = .019$$