Binomial stock model.

Let $S(t)$ be Value of stock for $t \in \mathbb{T}$

for now let us consider just one step, so $\mathbb{T} = \{0, 1\}$

$S(0)$ is fixed and $S(1)$ takes on some positive value

$$S(1) : \Omega \to \mathbb{R}^+.$$

Only 2 values: $\Omega = \{\omega_1, \omega_2\}$.

$\mathbb{P}$ is prob measure on $\Omega$

$$p_i = \mathbb{P}(\omega_i)$$

$$S^{\omega_i}(1) = S(0)(1 + m_i) \quad \text{for} \quad m_1, m_2 > -1.$$

Let the interest over one step be $r$

ie if we put $y_0$ in bond value at time 1 is

$$y_1 = y_0(1+r)$$

Let $g$ be the value of the option at time 1

~ eg $g(s) = (s-K)^+$ for eurocall.

Let $x_0$ be the amount of stocks purchased at time 0 and the $y_0$ the amount of money placed in bonds at time 0.

We attempt to 'replicate' the value of the option at time 1 w/ portfolio $(x_0, y_0)$

$\therefore$ at time 1

$$\omega_i: \quad x_0 S^{\omega_i}(1) + y_0(1+r) = g(S^{\omega_i}(1))$$

$$x_0 S(0)(1+m_1) + y_0(1+r) = g(S(0)(1+m_1))$$

$$x_0 S(0)(1+m_2) + y_0(1+r) = g(S(0)(1+m_2))$$

$$x_0 = \frac{g(S^{\omega_1}) - g(S^{\omega_2})}{S^{\omega_1} - S^{\omega_2}}$$

$$\therefore \quad x_0 = \frac{g(S^{\omega_1}) - g(S^{\omega_2})}{S(0)(m_1 - m_2)}$$

Amt put into stock

$$x_0 S_0 = \frac{g(S^{\omega_1}) - g(S^{\omega_2})}{m_1 - m_2}.$$

But

$$x_0 S^{\omega_1} + y_0(1+r) = g(S^{\omega_1})$$

$$y_0(1+r) = g(S^{\omega_1}) - x_0 S^{\omega_1}$$

$$= g(S^{\omega_1}) - (1+m_1)\left(\frac{g(S^{\omega_1}) - g(S^{\omega_2})}{m_1 - m_2}\right)$$

$$y_0(1+r) = \frac{(m_1-m_2)g(S^{\omega_1}) - (1+m_1)\left(g(S^{\omega_1}) - g(S^{\omega_2})\right)}{m_1 - m_2}$$

$$y_0 = \frac{-(1+m_2)\,g(S^{\omega_1}) + (1+m_1)\,g(S^{\omega_2})}{(m_1 - m_2)(1+r)}$$

The time zero value of the portfolio $(x_0, y_0)$ is

$$V_0 = x_0 S(0) + y_0$$

Since $V_1 = g(S(1))$
for any outcome $\omega \in \Omega$
the value ~~the~~ an option at time 0 with value $g(S(1))$ at time 1 must be $V_0$.

Eg: Call option at time 0 w/ maturity at time 1; price @ time 0 is $V_0$.

$$V_0 = \frac{g(S^{\omega_1}) - g(S^{\omega_2})}{m_1 - m_2} - \frac{(1+m_2) g(S^{\omega_1}) - (1+m_1) g(S^{\omega_2})}{(m_1 - m_2)(1+r)}$$

$$= \frac{r}{1+r} \frac{g(S^{\omega_1}) - g(S^{\omega_2})}{m_1 - m_2} - \frac{m_2 g(S^{\omega_1}) - m_1 g(S^{\omega_2})}{(m_1 - m_2)(1+r)}$$

Eg Consider Call optim price @ time 0 is

$$V_0 = \frac{r}{1+r} \frac{(S^{\omega_1}-K)^+ - (S^{\omega_2}-K)^+}{m_1 - m_2} - \frac{m_2 (S^{\omega_1}-K)^+ - m_1 (S^{\omega_2}-K)^+}{(m_1 - m_2)(1+r)}$$

<u>Eg</u> Strike, $K=110$, $m_2 = .05$, $m_1 = .15$

$S_0 = 100$, $r = .1$

$$V_0 = \frac{.1}{1.1} \frac{(115-110)}{.15-.05} - \frac{(.05)(115-110)}{(.15-.05)(1.1)}$$

$$= 5/1.1 - \frac{1}{2}(5/1.1) = \frac{1}{2}(5/1.1) = \frac{225}{99}$$

$$= 2.\overline{27}$$

Suppose we wish to buy the stock at time 1
But do not have funds. Instead, buy call.
price of call at time 0 is \$ 2.27

At time 1, w/prob $p_1$ value is 115
— $p_2$ — 105

Say $p_1 = p_2 = \frac{1}{2}$.

Variance of prices at time $t=1$ } is $\mathbb{E}\left((S(\omega)-110)^2\right) = 5^2$

Variance of cost when hedging w/ call } in $\omega_1$ we pay
$(2.27)(1.1) + 110$ @ time 1
in $\omega_2$ we pay
$(2.27)(1.1) + 105$ @ time 1.

$D(\omega_1) = 2.5 + 110 = 112.5$
$D(\omega_2) = 2.5 + 105 = 107.5$

Variance of cost is } $\mathbb{E}\left(D(\omega) - 110\right)^2 = (2.5)^2$

$\therefore$ much less risky to purchase call on stock we plan to buy.

Note: we can rewrite formula for $V_0$ as:

$$V_0 = \frac{1}{1+r}\left\{ \frac{r-m_2}{m_1-m_2}\, g(S^{\omega_1}) + \frac{m_1-r}{m_1-m_2}\, g(S^{\omega_2}) \right\}$$

this probability measure

$$\cancel{p_1 = \mathbb{P}^*(\omega_1) = \frac{r-m_2}{m_1-m_2}}$$

$$\cancel{p_2 = \mathbb{P}^*(\omega_2)}$$

$$p_1 = \mathbb{P}^*(\omega_1) = \frac{r-m_2}{m_1-m_2}$$

$$p_2 = \mathbb{P}^*(\omega_2) = \frac{m_1-r}{m_1-m_2}$$

is called the arbitrage measure.

let us define $\tau(\omega) = \dfrac{\mathbb{P}^*(\omega)}{\mathbb{P}(\omega)}$.

then

$$\mathbb{E} f\tau = \sum f(\omega)\,\tau(\omega)\,\mathbb{P}(\omega) =$$

$$= \sum f(\omega)\,\mathbb{P}^*(\omega) = \mathbb{E}^*(f)$$

$\therefore$ $\tau$ transforms $\mathbb{P}$ into $\mathbb{P}^*$.

Note we have $V_0 = \frac{1}{1+r}\mathbb{E}\, g\tau = \frac{1}{1+r}\mathbb{E}^* g$.

Notice the model only 'works'

if $m_2 \le r \le m_1$, $m_1 \ne m_2$

otherwise if, say, $r < m_2 < m_1$

then $p_1 < 0$

and we can contract payoff g

st $g(S^{(1)}) \ge 0$

but $V_0(g(S(0))) < 0$

which contradicts no arbitrage.